MULTILINEAR MAXIMAL FUNCTIONS AND FRACTIONAL INTEGRALS ON GENERALIZED MORREY SPACES

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Abstract. We revisit multisublinear maximal functions and multilinear fractional integrals which were initially studied by L. Grafakos [6]. We are particularly interested in their boundedness on Morrey spaces on $\mathbb{R}^d$. We extend existing results to generalized fractional integrals on generalized Morrey spaces. An extension to the non-homogeneous setting is also discussed.

1. Introduction

In 1992, L. Grafakos [6] studied the multisublinear maximal function $M_m\mathbf{f}$ on $\mathbb{R}^d$ given by the formula

$$M_m\mathbf{f}(x) := \sup_{r>0} \frac{1}{|B(0,r)|} \int_{B(0,r)} \prod_{j=1}^m |f_j(x-\theta_j y)|\, dy,$$

where $\mathbf{f} := (f_1, \dots, f_m)$, f_j's are locally integrable functions on $\mathbb{R}^d$, and θ_j's are nonzero distinct constants. He also discussed the multilinear fractional integral $I_{m,\alpha}\mathbf{f}$ by

$$I_{m,\alpha}\mathbf{f}(x) := \int_{\mathbb{R}^d} |y|^{\alpha-d} \prod_{j=1}^m f_j(x-\theta_j y)\, dy,$$

where $0 < \alpha < d$. Note that the two operators are different from the following versions:

$$\mathcal{M}_m\mathbf{f}(x) := \sup_{r>0} \frac{1}{|B(0,r)|^m} \int_{(B(0,r))^m} \prod_{j=1}^m |f_j(x-\theta_j y_j)|\, dy_j$$

and

$$\mathcal{I}_{m,\alpha}\mathbf{f}(x) := \int_{(\mathbb{R}^d)^m} |\mathbf{y}|^{\alpha-md} \prod_{j=1}^m f_j(x-\theta_j y_j)\, dy_j,$$

where $\mathbf{y} := (y_1, \dots, y_m)$ and $|\mathbf{y}| := \sqrt{\sum_{j=1}^m |y_j|^2}$. However, in both versions, the case $m = 1$ and $\theta_1 = 1$ reduces to the Hardy-Littlewood maximal function $M_1 f_1$ and fractional integral $I_\alpha f_1$. See [24] for the two classical operators M_1 and I_α.

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It is easy to see that the multisublinear maximal operator $\mathcal{M}_m$ is bounded from $L^{p_1} \times \cdots \times L^{p_m}$ to L^p, where $\frac{1}{p} := \sum_{j=1}^m \frac{1}{p_j}$ and $1 < p_j < \infty$ for $j = 1, \ldots, m$. Indeed, one may observe that

$$\mathcal{M}_m \mathbf{f}(x) = \sup_{r>0} \prod_{j=1}^m \frac{1}{|B(0,r)|} \int_{B(0,r)} |f_j(x - \theta_j y_j)| \, dy_j \leq \prod_{j=1}^m M_1 f_j(x),$$

and so it follows from Hólder's inequality that

$$\|\mathcal{M}_m \mathbf{f}\|_p \leq \prod_{j=1}^m \|M_1 f_j\|_{p_j} \leq C \prod_{j=1}^m \|f_j\|_{p_j},$$

provided that $f_j \in L^{p_j}$ for each $j = 1, \ldots, m$. Here we have used the boundedness of the Hardy-Littlewood maximal operator M_1 on L^{p_j}'s. We shall not discuss further the operators $\mathcal{M}_m$ and $\mathcal{I}_{m,\alpha}$ in this paper. We refer the readers who are interested in these operators to the works of T. Iida *et al.* [10, 11, 12, 13]. See also [2, 14, 16] for related results.

