L^2-ESTIMATES FOR SOME MAXIMAL FUNCTIONS

By Hendra Gunawan

SUNTO. Si presentono alcuni risultati sugli operatori massimali associati alle misure superficiali in $\mathbf{R}^n$. E. M. Stein [9] ha iniziato lo studio di quest'argomento, dimostrando una disuguaglianza *a priori* per la funzione massimale sferica, mediante l'utilizzo delle funzioni "g". In [3], M. Cowling e G. Mauceri hanno generalizzato il risultato di Stein. Qui si dà una dimostrazione diversa e forse più semplice della stima *a priori*, utilizzando la transformata di Mellin, come suggerito in un altro lavoro di Cowling e Mauceri [2].

INTRODUCTION

For a locally integrable function f on $\mathbf{R}^n$, one can define

$$\mathcal{M}f(x) \;=\; \sup_{r\in\mathbf{R}^+} \frac{1}{m(B^n)} \int_{B^n} |f(x-ry)|\, dy, \quad x \in \mathbf{R}^n$$

where B^n is the unit ball in $\mathbf{R}^n$ and $m(B^n)$ is its measure. This is known as the maximal function of f. One basic property of $\mathcal{M}f$ is that it satisfies the L^p-inequality

$$\| \mathcal{M}f \|_p \;\leq\; C_p \| f \|_p$$

whenever $f \in L^p(\mathbf{R}^n)$, $1 < p \leq \infty$. This was first studied by G. H. Hardy and J. E. Littlewood [4] for $n = 1$ and, for the general case, by N. Wiener [12] and by J. Marcinkiewicz

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and A. Zygmund [7]. Covering lemmas were used to prove the result. Marcinkiewicz [6] later proved this via an interpolation theorem. For a brief introduction to the theory of maximal functions we refer the reader to Stein [8, Ch. 1].

If f is smooth and rapidly decreasing, then one can replace B^n in the definition of $\mathcal{M}f$ by the unit sphere S^{n-1} and thus obtain the spherical maximal function of f. Stein [9] developed the theory and proved that, for $n \geq 3$, the *a priori* inequality still holds provided $f \in L^p(\mathbf{R}^n)$, $\frac{n}{n-1} < p \leq \infty$. The detailed proof of this can be found in Stein and S. Wainger [10]; it involves the theory of the Fourier transform and an argument using g-functions.

Cowling and Mauceri [2] reproved Stein's result and gave a new approach to the study of maximal functions. Here the maximal function of f, denoted now by $\mathcal{M}_\phi f$, is defined by

$$\mathcal{M}_\phi f(x) \;=\; \sup_{r\in\mathbf{R}^+} \, |\phi_r * f(x)|, \quad x \in \mathbf{R}^n$$

where ϕ is a distribution on $\mathbf{R}^n$ and ϕ_r is its dilate; Mellin transform techniques were used to tackle the problems.

A generalisation of Stein's treatment of spherical maximal functions was also found by Cowling and Mauceri [3]. One of their results, namely the L^2-estimate, asserts that for any smooth function f on $\mathbf{R}^n$, the inequality

$$\| \mathcal{M}_\phi f \|_2 \leq \; C \; \| f \|_2$$

holds whenever ϕ is compactly supported in $\mathbf{R}^n$ and $|\widehat{\phi}(r\xi)| \leq C\,(1+r)^{-\alpha}$, $r \in \mathbf{R}^+$, $\xi \in S^{n-1}$, for some $\alpha > \frac{1}{2}$. This was proved via a study of Riesz operators, the theory of fractional differentiation, some properties of Bessel functions, and an argument of g-functions.

The aim of this note is to prove a similar result for surface measures, using Mellin transform techniques.

Throughout this note we shall use the following notation. The Fourier transform of a function f on $\mathbf{R}^n$ is given by

$$\widehat{f}(\xi) \;=\; \int_{\mathbf{R}^n} f(x)\, e^{i\xi x} dx, \quad \xi \in \mathbf{R}^n.$$

$\mathcal{S}(\mathbf{R}^n)$ denotes the space of all smooth and rapidly decreasing functions on $\mathbf{R}^n$. By expressions $C,\ C_k,\ C_{k,l}$ etc. we mean various constants which may vary from line to line. These constants usually depend on n — the dimension of the ambient space.

Let ϕ be a distribution on $\mathbf{R}^n$. For $r \in \mathbf{R}^+$, we define the dilate ϕ_r by duality

$$\int_{\mathbf{R}^n} \phi_r(x)\, f(x)\; dx \;=\; \int_{\mathbf{R}^n} \phi(x)\, f(rx)\; dx, \quad f \in \mathcal{S}(\mathbf{R}^n).$$

One may observe that

$$\widehat{\phi}_r(\xi) \;=\; \widehat{\phi}(r\xi), \quad r \in \mathbf{R}^+,\ \xi \in S^{n-1}.$$

Now, for each $f \in \mathcal{S}(\mathbf{R}^n)$, define the associated maximal function $\mathcal{M}_\phi f$ by

$$\mathcal{M}_\phi f(x) \;=\; \sup_{r \in \mathbf{R}^+} |\phi_r * f(x)|, \quad x \in \mathbf{R}^n.$$

The focus of our interest is the L^2-inequality

$$\|\, \mathcal{M}_\phi f\, \|_2 \;\le\; C \|\, f\, \|_2, \quad f \in \mathcal{S}(\mathbf{R}^n).$$

Using the Mellin transform, we write

$$\widehat{\phi}_r(\xi) \;=\; \int_{\mathbf{R}} \psi(u,\xi)\; r^{iu}\; du, \quad r \in \mathbf{R}^+,$$

where

$$\psi(u,\xi) \;=\; \frac{1}{2\pi} \int_0^\infty \widehat{\phi}_r(\xi)\; r^{-1-iu}\; dr, \quad u \in \mathbf{R}.$$

(See [2] for more about the techniques.) For $s > 0$, one can observe that

$$\psi(u, s\xi) \;=\; s^{iu}\; \psi(u,\xi), \quad u \in \mathbf{R},\ \xi \in \mathbf{R}^n.$$

