A NOTE ON THE GENERALIZED FRACTIONAL INTEGRAL OPERATORS

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Abstract. We prove that the generalized fractional integral operators are bounded from a generalized Morrey space to another. Our result generalizes that of Chiarenza-Frasca [1] and links the results of Nakai [7] and of Eridani and Gunawan [3].

1. Introduction

The fractional integral operator or the Riesz potential $I_\alpha,\ 0 < \alpha < n$, defined by

$$I_\alpha f(x) = \int_{\mathbf{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy,$$

is known to be bounded from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$ provided that $\alpha = n/p - n/q,\ 1 < p < q < \infty$. The associated inequality

$$\|I_\alpha f\|_{L^q} \le C_{p,q}\|f\|_{L^p}$$

is known as the Hardy-Littlewood-Sobolev inequality. A proof of this inequality may be found in [9], p. 354. Recent applications are studied in [4].

In [8] (and in [1]), it is shown that I_α is bounded from the Morrey space $\mathcal{E}^{p,\beta}(\mathbf{R}^n)$ to $\mathcal{E}^{q,\gamma}(\mathbf{R}^n)$ where $\alpha = n/p - n/q, \alpha + \beta = \gamma,\ -n/p \le \beta < \alpha$. The Morrey space $\mathcal{E}^{p,\beta}(\mathbf{R}^n)$ consists of all locally integrable functions f on $\mathbf{R}^n$ for which

$$\sup_B \frac{1}{r^\beta}\left(\frac{1}{|B|}\int_B |f(y)|^p dy\right)^{1/p} < \infty$$

where the supremum is taken over all balls $B = B(a,r)$ in $\mathbf{R}^n$ and $|B|$ denotes the Lebesgue measure of B. Note that if $\beta = -n/p$, then $\mathcal{E}^{p,\beta}(\mathbf{R}^n) = L^p(\mathbf{R}^n)$, and so this result may be considered as a generalization of the earlier.

A further generalization of the above result is obtained by Nakai [5], in which I_α is shown to be bounded from the generalized Morrey space $\mathcal{M}_{p,\phi}(\mathbf{R}^n)$ to $\mathcal{M}_{q,\psi}(\mathbf{R}^n)$

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for $\alpha = n/p - n/q,\ 1 < p < q < \infty$ and appropriate functions ϕ and $p\psi$ with $\psi(r) = r^\alpha \phi(r)$. Here the generalized Morrey space $\mathcal{M}_{p,\phi} = \mathcal{M}_{p,\phi}(\mathbf{R}^n)$ is defined by

$$\mathcal{M}_{p,\phi} := \{f \in L^p_{\text{loc}}(\mathbf{R}^n) : \|f\|_{\mathcal{M}_{p,\phi}} < \infty\}$$

where

$$\|f\|_{\mathcal{M}_{p,\phi}} := \sup_B \frac{1}{\phi(r)} \left(\frac{1}{|B|} \int_B |f(y)|^p dy \right)^{1/p}.$$

In this case the classical result can be recovered by taking $\phi(r) = r^{-n/p}$.

In this note, we shall consider the generalized fractional integral operator T_ρ, defined for a suitable function $\rho : \mathbf{R}^+ \to \mathbf{R}^+$ by

$$T_\rho f(x) := \int_{\mathbf{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y)\, dy,$$

whenever this integral makes sense. Note that if $\rho(t) = t^\alpha,\ 0 < \alpha < n$, then $T_\rho = I_\alpha$ — the fractional integral operator discussed earlier.

In particular, we shall prove that I_ρ is bounded from the generalized Morrey space $\mathcal{M}_{p,\phi}$ to $\mathcal{M}_{q,\phi^{p/q}}$. This result generalizes that of Chiarenza-Frasca [1] and, at the same time, links the results of Nakai [7] and of Eridani and Gunawan [3], especially when one compares their assumptions on the functions $\rho,\ \phi$ and ψ.