From now on, we are interested in the operators M_m and $I_{m,\alpha}$. Regarding M_m, it follows from Hölder's inequality that

$$\frac{1}{|B(0,r)|} \int_{B(0,r)} \prod_{j=1}^m |f_j(x - \theta_j y)| \, dy \leq \prod_{j=1}^m \Big[\frac{1}{|B(0,r)|} \int_{B(0,r)} |f_j(x - \theta_j y)|^{p_j/p} dy \Big]^{p/p_j},$$

for every $r > 0$. Here $\frac{1}{p} := \sum_{j=1}^m \frac{1}{p_j}$ as before. Hence, we get the pointwise estimate

$$M_m \mathbf{f}(x) \leq \prod_{j=1}^m \big[M_1(f_j^{p_j/p})(x) \big]^{p/p_j}, \quad x \in \mathbb{R}^d. \tag{1.1}$$

Applying Hölder's inequality once again, we get

$$\int_{\mathbb{R}^d} |M_m \mathbf{f}(x)|^p dx \leq \prod_{j=1}^m \Big[\int_{\mathbb{R}^d} \big[M_1(f_j^{p_j/p})(x) \big]^p dx \Big]^{p/p_j} = \prod_{j=1}^m \big\| M_1(f_j^{p_j/p}) \big\|_p^{p^2/p_j},$$

whence

$$\|M_m \mathbf{f}\|_p \leq \prod_{j=1}^m \big\| M_1(f_j^{p_j/p}) \big\|_p^{p/p_j}.$$

Assuming that $p > 1$ and noting that $f_j \in L^{p_j}$ is equivalent to $f_j^{p_j/p} \in L^p$ with $\|f_j^{p_j/p}\|_p = \|f_j\|_{p_j}^{p_j/p}$, we have

$$\|M_1(f_j^{p_j/p})\|_p \leq C \, \|f_j^{p_j/p}\|_p = C \, \|f_j\|_{p_j}^{p_j/p}.$$

Accordingly, we obtain the following inequality

$$\|M_m \mathbf{f}\|_p \leq C \prod_{j=1}^m \|f_j\|_{p_j},$$

which tells us that M_m is a bounded operator from $L^{p_1} \times \cdots \times L^{p_m}$ to L^p. In general, one can take p to be the harmonic mean of p_j's, that is, $\frac{m}{p} := \sum_{j=1}^m \frac{1}{p_j}$, to make sure that $p > 1$. In this case M_m is bounded from $L^{mp_1} \times \cdots \times L^{mp_m}$ to L^p, with

$$\|M_m \mathbf{f}\|_p \le C \prod_{j=1}^m \|f_j\|_{mp_j}.$$

Note that C denotes a positive constant whose value may vary from line to line, but is always independent of the functions f_j's and the variables x or y.

In the following sections, we study the boundedness of the multisublinear maximal operator M_m on the (classical) Morrey spaces and on the generalized Morrey spaces. We shall also discuss the boundedness of the multilinear fractional integral operator $I_{m,\alpha}$ from $L^{p_1,\lambda} \times \cdots \times L^{p_m,\lambda}$ to $L^{q,\lambda}$ where $\frac{1}{q} := \frac{1}{p} - \frac{\alpha}{d-\lambda}$, and the boundedness of the generalized multilinear fractional integral operator $I_{m,\rho}$ (which will be defined in Section 3) from $L^{p_1,\phi^{p/p_1}} \times \cdots \times L^{p_m,\phi^{p/p_m}}$ to $L^{q,\phi^{p/q}}$, where $p < q < \infty$. Here $L^{q,\lambda}$ and $L^{q,\psi}$ denote the classical and the generalized Morrey spaces, respectively (which will also be defined soon). In the last section, an extension to the non-homogeneous setting will also be presented.

Our results can be viewed as a refinement and also an extension of the results obtained recently by W. Setya Budhi and J. Lindiarti [22].