Moreover, from the theory of the Fourier transform, for all f in $\mathcal{S}(\mathbf{R}^n)$,

$$(\phi_r * f)\,\widehat{}\;\; = \;\widehat{\phi}_r\, \widehat{f}$$

whence

$$\begin{aligned} \phi_r * f \;&=\; (\widehat{\phi}_r\, \widehat{f}\,)\,\check{} \\ &=\; \left\{ \int_{\mathbf{R}} \psi(u,.)\, \widehat{f}\; r^{iu}\; du \right\}\check{} \\ &=\; \int_{\mathbf{R}} \left\{ \check{\psi}(u,.) * f \right\} r^{iu}\; du. \end{aligned}$$

(Here ˇ denotes the inverse Fourier transform.) Hence we find that

$$\mathcal{M}_\phi f(x) \;\le\; \int_{\mathbf{R}} |\check{\psi}(u,.) * f(x)|\, du, \quad x \in \mathbf{R}^n.$$

Applying Minkowski's inequality and Plancherel's theorem, we obtain

$$\begin{aligned}\| \mathcal{M}_\phi f \|_2 \;&\le\; \int_{\mathbf{R}} \| \check{\psi}(u,.) * f \|_2 \, du \\ &=\; \int_{\mathbf{R}} \| \psi(u,.)\, \widehat{f} \|_2 \, du \\ &\le\; \int_{\mathbf{R}} \| \psi(u,.) \|_\infty \, \| \widehat{f} \|_2 \, du \\ &=\; (2\pi)^{\frac{1}{2}} \| f \|_2 \int_{\mathbf{R}} \| \psi(u,.) \|_\infty \, du\end{aligned}$$

where

$$\| \psi(u,.) \|_\infty \;=\; \sup_{\xi \in S^{n-1}} |\psi(u,\xi)|, \quad u \in \mathbf{R}.$$

Thus, to have the L^2-estimate, it suffices to verify

$$\int_{\mathbf{R}} \| \psi(u,.) \|_\infty \;\, du \;<\; \infty.$$

But this would be satisfied when

$$|\psi(u,\xi)| \;\le\; C(1+|u|)^{-1-\delta}, \quad u \in \mathbf{R},\ \xi \in S^{n-1},$$

for some $\delta > 0$. With this approach, therefore, we attempt to find conditions on ϕ such that this inequality holds.

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I

In this part, we are concerned with some maximal operators associated to distributions on $\mathbf{R}^n$. Our results here are based on a study of Riesz operators, due to Cowling and Mauceri [3].

For a, b in $\mathbf{C}$, with $\mathrm{Re}(a) > 0$, and a distribution ϕ on $\mathbf{R}^n$, we define the Riesz operator $R_{a,b}$ via

$$(R_{a,b}\phi)\widehat{\ }(\xi) \;=\; \frac{2}{\Gamma(b)} \int_0^1 s^{a-1}\,(1-s^2)^{b-1}\,\widehat{\phi}(s\xi)\;ds, \quad \xi \in \mathbf{R}^n.$$

This clearly makes sense for $\mathrm{Re}(b) > 0$. Moreover, by analytic continuation, it extends to $\mathrm{Re}(b) > -N$ for any $N \in \mathbf{Z}^+$ (see [3] for justification).

The following lemma generalises [3, Lemma 1.2].

Lemma 1.1 Suppose ϕ is a compactly supported distribution on $\mathbf{R}^n$ and for $k = 0, 1, 2, \ldots$ there exist $\alpha_k > 0$ such that

$$\left|\frac{\partial^k}{\partial r^k}\widehat{\phi}(r\xi)\right| \;\leq\; C_k\,(1+r)^{-\alpha_k}, \quad r \in \mathbf{R}^+,\; \xi \in S^{n-1}.$$

Then for $\mathrm{Re}(a_k) > \alpha_k - k,\; k = 0, 1, 2, \ldots$, and any $\delta > 0$, we have

$$\left|\frac{\partial^k}{\partial r^k}(R_{a_k,b}\phi)\widehat{\ }(r\xi)\right| \;\leq\; C_k\,(1+r)^{-\alpha_k+m+\delta}, \quad r \in \mathbf{R}^+,\; \xi \in S^{n-1},$$

where $m = \max(0, -\mathrm{Re}(b))$.

Proof. For each $k = 0, 1, 2, \ldots$, we have, whenever $r \in \mathbf{R}^+,\; \xi \in S^{n-1}$,

$$\begin{aligned}\frac{\partial^k}{\partial r^k}(R_{a_k,b}\phi)\widehat{\ }(r\xi) \;&=\; \frac{2}{\Gamma(b)} \int_0^1 s^{a_k-1}\,(1-s^2)^{b-1}\,\frac{\partial^k}{\partial r^k}\widehat{\phi}(sr\xi)\;ds \\ &=\; \frac{2}{\Gamma(b)} \int_0^1 s^{a_k+k-1}\,(1-s^2)^{b-1}\,\frac{\partial^k}{\partial (sr)^k}\widehat{\phi}(sr\xi)\;ds.\end{aligned}$$

As in the proof of [3, Lemma 1.2], we find that for $\mathrm{Re}(a_k) > \alpha_k - k$, and $\delta > 0$,

$$\left|\frac{\partial^k}{\partial r^k}(R_{a_k,b}\phi)\widehat{\ }(r\xi)\right| \;\leq\; C_k\,(1+r)^{-\alpha_k+m+\delta}, \quad r \in \mathbf{R}^+,\; \xi \in S^{n-1},$$

where $m = \max(0, -\mathrm{Re}(b))$. $\square$

Now we have the following theorem.

Theorem 1.2 Suppose ϕ is a compactly supported distribution on $\mathbf{R}^n$ with $\widehat{\phi}(0) = 0$ such that

$$\left|\widehat{\phi}(r\xi)\right| \;\le\; C\,(1+r)^{-\epsilon}, \quad r \in \mathbf{R}^+,\ \xi \in S^{n-1}$$

and

$$\left|\frac{\partial}{\partial r}\widehat{\phi}(r\xi)\right| \;\le\; C\,(1+r)^{-1-\epsilon}, \quad r \in \mathbf{R}^+,\ \xi \in S^{n-1}$$

for some $\epsilon > 0$. Then we have the L^2-estimate

$$\|\, \mathcal{M}_\phi f \,\|_2 \;\le\; C \,\|\, f \,\|_2, \quad f \in \mathcal{S}(\mathbf{R}^n).$$

Proof. For each ξ in S^{n-1}, write

$$\widehat{\phi}(r\xi) \;=\; \int_{\mathbf{R}} \psi(u,\xi)\, r^{iu}\, du, \quad r \in \mathbf{R}^+,$$

where

$$\psi(u,\xi) \;=\; \frac{1}{2\pi}\int_0^\infty \widehat{\phi}(r\xi)\, r^{-1-iu}\, dr, \quad u \in \mathbf{R}.$$

We need to show that $|\psi(u,\xi)| \;\le\; C\,(1+|u|)^{-1-\delta}$, $u \in \mathbf{R}$, for some $\delta > 0$.