2. Main Results

The generalized fractional integral operator I_ρ was first studied by Nakai [6]. The function ρ considered here is one that satisfies the doubling condition

$$\frac{1}{C} \le \frac{\rho(r)}{\rho(s)} \le C, \quad \text{for } \frac{1}{2} \le \frac{r}{s} \le 2.$$

Under appropriate conditions on ϕ and ψ, particularly the assumption that

$$\phi(r) \int_0^r \frac{\rho(t)}{t} dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t} dt \le C\psi(r), \quad \text{for all } r > 0,$$

Nakai [7] proved that I_ρ is bounded from $\mathcal{M}_{1,\phi}$ to $\mathcal{M}_{1,\psi}$. Later, Eridani [2] showed that, under similar assumptions on $\rho,\ \phi$ and ψ, I_ρ is bounded from $\mathcal{M}_{p,\phi}$ to $\mathcal{M}_{p,\psi}$ for $1 < p < \infty$. These results, however, cannot be viewed as a generalization of the known results for I_α.

Recently, Eridani and Gunawan [3] proved that I_ρ is bounded from $\mathcal{M}_{p,\phi}$ to $\mathcal{M}_{p,\phi^{p/q}}$ for $1 < p < q < \infty$, under some assumptions on ρ and ϕ. Precisely, they proved the following theorem.

Theorem A (Eridani & Gunawan). *Suppose that ρ is surjective and satisfies the doubling condition. Suppose also that ϕ satisfies the doubling condition, $\int_r^\infty \frac{\phi(t)^p}{t} dt \le C\phi(r)^p$, and*

$$\int_0^r \frac{\rho(t)}{t} dt + \rho(r)^{q/(q-p)} \int_r^\infty \frac{\rho(t)\phi(t)}{t} dt \le C\rho(r), \quad \text{for all } r > 0.$$

Then there exists $C_{p,q} > 0$ such that

$$\|T_\rho f\|_{\mathcal{M}_{q,\phi^{p/q}}} \le C_{p,q} \|f\|_{\mathcal{M}_{p,\phi}}$$

that is, T_ρ is bounded from $\mathcal{M}_{p,\phi}$ to $\mathcal{M}_{q,\phi^{p/q}}$, for $1 < p < q < \infty$.

Although Theorem A generalizes the result for I_α, the assumptions on ρ and ϕ seem to be different from those made by Nakai [7]. Our theorem below serves as a link between the above result of Eridani and Gunawan and that of Nakai.

Theorem B. *Suppose that ρ and ϕ satisfies the doubling condition. Suppose also that ϕ is surjective and satisfies $\int_r^\infty \frac{\phi(t)^p}{t} dt \le C\phi(r)^p$ and*

$$\phi(r) \int_0^r \frac{\rho(t)}{t} dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t} dt \le C\phi(r)^{p/q}, \quad \text{for all } r > 0,$$

Then there exists $C_{p,q} > 0$ such that

$$\|T_\rho f\|_{\mathcal{M}_{q,\phi^{p/q}}} \le C_{p,q} \|f\|_{\mathcal{M}_{p,\phi}}$$

that is, T_ρ is bounded from $\mathcal{M}_{p,\phi}$ to $\mathcal{M}_{q,\phi^{p/q}}$, for $1 < p < q < \infty$.

Note that if ρ satisfies the doubling condition, then for every integer k and $r > 0$ we have

$$\int_{2^k r}^{2^{k+1} r} \frac{\rho(t)}{t} \, dt \sim \rho(2^k r).$$

Further, it follows from the doubling condition that

$$\rho(r) \le C \int_0^r \frac{\rho(t)}{t} \, dt,$$

for every $r > 0$.

As in [3], we shall involve the well-known Hardy-Littlewood maximal operator M defined by

$$Mf(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| \, dy.$$

The operator M is known to be bounded on L^p for $1 < p \le \infty$ (see e.g. [9]). For our purpose, we shall use the fact that if ϕ satisfies the doubling condition and, for $1 < p < \infty$, $\int_r^\infty \frac{\phi(t)^p}{t} dt \le C\phi(r)^p$ for all $r > 0$, then there exists $C_p > 0$ such that

$$\|Mf\|_{\mathcal{M}_{p,\phi}} \le C_p \|f\|_{\mathcal{M}_{p,\phi}},$$

that is, M is bounded on $\mathcal{M}_{p,\phi}$ (see [5]).

Now here is the proof of the theorem. As usual, the letter C denotes positive constants, which may vary from line to line.