2. The boundedness of M_m and $I_{m,\alpha}$ on the (classical) Morrey spaces

For $1 \le p < \infty$ and $0 \le \lambda < d$, the Morrey space $L^{p,\lambda} = L^{p,\lambda}(\mathbb{R}^d)$ consists of all locally integrable functions f on $\mathbb{R}^d$ for which

$$\|f\|_{p,\lambda} := \sup_{a,\,r} \Big[\frac{1}{r^\lambda} \int_{B(a,r)} |f(x)|^p dx\Big]^{\frac{1}{p}} = \sup_{a,\,r} \frac{1}{r^{(\lambda-d)/p}} \Big[\frac{1}{|B(a,r)|} \int_{B(a,r)} |f(x)|^p dx\Big]^{\frac{1}{p}} < \infty.$$

Here the supremum is taken over all $a \in \mathbb{R}^d$ and $r > 0$. Note that $L^{p,0} = L^p$. See [19] for more information about these spaces.

We shall now verify the boundedness of M_m on these spaces. We already have the pointwise estimate (1.1):

$$M_m \mathbf{f}(x) \le \prod_{j=1}^m \big[M_1(f_j^{p_j/p})(x)\big]^{p/p_j}, \quad x \in \mathbb{R}^d,$$

where $\frac{1}{p} := \sum_{j=1}^m \frac{1}{p_j}$. Unless otherwise stated, we shall always assume that p satisfies the relationship $\frac{1}{p} = \sum_{j=1}^m \frac{1}{p_j} < 1$ where $1 < p_j < \infty$ for $j = 1, \ldots, m$.

It then follows from the above estimate and Hölder's inequality that

$$\frac{1}{|B(a,r)|}\int_{B(a,r)}|M_m\mathbf{f}(x)|^p dx \le \frac{1}{|B(a,r)|}\int_{B(a,r)}\prod_{j=1}^m [M_1(f_j^{p_j/p})(x)]^{p^2/p_j} dx$$
$$\le \prod_{j=1}^m \Big[\frac{1}{|B(a,r)|}\int_{B(a,r)}[M_1(f_j^{p_j/p})(x)]^p dx\Big]^{p/p_j},$$

for every $a \in \mathbb{R}^d$ and $r > 0$. Taking the p-th root of both sides, we get

$$\Big[\frac{1}{|B(a,r)|}\int_{B(a,r)}|M_m\mathbf{f}(x)|^p dx\Big]^{1/p} \le \prod_{j=1}^m \Big[\frac{1}{|B(a,r)|}\int_{B(a,r)}[M_1(f_j^{p_j/p})(x)]^p dx\Big]^{1/p_j}.$$

Next, we divide both sides by $r^{(\lambda-d)/p} = \prod_{j=1}^m \big[r^{(\lambda-d)/p}\big]^{p/p_j}$ to get

$$\frac{1}{r^{(\lambda-d)/p}}\Big[\frac{1}{|B(a,r)|}\int_{B(a,r)}|M_m\mathbf{f}(x)|^p dx\Big]^{1/p}$$
$$\le \prod_{j=1}^m \left[\frac{1}{r^{(\lambda-d)/p}}\Big[\frac{1}{|B(a,r)|}\int_{B(a,r)}[M_1(f_j^{p_j/p})(x)]^p dx\Big]^{1/p}\right]^{p/p_j}$$
$$\le \prod_{j=1}^m \|M_1(f_j^{p_j/p})\|_{p,\lambda}^{p/p_j} \le C\prod_{j=1}^m \|f_j^{p_j/p}\|_{p,\lambda}^{p/p_j} = C\prod_{j=1}^m \|f_j\|_{p_j,\lambda},$$

assuming that $f_j \in L^{p_j,\lambda}$ for each $j = 1,\dots,m$.

Since this is true for all $a \in \mathbb{R}^d$ and $r > 0$, we obtain

$$\|M_m\mathbf{f}\|_{p,\lambda} \le C\prod_{j=1}^m \|f_j\|_{p_j,\lambda}.$$

We summarize this in the following theorem.