First, since $\widehat{\phi}(0) = 0$ and $\widehat{\phi}$ is differentiable, we have

$$|\widehat{\phi}(r\xi)| \;\le\; C\,r, \quad 0 \le r \le 1,$$

and thus, whenever $u \in \mathbf{R}$, we obtain

$$\begin{aligned} |\psi(u,\xi)| &\le \int_0^\infty |\widehat{\phi}(r\xi)|\; r^{-1}\, dr \\ &\le \int_0^1 C\; dr + \int_1^\infty C\, r^{-1-\epsilon}\, dr \\ &\le C. \end{aligned}$$

Next, we invoke the identity

$$\phi \;=\; R_{a+2b,-b} R_{a,b}\phi$$

where $\mathrm{Re}(a) > 0$, $\mathrm{Re}(a+2b) > 0$ (see [3, Lemma 1.3]), to have

$$\frac{\partial}{\partial r}\widehat{\phi}(r\xi) \;=\; \frac{2}{\Gamma(-b)}\int_0^1 s^{a+2b-1}\,(1-s^2)^{-b-1}\,\frac{\partial}{\partial r}(R_{a,b}\phi)\widehat{\ }(sr\xi)\; ds, \quad r \in \mathbf{R}^+.$$

Applying Fubini's theorem, we obtain

$$
\begin{aligned}
2\pi\, iu\, \psi(u,\xi) &= -\int_0^\infty \widehat{\phi}(r\xi)\,(-iu)\, r^{-1-iu}\; dr \\
&= -\int_0^\infty \widehat{\phi}(r\xi)\, \frac{\partial}{\partial r} r^{-iu}\; dr \\
&= \int_0^\infty \frac{\partial}{\partial r}\widehat{\phi}(r\xi)\; r^{-iu}\; dr \\
&= \int_0^\infty \frac{2}{\Gamma(-b)} \int_0^1 s^{a+2b-1}\,(1-s^2)^{-b-1}\,\frac{\partial}{\partial r}(R_{a,b}\phi)\widehat{\ }(sr\xi)\; ds\; r^{-iu}\; dr \\
&= \frac{2}{\Gamma(-b)} \int_0^1 \int_0^\infty \frac{\partial}{\partial(sr)}(R_{a,b}\phi)\widehat{\ }(sr\xi)\,(sr)^{-iu} d(sr)\, s^{a+2b-1+iu}(1-s^2)^{-b-1} ds \\
&= \frac{2}{\Gamma(-b)} \int_0^\infty \frac{\partial}{\partial t}(R_{a,b}\phi)\widehat{\ }(t\xi)\; t^{-iu}\; dt\; \int_0^1 s^{a+2b-1+iu}\,(1-s^2)^{-b-1}\; ds \\
&= \frac{\Gamma(\frac{a+2b+iu}{2})}{\Gamma(\frac{a+iu}{2})} \int_0^\infty \frac{\partial}{\partial t}(R_{a,b}\phi)\widehat{\ }(t\xi)\; t^{-iu}\; dt.
\end{aligned}
$$

For $\mathrm{Re}(a) > \epsilon$ and $b = -\frac{\epsilon}{4}$, we have

$$
|u\,\psi(u,\xi)| \;\le\; \left|\frac{\Gamma(-\frac{\epsilon}{4}+\frac{a+iu}{2})}{\Gamma(\frac{a+iu}{2})}\right| \int_0^\infty \left|\frac{\partial}{\partial t}(R_{a,-\frac{\epsilon}{4}}\phi)\widehat{\ }(t\xi)\right| dt \;\le\; C\,|u|^{-\frac{\epsilon}{4}}, \quad |u|>1,
$$

since $|\Gamma(c+id)| \sim C\, e^{-\frac{\pi}{2}|d|}\, |d|^{c-\frac{1}{2}}$ as $|d| \to \infty$ (see [11, p. 151]) and $\left|\frac{\partial}{\partial t}(R_{a,-\frac{\epsilon}{4}}\phi)\widehat{\ }(t\xi)\right| \le C\,(1+t)^{-1-\frac{\epsilon}{2}}$ for $t \in \mathbf{R}^+$ (by Lemma 1.1, with $\delta = \frac{\epsilon}{4}$). We therefore find that

$$
|\psi(u,\xi)| \;\le\; C\,|u|^{-1-\frac{\epsilon}{4}}, \quad |u|>1.
$$

Combining this with the previous inequality, we obtain

$$
|\psi(u,\xi)| \;\le\; C\,(1+|u|)^{-1-\frac{\epsilon}{4}}, \quad u \in \mathbf{R},\; \xi \in S^{n-1},
$$

as desired. □

Remark. The condition $\widehat{\phi}(0) = 0$ in the theorem can in fact be removed. When ϕ does not satisfy this condition, we can set $g = \phi - \varphi$, where $\varphi \in \mathcal{S}(\mathbf{R}^n)$ with $\widehat{\varphi}(0) = \widehat{\phi}(0)$. Thus g satisfies the hypothesis, and hence the conclusion holds for $\mathcal{M}_g f$. But this implies that the conclusion also holds for $\mathcal{M}_\phi f$ since $\mathcal{M}_\varphi f$ is known to be dominated by the standard maximal function of f. Moreover, the theorem is also true for a distribution ϕ which is not compactly supported but such that $\widehat{\phi}$ is C^1.

II

Here we deal with some maximal operators associated to surface measures on $\mathbf{R}^n$.