Proof of Theorem B. For every $x \in \mathbf{R}^n$ and $R > 0$, write

$$T_\rho f(x) = \int_{|x-y|<R} \frac{\rho(|x-y|)}{|x-y|^n} f(y) \, dy + \int_{|x-y|\ge R} \frac{\rho(|x-y|)}{|x-y|^n} f(y) \, dy = I_1(x) + I_2(x).$$

For $I_1(x)$, we have the following estimate:

$$\begin{aligned} |I_1(x)| &\le \int_{|x-y|<R} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| \, dy \\ &\le \sum_{k=-\infty}^{1} \int_{2^k R \le |x-y| < 2^{k+1} R} \frac{\rho(|x-y|)}{|x-y|^n} |f(y)| \, dy \\ &\le C \sum_{k=-\infty}^{1} \frac{\rho(2^k R)}{(2^k R)^n} \int_{|x-y|<2^{k+1}R} |f(y)| \, dy \\ &\le C \, Mf(x) \sum_{k=-\infty}^{1} \rho(2^k R) \\ &\le C \, Mf(x) \sum_{k=-\infty}^{1} \int_{2^k R}^{2^{k+1} R} \frac{\rho(t)}{t} \, dt \\ &= C \, Mf(x) \int_0^R \frac{\rho(t)}{t} \, dt \\ &\le C \, Mf(x) \, \phi(R)^{(p-q)/q}. \end{aligned}$$

Meanwhile, for $I_2(x)$, we have:

$$\begin{aligned}
|I_2(x)| &\le \int_{|x-y|\ge R} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\,dy \\
&\le \sum_{k=0}^{\infty}\int_{2^kR\le|x-y|<2^{k+1}R} \frac{\rho(|x-y|)}{|x-y|^n}|f(y)|\,dy \\
&\le C\sum_{k=0}^{\infty}\frac{\rho(2^kR)}{(2^kR)^n}\int_{|x-y|<2^{k+1}R}|f(y)|\,dy \\
&\le C\sum_{k=0}^{\infty}\frac{\rho(2^kR)}{(2^kr)^{n/p}}\left(\int_{|x-y|<2^{k+1}R}|f(y)|^p\,dy\right)^{1/p} \\
&\le C\,\|f\|_{\mathcal{M}_{p,\phi}}\sum_{k=0}^{\infty}\rho(2^{k+1}R)\phi(2^{k+1}R) \\
&\le C\,\|f\|_{\mathcal{M}_{p,\phi}}\sum_{k=0}^{\infty}\int_{2^kR}^{2^{k+1}R}\frac{\rho(t)\phi(t)}{t}\,dt \\
&= C\,\|f\|_{\mathcal{M}_{p,\phi}}\int_R^{\infty}\frac{\rho(t)\phi(t)}{t}\,dt \\
&\le C\,\|f\|_{\mathcal{M}_{p,\phi}}\phi(R)^{p/q}.
\end{aligned}$$

Summing the two estimates, we get

$$|T_\rho f(x)| \le C[Mf(x)\,\phi(R)^{(p-q)/q} + \|f\|_{\mathcal{M}_{p,\phi}}\phi(R)^{p/q}].$$

Assuming that f is not identically 0 and that Mf is finite everywhere, we can choose $R > 0$ such that $\phi(R) = Mf(x).\|f\|_{\mathcal{M}_{p,\phi}}^{-1}$, because ϕ is surjective. Hence, for every $x \in \mathbf{R}^n$, we have

$$|T_\rho f(x)|^q \le C\,Mf(x)^p\|f\|_{\mathcal{M}_{p,\phi}}^{q-p}.$$

The desired inequality then follows from this and the fact that the maximal operator M is bounded on $\mathcal{M}_{p,\phi}$. (QED)

We shall end our paper with a brief example. Let $1 < p < q < \infty$. Take $\rho(r) = r^\alpha l(r)^\beta$, where $\alpha = n/p - n/q$, $\beta > 0$, and $l(r) = -1/\log r$ for small r and $l(r) = \log r$ for large r, so that ρ satisfies the doubling condition. Then $\int_0^r \frac{\rho(t)}{t}\,dt \sim \rho(r)$ (see [6]). Now take $\phi(r) = r^{-n/p}l(r)^{\beta q/(p-q)}$. Then $\phi(r)^{(p-q)/q} = \rho(r)$, and one may check that ρ and ϕ satisfy the assumptions in Theorem B. Hence the associated operator T_ρ is bounded from $\mathcal{M}_{p,\phi}$ to $\mathcal{M}_{q,\phi^{p/q}}$.

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