Theorem 2.1 *The operator M_m is bounded from $L^{p_1,\lambda}\times\cdots\times L^{p_m,\lambda}$ to $L^{p,\lambda}$, where $0 \le \lambda < d$.*

Let us now consider the multilinear fractional integral $I_{m,\alpha}\mathbf{f}$. By Hölder's inequality, we observe that

$$|I_{m,\alpha}\mathbf{f}(x)| \le \int_{\mathbb{R}^d}|y|^{\alpha-d}\prod_{j=1}^m |f_j(x-\theta_j y)|\, dy$$
$$\le \prod_{j=1}^m \Big[\int_{\mathbb{R}^d}|y|^{\alpha-d}|f_j(x-\theta_j y)|^{p_j/p}dy\Big]^{p/p_j}$$
$$= C\prod_{j=1}^m \big[I_\alpha(|f_j|^{p_j/p})(x)\big]^{p/p_j}.$$

Consequently, for every $a \in \mathbb{R}^d$ and $r > 0$, we have

$$\frac{1}{|B(a,r)|}\int_{B(a,r)} |I_{m,\alpha}\mathbf{f}(x)|^q dx \le \frac{C}{|B(a,r)|}\int_{B(a,r)} \prod_{j=1}^m \left[I_\alpha(|f_j|^{p_j/p})(x)\right]^{pq/p_j} dx$$
$$\le C\prod_{j=1}^m \Big[\frac{1}{|B(a,r)|}\int_{B(a,r)} \left[I_\alpha(|f_j|^{p_j/p})(x)\right]^q dx\Big]^{p/p_j}.$$

With $\frac{1}{q} := \frac{1}{p} - \frac{\alpha}{d-\lambda}$, we take the q-th root of both sides and then divide them by $r^{(\lambda-d)/q} = \prod_{j=1}^m \left[r^{(\lambda-d)/q}\right]^{p/p_j}$ to obtain

$$\frac{1}{r^{(\lambda-d)/q}}\Big[\frac{1}{|B(a,r)|}\int_{B(a,r)} |I_{m,\alpha}\mathbf{f}(x)|^q dx\Big]^{1/q}$$
$$\le C\prod_{j=1}^m \left[\frac{1}{r^{(\lambda-d)/q}}\Big[\frac{1}{|B(a,r)|}\int_{B(a,r)} \left[I_\alpha(|f_j|^{p_j/p})(x)\right]^q dx\Big]^{1/q}\right]^{p/p_j}$$
$$\le C\prod_{j=1}^m \|I_\alpha(|f_j|^{p_j/p})\|_{q,\lambda}^{p/p_j} \le C\prod_{j=1}^m \|f_j^{p_j/p}\|_{p,\lambda}^{p/p_j} = C\prod_{j=1}^m \|f_j\|_{p_j,\lambda}.$$

Here we have used the fact that the (linear) fractional integral operator I_α is bounded from $L^{p,\lambda}$ to $L^{q,\lambda}$.

The last estimate holds for all $a \in \mathbb{R}^d$ and $r > 0$, and so we conclude that

$$\|I_{m,\alpha}\mathbf{f}\|_{q,\lambda} \le C\prod_{j=1}^m \|f_j\|_{p_j,\lambda}. \tag{2.1}$$

Therefore we have the following theorem. (Note that the case $m = 1$ reduces to the result of Adams [1]. Also, the case $\lambda = 0$ gives the boundedness of $I_{m,\alpha}$ on Lebesgue spaces.)