We first outline the result of Cowling *et al* [1]. Let $\mathcal{H}$ be a smooth compact convex hypersurface of class C^∞ in $\mathbf{R}^n$ whose tangent lines have order of contact at most $m < \infty$. Suppose μ is a smooth surface measure on $\mathcal{H}$. Then, it is shown in [1] that for all $r \in \mathbf{R}^+,\ \xi \in S^{n-1}$, we have

$$\widehat{\mu}(r\xi) \;=\; F(r,\xi)\, e^{-irp(\xi)\cdot\xi} + F(r,-\xi)\, e^{-irp(-\xi)\cdot\xi} + E(r,\xi)$$

where

$$\begin{aligned}
|p(\pm\xi)\cdot\xi| \;&\leq\; C,\\
\left|\frac{\partial^k}{\partial r^k} F(r,\pm\xi)\right| \;&\leq\; C_k\, r^{-k}\, K(r), \quad k = 0,1,2,\ldots\\
&\qquad \text{with } K(r) \;=\; O(r^{-\alpha}) \text{ as } r \;\to\; \infty, \text{ for some } \alpha > 0,\\
\text{and } \left|\frac{\partial^k}{\partial r^k} E(r,\xi)\right| \;&\leq\; C_{k,l}\, r^{-l}, \quad k,l = 0,1,2,\ldots .
\end{aligned}$$

By rearranging F and E for $0 \leq r \leq 1$, we may assume that $F(0,\pm\xi) = 0$ (or even $F(r,\pm\xi) = 0$, for $0 \leq r < s < 1$). If, in addition, we have for all $r \in \mathbf{R}^+,\ \xi \in S^{n-1}$,

$$\left|\frac{\partial^k}{\partial r^k} F(r,\pm\xi)\right| \;\leq\; C_k\, r^{-k-\alpha}, \quad k = 0,1,2,\ldots,$$

for some $\alpha > \frac{1}{2}$, we then say that μ is 'of good decay type'.

Our results are the following.

Theorem 2.1 Suppose that $\mathcal{H}$ is a hypersurface in $\mathbf{R}^n$, and that μ is a surface measure of good decay type on $\mathcal{H}$. Then we have the L^2-estimate

$$\| \mathcal{M}_\mu f \|_2 \;\leq\; C \| f \|_2, \quad f \in \mathcal{S}(\mathbf{R}^n).$$

Theorem 2.2 Let μ be a measure on $\mathbf{R}^n$, p be a bounded function on S^{n-1}, and $0 < \epsilon < 3$. If, for each ξ in S^{n-1}, there exists a measure $\nu = \nu_\xi$ on $\mathbf{R}$ such that

$$\begin{array}{lll} (i) & \widehat{\nu}(r) \;=\; \widehat{\mu}(r\xi)\, e^{irp(\xi)\cdot\xi}\,(1+r^2)^{\frac{1+\epsilon}{4}}, & r \in \mathbf{R}^+, \\ (ii) & \displaystyle\int_{\mathbf{R}} (1+s^2)\;|d\nu(s)| \;<\; C, & \end{array}$$

with C being independent of ξ, then we have the L^2-estimate

$$\| \mathcal{M}_\mu f \|_2 \;\leq\; C \| f \|_2, \quad f \in \mathcal{S}(\mathbf{R}^n).$$

We shall prove later that Theorem 2.1 is in fact a special case of Theorem 2.2. Now, Theorem 2.2 follows from the lemma below. We are indebted to Prof. G. Mauceri for suggesting its proof.

Lemma 2.3 Let μ be a measure on $\mathbf{R}^n$ such that $\widehat{\mu}(0) = 0$ and $\nabla\widehat{\mu}(0) = 0$, p be a bounded function on S^{n-1}, and $0 < \epsilon < 3$. Suppose that for each ξ in S^{n-1} there exists a measure $\nu = \nu_\xi$ on $\mathbf{R}$ such that

$$\begin{array}{lll} (i) & \widehat{\nu}(r) \;=\; \widehat{\mu}(r\xi)\, e^{irp(\xi)\cdot\xi}\,(1+r^2)^{\frac{1+\epsilon}{4}}, & r \in \mathbf{R}^+, \\ (ii) & \displaystyle\int_{\mathbf{R}} (1+s^2)\;|d\nu(s)| \;<\; C, & \end{array}$$

with C being independent of ξ. Define $\psi = \psi_\xi$ by

$$\psi(u) \;=\; \frac{1}{2\pi}\int_0^\infty \widehat{\mu}(r\xi)\, r^{-1-iu}\; dr, \quad u \in \mathbf{R}.$$

Then we have

$$|\psi(u)| \;\leq\; C\,(1+|u|)^{-1-\frac{\epsilon}{2}}, \quad u \in \mathbf{R},$$

with $C = C(\epsilon, p)\int_{\mathbf{R}}(1+s^2)\;|d\nu(s)|$.

Proof. We note first that $\widehat{\nu}$ is C^2 from (ii). We also have $\widehat{\nu}(0) = 0$ and $\widehat{\nu}\,'(0) = 0$. And, for $0 \le r \le 1$, we have

$$\begin{aligned}
|\widehat{\nu}(r)| &= |\widehat{\nu}(r) - \widehat{\nu}(0)| \qquad\quad (\text{as } \widehat{\nu}(0) = 0) \\
&= \left| r \, \frac{\partial \widehat{\nu}}{\partial r}(\rho) \right| \quad (\text{for some } 0 < \rho < r) \\
&\le r \int_{\mathbf{R}} |s| \; |d\nu(s)| \\
&\le r \int_{\mathbf{R}} (1 + s^2) \; |d\nu(s)| \\
&\le C\, r.
\end{aligned}$$

Now, from (i), we have

$$\widehat{\mu}(r\xi) \;=\; \widehat{\nu}(r)\, e^{-irq}\, (1 + r^2)^{-\frac{1+\epsilon}{4}}, \quad r \in \mathbf{R}^+,$$

with $q = p(\xi) \cdot \xi$. Thus, whenever $u \in \mathbf{R}$,

$$\begin{aligned}
|\psi(u)| &\le \int_0^\infty |\widehat{\mu}(r\xi)|\, r^{-1} \; dr \\
&\le \int_0^1 |\widehat{\nu}(r)|\, r^{-1} \; dr + \int_1^\infty |\widehat{\nu}(r)|\, (1 + r^2)^{-\frac{1+\epsilon}{4}}\, r^{-1} \; dr \\
&\le \int_0^1 C \; dr + \int_1^\infty \|\widehat{\nu}\|_\infty\, r^{-\frac{3}{2}} \; dr \\
&\le C \int_{\mathbf{R}} (1 + s^2) \; |d\nu(s)| \\
&< \infty.
\end{aligned}$$

From here on we assume $|u| > 1$. Take a smooth function φ on $\mathbf{R}^+$ such that $\varphi(r) = 0$ if $0 \le r \le \frac{3}{2}$ and $\varphi(r) = 1$ if $r \ge 2$. We then write

$$\begin{aligned}
2\pi\, \psi(u) &= \int_0^\infty \widehat{\nu}(r)\, e^{-irq}\, (1 + r^2)^{-\frac{1+\epsilon}{4}}\, r^{-1-iu} \; dr \\
&= \int_0^\infty \widehat{\nu}(r)\, e^{-irq}\, \{1 - \varphi(r)\}\, (1 + r^2)^{-\frac{1+\epsilon}{4}}\, r^{-1-iu} \; dr \\
&\quad + \int_0^\infty \widehat{\nu}(r)\, e^{-irq}\, \varphi(r)\, \left\{ (1 + r^2)^{-\frac{1+\epsilon}{4}} - r^{-\frac{1+\epsilon}{2}} \right\} r^{-1-iu} \; dr \\
&\quad + \int_0^\infty \widehat{\nu}(r)\, e^{-irq}\, \varphi(r)\, r^{-\frac{1+\epsilon}{2}}\, r^{-1-iu} \; dr \\
&= \psi_1(u) + \psi_2(u) + \psi_3(u), \quad \text{say.}
\end{aligned}$$

For $\psi_1(u)$, we have

$$-iu\,\psi_1(u) \;=\; \int_0^\infty \widehat{\nu}(r)\, e^{-irq}\, \{1-\varphi(r)\}\, (1+r^2)^{-\frac{1+\epsilon}{4}}\, \frac{\partial}{\partial r} r^{-iu}\; dr$$
$$\;=\; -\int_0^\infty \frac{\partial}{\partial r}\left[\widehat{\nu}(r)\, e^{-irq}\, \{1-\varphi(r)\}\, (1+r^2)^{-\frac{1+\epsilon}{4}}\right]\; r^{-iu}\; dr,$$

and similarly

$$-iu(1-iu)\,\psi_1(u) \;=\; \int_0^\infty \frac{\partial^2}{\partial r^2}\left[\widehat{\nu}(r)\, e^{-irq}\, \{1-\varphi(r)\}\, (1+r^2)^{-\frac{1+\epsilon}{4}}\right]\; r^{1-iu}\; dr.$$

But $1-\varphi(r) = 0$ for $r \geq 2$, and $\frac{\partial^2}{\partial r^2}\left[\widehat{\nu}(r)\, e^{-irq}\, \{1-\varphi(r)\}\, (1+r^2)^{-\frac{1+\epsilon}{4}}\right]$ is in fact a linear combination of products of derivatives of the four factors, which are all bounded for $0 \leq r \leq 2$ (in particular, the derivatives of $\widehat{\nu}$ are dominated by $\int_{\mathbf{R}}(1+s^2)\, |d\nu(s)|$). Thus we obtain

$$|iu(1-iu)\,\psi_1(u)| \;\leq\; \int_0^2 \left|\frac{\partial^2}{\partial r^2}\left[\widehat{\nu}(r)\, e^{-irq}\, \{1-\varphi(r)\}\, (1+r^2)^{-\frac{1+\epsilon}{4}}\right]\right|\; r\; dr \;\leq\; C,$$

which gives

$$|\psi_1(u)| \;\leq\; C\, |u|^{-2}.$$

For $\psi_2(u)$, we also have

$$-iu(1-iu)\,\psi_2(u) \;=\; \int_0^\infty \frac{\partial^2}{\partial r^2}\left[\widehat{\nu}(r)\, e^{-irq}\, \varphi(r)\, \left\{(1+r^2)^{-\frac{1+\epsilon}{4}} - r^{-\frac{1+\epsilon}{2}}\right\}\right]\; r^{1-iu}\; dr.$$

Put

$$R(r) \;=\; (1+r^2)^{-\frac{1+\epsilon}{4}} - r^{-\frac{1+\epsilon}{2}} \;=\; r^{-\frac{1+\epsilon}{2}}\, \left\{(1+r^{-2})^{-\frac{1+\epsilon}{4}} - 1\right\}, \quad r \in \mathbf{R}^+.$$

For $r \geq \frac{3}{2}$, we can expand $(1+r^{-2})^{-\frac{1+\epsilon}{4}}$ into its convergent Taylor series to get

$$(1+r^{-2})^{-\frac{1+\epsilon}{4}} \;=\; 1 - \frac{1+\epsilon}{4}\, r^{-2} + \frac{1+\epsilon}{4}\,\frac{5+\epsilon}{4}\,\frac{r^{-4}}{2!} - \cdots.$$

Hence

$$R(r) \;=\; -\frac{1+\epsilon}{4}\, r^{-\frac{5+\epsilon}{2}} + \text{terms of a lower power of } r, \quad r \geq \frac{3}{2}.$$

We therefore find that $|R(r)|$, $|\frac{\partial}{\partial r} R(r)|$ and $\left|\frac{\partial^2}{\partial r^2} R(r)\right| \;\leq\; C\, r^{-\frac{5+\epsilon}{2}}$ for $r \geq \frac{3}{2}$. And thus, since the derivatives of $\widehat{\nu}$, e^{-irq} and φ are bounded, we obtain

$$|iu(1-iu)\,\psi_2(u)| \;\leq\; \int_{\frac{3}{2}}^{\infty} C\, r^{-\frac{5+\epsilon}{2}}\; dr \;\leq\; C,$$

which gives

$$|\psi_2(u)| \;\leq\; C\,|u|^{-2}.$$

For $\psi_3(u)$, we write

$$\begin{aligned}
\psi_3(u) \;&=\; \int_0^{\infty} \widehat{\nu}(r)\, e^{-irq}\, \varphi(r)\, r^{-\frac{3+\epsilon}{2}-iu}\; dr \\
&=\; \int_0^{\infty} \widehat{\nu}(r)\, e^{-irq}\, \varphi(r)\, r^{z-1}\; dr \qquad\qquad (\text{with } z = -\frac{1+\epsilon}{2} - iu) \\
&=\; \int_{\mathbf{R}} \int_0^{\infty} \varphi(r)\, r^{z-1}\, e^{-ir(s+q)}\; dr\; d\nu(s) \quad (\text{as } \varphi(r) = 0 \text{ for } 0 \leq r \leq \frac{3}{2}).
\end{aligned}$$