Theorem 2.2 *The operator $I_{m,\alpha}$ is bounded from $L^{p_1,\lambda} \times \cdots \times L^{p_m,\lambda}$ to $L^{q,\lambda}$, where $\frac{1}{q} := \frac{1}{p} - \frac{\alpha}{d-\lambda}$ and $0 \le \lambda < d - \alpha p$.*

`Remark`. As one might think, Theorem 2.2 may also be proved by establishing the following Hedberg's type estimate:

$$|I_{m,\alpha}\mathbf{f}(x)| \le C\,[M_m\mathbf{f}(x)]^{1-\alpha p/(d-\lambda)} \prod_{j=1}^m \|f_j\|_{p_j,\lambda}^{\alpha p/(d-\lambda)}, \quad x \in \mathbb{R}^d. \tag{2.2}$$

The idea is to split the integral into two parts:

$$I_{m,\alpha}\mathbf{f}(x) = \int_{|y|<R} |y|^{\alpha-d}\prod_{j=1}^m f_j(x-\theta_j y)\,dy + \int_{|y|\ge R} |y|^{\alpha-d}\prod_{j=1}^m f_j(x-\theta_j y)\,dy,$$

and then estimate each part separately. The value of $R = R(x)$ is chosen so that both estimates balance. Using the estimate (2.2) and the boundedness of the operator M_m

on the Morrey spaces, the inequality (2.1) follows. We leave the details to the readers (see [8] for the case $m = 1$).

From Theorem 2.2, we obtain an Olsen's type inequality as in the following corollary (see [15, 23] for the case $m = 1$).

Corollary 2.3 *For $0 \le \lambda < d - \alpha p$, we have*

$$\|W \cdot I_{m,\alpha}\mathbf{f}\|_{p,\lambda} \le C\,\|W\|_{(d-\lambda)/\alpha,\lambda} \prod_{j=1}^{m} \|f_j\|_{p_j,\lambda}.$$

3. The boundedness of M_m and $I_{m,\rho}$ on the generalized Morrey spaces

For $\rho : (0,\infty) \to (0,\infty)$ with $\int_0^1 \frac{\rho(t)}{t}\,dt < \infty$, we define the multilinear fractional integral $I_{m,\rho}\mathbf{f}$, for suitable $\mathbf{f} = (f_1, \dots, f_m)$, by

$$I_{m,\rho}\mathbf{f}(x) := \int_{\mathbb{R}^d} \frac{\rho(|y|)}{|y|^d} \prod_{j=1}^{m} f_j(x - \theta_j y)\,dy,$$

Here we assume that ρ satisfies the *doubling condition*: there exists a constant $C_1 > 1$ such that if $1 \le \frac{r}{t} \le 2$, then $\frac{1}{C_1} \le \frac{\rho(r)}{\rho(t)} \le C_1$. (Actually, we may assume that ρ satisfies a weaker growth condition: there exist constants $C_2, \delta > 0$ and $0 \le \epsilon < 1$ such that

$$\sup_{r/2<t\le r} \rho(t) \le C_2 \int_{\delta(1-\epsilon)r/2}^{\delta(1+\epsilon)r} \frac{\rho(t)}{t}\,dt, \quad r > 0, \tag{3.1}$$

as in [21].) Note that if $\rho(r) := r^{\alpha}$, then $I_{m,\rho} = I_{m,\alpha}$. For $m = 1$, the operator I_ρ was studied by E. Nakai [18]. See also [3, 7, 8, 9, 21] for more results on the operator I_ρ.

We shall be interested in the boundedness of the operators M_m and $I_{m,\rho}$ on generalized Morrey spaces. We say that a function f belongs to the generalized Morrey space $L^{p,\phi} = L^{p,\phi}(\mathbb{R}^d)$, where $1 \le p < \infty$, if

$$\|f\|_{p,\phi} := \sup_{a,\,r} \frac{1}{\phi(r)} \left[\frac{1}{|B(a,r)|} \int_{B(a,r)} |f(x)|^p dx\right]^{1/p} < \infty.$$

See [17] for the original idea on these spaces. Here we assume that ϕ is *almost decreasing* [that is, if $r \le t$, then $\phi(r) \ge C_3\phi(t)$], and that $r^d\phi^p(r)$ is *almost increasing.* These two conditions implies that ϕ also satisfies the doubling condition. In addition, we also assume that $\phi(r) \to 0$ as $r \to \infty$. We note that if $\phi(r) := r^{(\lambda-d)/p}$ with $0 \le \lambda < d$, then $L^{p,\phi} = L^{p,\lambda}$ — the classical Morrey space.