Then, for fixed $s \in \mathbf{R}$, consider

$$I(z) \;=\; \int_0^{\infty} \varphi(r)\, r^{z-1}\, e^{-ir(s+q)}\; dr, \quad z \in \mathbf{C}.$$

We note that $I(z)$ continues analytically into $\mathrm{Re}(z) \leq 1$. For $0 < \mathrm{Re}(z) < 1$, we write

$$\begin{aligned}
I(z) \;&=\; \int_0^{\infty} \{\varphi(r) - 1\}\, r^{z-1}\, e^{-ir(s+q)}\; dr + \int_0^{\infty} r^{z-1}\, e^{-ir(s+q)}\; dr \\
&=\; I_4(z) + I_5(z), \quad \text{say.}
\end{aligned}$$

Corresponding to $I_4(z)$, we have

$$\psi_4(z) \;=\; \int_{\mathbf{R}} I_4(z)\; d\nu(s) \;=\; \int_0^{\infty} \widehat{\nu}(r)\, e^{-irq}\, \{\varphi(r) - 1\}\, r^{z-1}\; dr,$$

which continues analytically into $-2 < \mathrm{Re}(z) < 1$, since $\widehat{\nu}(0) = 0$ and $\widehat{\nu}\,'(0) = 0$. Hence

$$\begin{aligned}
\psi_4(u) \;&=\; \int_0^{\infty} \widehat{\nu}(r)\, e^{-irq}\, \{\varphi(r) - 1\}\, r^{-\frac{3+\epsilon}{2}-iu}\; dr \\
&=\; \int_0^{\infty} \left[\widehat{\nu}(r)\, e^{-irq}\, \{\varphi(r) - 1\}\, r^{-\frac{1+\epsilon}{2}}\right]\, r^{-1-iu}\; dr.
\end{aligned}$$

Integrating by parts twice, we obtain

$$-iu(1-iu)\,\psi_4(u) \;=\; \int_0^{\infty} \frac{\partial^2}{\partial r^2}\left[\widehat{\nu}(r)\, e^{-irq}\, \{\varphi(r) - 1\}\, r^{-\frac{1+\epsilon}{2}}\right]\; r^{1-iu}\; dr.$$

But $\varphi(r)-1=0$ for $r\geq 2$, and $\left|\frac{\partial^2}{\partial r^2}\left[\widehat{\nu}(r)\,e^{-irq}\,\{\varphi(r)-1\}\,r^{-\frac{1+\epsilon}{2}}\right]\right| \;\leq\; C\,r^{-\frac{1+\epsilon}{2}}$ for $0\leq r\leq 2$ since again the derivatives of $\widehat{\nu}$, e^{-irq} and φ are bounded. Hence we find that

$$|iu(1-iu)\,\psi_4(u)| \;\leq\; \int_0^2 C\,r^{\frac{1-\epsilon}{2}}\,dr \;<\; \infty \quad (\text{as } 0<\epsilon<3),$$

whence

$$|\psi_4(u)| \;\leq\; C\,|u|^{-2}.$$

It remains to estimate $\psi_5(u) \;=\; \int_{\mathbf{R}} I_5(z)\,d\nu(s)$, where $z=-\frac{1+\epsilon}{2}-iu$. Let h be the function on $\mathbf{C}$ defined by

$$h(w) \;=\; w^{z-1}\,e^{-wt}, \quad z=-\frac{1+\epsilon}{2}-iu,\; t\in\mathbf{R}.$$

By integrating h around the contour γ_1 (see `Fig. 1`) for $t>0$, or around the contour γ_2 (see `Fig. 2`) for $t<0$, one can observe that, as $R\to\infty$,

$$\int_0^\infty r^{z-1}\,e^{-irt}\,dr \;=\; (it)^{-z}\,\Gamma(z), \quad t\neq 0.$$

`Fig. 1` The contour γ_1

`Fig. 2` The contour γ_2

Hence

$$\begin{aligned}
|\psi_5(u)| &= \left|\int_{\mathbf{R}} I_5(z)\,d\nu(s)\right| && (\text{with } z=-\tfrac{1+\epsilon}{2}-iu)\\
&= \left|\int_{\mathbf{R}}\int_0^\infty r^{z-1}\,e^{-ir(s+q)}\,dr\,d\nu(s)\right|\\
&= \left|\int_{\mathbf{R}} \{i(s+q)\}^{-z}\,\Gamma(z)\,d\nu(s)\right|\\
&\leq e^{-\frac{\pi}{2}u}\,|\Gamma(-\tfrac{1+\epsilon}{2}-iu)|\int_{\mathbf{R}} |s+q|^{\frac{1+\epsilon}{2}}\,|d\nu(s)|\\
&\leq C\,|u|^{-1-\frac{\epsilon}{2}}\int_{\mathbf{R}} |s+q|^{\frac{1+\epsilon}{2}}\,|d\nu(s)| && (\text{as } |u|>1)\\
&\leq C\,|u|^{-1-\frac{\epsilon}{2}}\int_{\mathbf{R}} \{1+(s+q)^2\}\,|d\nu(s)| && (\text{as } 0<\epsilon<3)\\
&\leq C\,|u|^{-1-\frac{\epsilon}{2}}\,(1+q^2)\int_{\mathbf{R}} (1+s^2)\,|d\nu(s)|\\
&\leq C\,|u|^{-1-\frac{\epsilon}{2}}.
\end{aligned}$$

Combining this with the previous results, we obtain

$$|\psi(u)| \;\leq\; C\,|u|^{-1-\frac{\epsilon}{2}}, \quad |u| > 1,$$

with $C = C(\epsilon, p) \int_{\mathbf{R}} (1+s^2)\; |d\nu(s)|$. This completes the proof. □

To prove Theorem 2.1, we adapt a result of Hörmander [5, pp. 121-122].

Proposition 2.4 Suppose that Φ is a smooth function on $\mathbf{R}$, and that for $k = 0, 1, 2, \ldots$

$$\left|\frac{\partial^k}{\partial r^k}\Phi(r)\right| \;\leq\; C_k\,(1+|r|)^{-k-\epsilon}, \quad r \in \mathbf{R},$$

for some $\epsilon > 0$. It then follows that $\check{\Phi} \in L^1(\mathbf{R})$. In general, $s^k\,\check{\Phi}(s) \in L^1(\mathbf{R})$, for $k = 0, 1, 2, \ldots$.

Proof. First of all (see [5, p. 121]), there exists a function $\varphi \in C_0^\infty(\mathbf{R})$ supported in $\{r \;:\; \frac{1}{2} < |r| < 2\}$ such that

$$\sum_{j=-\infty}^{\infty} \varphi(2^{-j}r) \;=\; 1, \quad r \neq 0.$$

For this φ, we have

$$\sum_{j=1}^{\infty} \varphi(2^{-j}r) \;=\; 0, \quad 0 < |r| \leq \frac{1}{2}$$

and

$$\sum_{j=1}^{\infty} \varphi(2^{-j}r) \;=\; 1, \quad |r| > 2.$$

Let us put

$$\varphi_0(r) \;=\; \begin{cases} 1 - \sum_{j=1}^{\infty} \varphi(2^{-j}r), & \text{if } r \neq 0, \\ 1, & \text{if } r = 0. \end{cases}$$

It is clear that $\varphi_0 \in C_0^\infty(\mathbf{R})$ and that

$$\varphi_0(r) + \sum_{j=1}^{\infty} \varphi(2^{-j}r) \;=\; 1, \quad r \in \mathbf{R}.$$

This enables us to decompose Φ into

$$\Phi \;=\; \Phi_0 + \sum_{j=1}^{\infty} \Phi_j$$

where $\Phi_0 = \varphi_0\, \Phi$ and $\Phi_j(r) = \varphi(2^{-j} r)\, \Phi(r),\ r \in \mathbf{R}$ $(j = 0, 1, 2, \dots)$.

We see that $\Phi_0 \;=\; \varphi_0\, \Phi$ is smooth and compactly supported on $\mathbf{R}$. Thus for all $s \in \mathbf{R}$

$$|\check{\Phi}_0(s)| \;\le\; \frac{1}{2\pi} \int_{\mathbf{R}} |\Phi_0(r)|\; dr \;\le\; C$$

and

$$|s^2\, \check{\Phi}_0(s)| \;\le\; \frac{1}{2\pi} \int_{\mathbf{R}} \Big| \frac{\partial^2}{\partial r^2} \Phi_0(r) \Big|\; dr \;\le\; C.$$

Hence

$$|\check{\Phi}_0(s)| \;\le\; C\,(1 + |s|)^{-2}, \quad s \in \mathbf{R},$$

yielding

$$\int_{\mathbf{R}} |\check{\Phi}_0(s)|\; ds \;<\; \infty.$$

Now, for $j = 1, 2, 3, \dots$, we have

$$\begin{aligned}
\int_{\mathbf{R}} |\Phi_j(r)|^2\; dr \;&=\; \int_{\mathbf{R}} |\varphi(2^{-j} r)\, \Phi(r)|^2\; dr \\
&\le\; C \int_{2^{j-1} \le |r| \le 2^{j+1}} |\Phi(r)|^2\; dr \\
&\le\; C \int_{2^{j-1} \le |r| \le 2^{j+1}} |r|^{-2\epsilon}\; dr \\
&\le\; C\, 2^{(1-2\epsilon) j},
\end{aligned}$$

and similarly

$$\begin{aligned}
&\int_{\mathbf{R}} \Big| 2^j\, \frac{\partial}{\partial r} \Phi_j(r) \Big|^2\; dr \\
&\quad=\; \int_{\mathbf{R}} \Big| \frac{\partial \varphi}{\partial r}(2^{-j} r)\, \Phi(r) + 2^j\, \varphi(2^{-j} r)\, \frac{\partial}{\partial r} \Phi(r) \Big|^2\; dr \\
&\quad\le\; C \int_{2^{j-1} \le |r| \le 2^{j+1}} \Big\{ |\Phi(r)|^2 + 2^j\, |\Phi(r)|\, \Big|\frac{\partial}{\partial r} \Phi(r)\Big| + 2^{2j}\, \Big|\frac{\partial}{\partial r} \Phi(r)\Big|^2 \Big\}\; dr \\
&\quad\le\; C \int_{2^{j-1} \le |r| \le 2^{j+1}} \Big\{ |r|^{-2\epsilon} + 2^j\, |r|^{-1-2\epsilon} + 2^{2j}\, |r|^{-2-2\epsilon} \Big\}\; dr \\
&\quad\le\; C\, 2^{(1-2\epsilon) j}.
\end{aligned}$$

Then, by Plancherel's theorem,

$$\int_{\mathbf{R}} |\check{\Phi}_j(s)|^2 \; ds \;=\; \int_{\mathbf{R}} |\Phi_j(r)|^2 \; dr \;\leq\; C\, 2^{(1-2\epsilon)j}$$

and

$$\int_{\mathbf{R}} \left|2^j\, s\, \check{\Phi}_j(s)\right|^2 \; ds \;=\; \int_{\mathbf{R}} \left|2^j \, \frac{\partial}{\partial r} \Phi_j(r)\right|^2 \; dr \;\leq\; C\, 2^{(1-2\epsilon)j}.$$

These give

$$\int_{\mathbf{R}} 2^{-j} \left(1 + 2^{2j}\, s^2\right) |\check{\Phi}_j(s)|^2 \; ds \;\leq\; C\, 2^{-2\epsilon j}.$$

Applying the Cauchy-Schwartz inequality, we obtain

$$\begin{aligned} \int_{\mathbf{R}} |\check{\Phi}_j(s)| \; ds \;&\leq\; \left\{ \int_{\mathbf{R}} 2^{-j} \left(1 + 2^{2j}\, s^2\right) |\check{\Phi}_j(s)|^2 \; ds \right\}^{\frac{1}{2}} \left\{ \int_{\mathbf{R}} 2^{j} \left(1 + 2^{2j}\, s^2\right)^{-1} \; ds \right\}^{\frac{1}{2}} \\ &\leq\; C\, 2^{-\epsilon j} \left\{ \int_{|s| \leq 2^{-j}} 2^j \; ds + \int_{2^{-j} \leq |s| \leq 1} ds + \int_{|s| \geq 1} 2^{-j}\, s^{-2} \; ds \right\}^{\frac{1}{2}} \\ &\leq\; C\, 2^{-\epsilon j}. \end{aligned}$$

Since $\check{\Phi} = \check{\Phi}_0 + \sum_{j=1}^{\infty} \check{\Phi}_j$, we therefore have

$$\| \, \check{\Phi} \, \|_1 \;\leq\; \| \, \check{\Phi}_0 \, \|_1 + \sum_{j=1}^{\infty} \| \, \check{\Phi}_j \, \|_1 \;\leq\; C(\epsilon) \;<\; \infty,$$

meaning that $\check{\Phi} \in L^1(\mathbf{R})$.