As on the (classical) Morrey spaces, we have the following result for M_m on the generalized Morrey spaces, which is a refinement of Theorem 1 in [22].

Theorem 3.1 *Let* $\phi_j := \phi^{p/p_j}$, $j = 1, \ldots, m$. *If* ϕ^p *satisfies the* integral condition:

$$\int_r^\infty \frac{\phi^p(t)}{t}\,dt \le C_4 \phi^p(r), \quad r > 0, \tag{3.2}$$

then M_m *is a bounded operator from* $L^{p_1,\phi_1} \times \cdots \times L^{p_m,\phi_m}$ *to* $L^{p,\phi}$, *with*

$$\|M_m \mathbf{f}\|_{p,\phi} \le C \prod_{j=1}^m \|f_j\|_{p_j,\phi_j}.$$

Remark. The integral condition on ϕ^p implies that on $\phi_j^{p_j}$, and so the Hardy-Littlewood maximal operator M_1 is bounded on L^{p_j,ϕ_j} for each $j = 1, \ldots, m$ (see Theorem 1 in [17]). With this in mind, the proof of Theorem 3.1 is very similar to that of Theorem 2.1, and so we leave it to the readers.

Analogous to Theorem 2.2, we have the following theorem about the boundedness of $I_{m,\rho}$ on the generalized Morrey spaces.

Theorem 3.2 *Suppose that* ϕ^p *satisfies the integral condition (3.2) and the inequality*

$$\phi(r) \int_0^r \frac{\rho(t)}{t}\,dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t}\,dt \le C_5\, \phi^{p/q}(r), \quad r > 0, \tag{3.3}$$

where $p < q < \infty$. *Then,* $I_{m,\rho}$ *is a bounded operator from* $L^{p_1,\phi_1} \times \cdots \times L^{p_m,\phi_m}$ *to* $L^{q,\phi^{p/q}}$, *with*

$$\|I_{m,\rho} \mathbf{f}\|_{q,\phi^{p/q}} \le C \prod_{j=1}^m \|f_j\|_{p_j,\phi_j},$$

where $\phi_j := \phi^{p/p_j}$, $j = 1, \ldots, m$.

Remark. The hypotheses are imposed to assure that the (linear) fractional integral operator I_ρ is bounded from $L^{p,\phi}$ to $L^{q,\phi^{p/q}}$ (see [4]). Here we do not assume that ϕ is surjective as we do in [7]. The proof of Theorem 3.2 is similar to that of Theorem 2.2.

Corollary 3.3 *With the same hypotheses as in Theorem 3.2, we have*

$$\|W \cdot I_{m,\rho} \mathbf{f}\|_{p,\phi} \le C\, \|W\|_{s,\phi^{p/s}} \prod_{j=1}^m \|f_j\|_{p_j,\phi_j},$$

where $\phi_j := \phi^{p/p_j}$, $j = 1, \ldots, m$, *and* $\frac{1}{s} := \frac{1}{p} - \frac{1}{q}$.

4. An extension to the non-homogeneous spaces

Suppose now that $\mathbb{R}^d$ is equipped with a Borel measure μ satisfying the *growth condition* of order n with $0 < n \le d$, that is, there exists a constant $C_6 > 0$ such that

$$\mu(B(a, r)) \le C_6\, r^n$$

for all $a \in \mathbb{R}^d$ and $r > 0$. This condition replaces the doubling property

$$\mu(B(a, 2r)) \sim \mu(B(a, r)),$$

that we enjoy in the homogeneous setting (especially when $\mathbb{R}^d$ is equipped with the usual Lebesgue measure).