In general, the proof also applies to $\frac{\partial^k}{\partial r^k}\Phi(r)$, and so we have $s^k \, \check{\Phi}(s) \in L^1(\mathbf{R})$, for all $k = 0, 1, 2, \ldots$. $\square$

We now come to the proof of Theorem 2.1.

Proof (of Theorem 2.1). For $r \in \mathbf{R}^+, \; \xi \in S^{n-1}$, we have

$$\widehat{\mu}(r\xi) \;=\; F(r, \xi)\, e^{-irp(\xi)\cdot\xi} + F(r, -\xi)\, e^{-irp(-\xi)\cdot\xi} + E(r, \xi),$$

with p, F, and E as prescribed. We assume here that $F(r, \pm\xi) = 0$, for $0 \leq r < s < 1$, to have $\frac{\partial^k}{\partial r^k} F(r, \xi)|_{r=0} = 0$, for all $k = 0, 1, 2, \ldots$.

For $u \in \mathbf{R},\ \xi \in S^{n-1}$, consider

$$\begin{aligned}
\psi(u,\xi) &= \frac{1}{2\pi}\int_0^\infty \widehat{\mu}(r\xi)\, r^{-1-iu}\, dr \\
&= \frac{1}{2\pi}\int_0^\infty F(r,\xi)\, e^{-irp(\xi)\cdot\xi}\, r^{-1-iu}\, dr \\
&\quad + \frac{1}{2\pi}\int_0^\infty F(r,-\xi)\, e^{-irp(-\xi)\cdot\xi}\, r^{-1-iu}\, dr \\
&\quad + \frac{1}{2\pi}\int_0^\infty E(r,\xi)\, r^{-1-iu}\, dr \\
&= \psi_1(u,\xi) + \psi_2(u,\xi) + \psi_3(u,\xi), \quad \text{say.}
\end{aligned}$$

Let us fix $\xi \in S^{n-1}$ hereafter. By Theorem 1.2, we have

$$|\psi_3(u,\xi)| \leq C\,(1+|u|)^{-1-\delta}, \quad u \in \mathbf{R},$$

for some $\delta > 0$, assuming that $E(0,\xi) = 0$.

It then remains to tackle ψ_1 and ψ_2. But since they are similar, it suffices to work with one of them, ψ_1 say. We put

$$\widehat{\mu}_1(r\xi) \;=\; F(r,\xi)\, e^{-irp(\xi)\cdot\xi},$$

and define $\widehat{\nu}$ on $\mathbf{R}$ by

$$\widehat{\nu}(r) \;=\; \begin{cases} \widehat{\mu}_1(r\xi)\, e^{irp(\xi)\cdot\xi}\,(1+r^2)^{\frac{1+\epsilon}{4}}, & \text{if } r \geq 0, \\ \widehat{\mu}_1((-r)(-\xi))\, e^{irp(-\xi)\cdot\xi}\,(1+r^2)^{\frac{1+\epsilon}{4}}, & \text{if } r < 0, \end{cases}$$

for some $0 < \epsilon < \min(3, \alpha - \frac{1}{2})$. Thus

$$\widehat{\nu}(r) \;=\; \begin{cases} F(r,\xi)\,(1+r^2)^{\frac{1+\epsilon}{4}}, & \text{if } r \geq 0, \\ F(-r,-\xi)\,(1+r^2)^{\frac{1+\epsilon}{4}}, & \text{if } r < 0. \end{cases}$$

Writing

$$\widehat{\nu}(r) \;=\; \begin{cases} F(r,\xi)\, r^{\frac{1+\epsilon}{2}}\,(1+r^{-2})^{\frac{1+\epsilon}{4}}, & \text{if } r \geq 0, \\ F(-r,-\xi)\,(-r)^{\frac{1+\epsilon}{2}}\,(1+r^{-2})^{\frac{1+\epsilon}{4}}, & \text{if } r < 0, \end{cases}$$

and applying Leibnitz's formula, we obtain that for $k = 0, 1, 2, \ldots$,

$$\left|\frac{\partial^k}{\partial r^k}\widehat{\nu}(r)\right| \;\leq\; C, \quad |r| \leq 1$$

and

$$\left|\frac{\partial^k}{\partial r^k}\widehat{\nu}(r)\right| \;\leq\; C\,|r|^{-k-\frac{\epsilon}{2}}, \quad |r| > 1.$$

Hence, for all $k = 0, 1, 2, \dots,$ we have

$$\left|\frac{\partial^k}{\partial r^k}\widehat{\nu}(r)\right| \;\leq\; C\,(1+|r|)^{-k-\frac{\epsilon}{2}}, \quad r \in \mathbf{R}.$$

It follows from Proposition 2.4 that $\nu = (\widehat{\nu})\check{} \in L^1(\mathbf{R})$ (which assures us that ν defines a measure on $\mathbf{R}$), and further $s^2\,\nu(s) \in L^1(\mathbf{R})$. So we find that there exists a measure ν on $\mathbf{R}$ which satisfies the hypothesis of Lemma 2.3. The proof is therefore complete. □

Remark. The result extends to surface measures of class C^K for some finite K; K is sufficiently large so that the hypothesis

$$\left|\frac{\partial^k}{\partial r^k}F(r, \pm\xi)\right| \;\leq\; C_k\, r^{-k-\alpha}, \quad r \in \mathbf{R}^+,\ \xi \in S^{n-1},$$

is satisfied for $k = 0, 1, 2, 3$ and 4.

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School of Mathematics, The University of New South Wales, Kensington, N.S.W. 2033, Australia