As in [5], for any locally integrable function f on $\mathbb{R}^d$, we define the maximal function $M^\mu f$ by the formula

$$M^\mu f(x) := \sup_{r>0} \frac{1}{r^n} \int_{B(0,r)} |f(x-y)| \, d\mu(y).$$

The operator M^μ is bounded on $L^p(\mu)$ for $1 < p \leq \infty$ [5]. We shall now discuss its boundedness on the generalized Morrey spaces $L^{p,\phi}(\mu)$ which is defined below.

For $1 \leq p < \infty$ and $\phi : (0, \infty) \to (0, \infty)$, the space $L^{p,\phi}(\mu) = L^{p,\phi}(\mathbb{R}^d, \mu)$ consists of all functions $f \in L^p_{\mathrm{loc}}(\mu)$ for which

$$\|f\|_{p,\phi,\mu} := \sup_{a,\, r} \frac{1}{\phi(r)} \Big[\frac{1}{r^n} \int_{B(a,r)} |f(x)|^p d\mu(x) \Big]^{1/p} < \infty.$$

As in the previous section, we assume that ϕ is almost decreasing and $r^n \phi^p(r)$ is almost increasing.

On these generalized Morrey spaces, we have the following result on the (sublinear) maximal operator M^μ.

Theorem 4.1 *The operator M^μ is bounded on $L^{p,\phi}(\mu)$ for $1 < p < \infty$.*

In the non-homogeneous setting, we define the (linear) fractional integral $I^\mu_\rho f$ by the formula

$$I^\mu_\rho f(x) := \int_{\mathbb{R}^d} \frac{\rho(|y|)}{|y|^n} f(x-y) \, d\mu(y).$$

Here $\rho : (0, \infty) \to (0, \infty)$ satisfies the usual doubling condition or the weaker growth condition (3.1).

The following theorem gives the boundedness of I^μ_ρ on the generalized Morrey spaces in the non-homogeneous setting.

Theorem 4.2 *Suppose that ϕ^p satisfies the integral condition (3.2) and the inequality*

$$\phi(r) \int_0^r \frac{\rho(t)}{t} \, dt + \int_r^\infty \frac{\rho(t)\phi(t)}{t} \, dt \leq C_7 \, \phi^{p/q}(r), \quad r > 0,$$

where $1 < p < q < \infty$. Then we have

$$\|I^\mu_\rho f\|_{q,\phi^{p/q},\mu} \leq C \, \|f\|_{p,\phi,\mu},$$

that is, I^μ_ρ is bounded from $L^{p,\phi}(\mu)$ to $L^{q,\phi^{p/q}}(\mu)$.

Corollary 4.3 *With the same hypotheses as in Theorem 4.2, we have*

$$\|W \cdot I^\mu_\rho f\|_{p,\phi,\mu} \le C\, \|W\|_{s,\phi^{p/s},\mu} \|f\|_{p,\phi,\mu},$$

where $\frac{1}{s} := \frac{1}{p} - \frac{1}{q}$.

We now move on to the corresponding multilinear operators we are interested in, namely the multisublinear maximal operator M^μ_m:

$$M^\mu_m \mathbf{f}(x) := \sup_{r>0} \frac{1}{r^n} \int_{B(0,r)} \prod_{j=1}^m |f_j(x - \theta_j y)|\, d\mu(y);$$

and the multilinear fractional integral operator $I^\mu_{m,\rho}$:

$$I^\mu_{m,\rho} \mathbf{f}(x) := \int_{\mathbb{R}^d} \frac{\rho(|y|)}{|y|^n} \prod_{j=1}^m f_j(x - \theta_j y)\, d\mu(y).$$

Let $\phi_j := \phi^{p/p_j}$, $j = 1, \dots, m$. Then we have analogous results to Theorems 3.1–3.3.

Theorem 4.4 *If ϕ^p satisfies the integral condition (3.2), then M^μ_m is a bounded operator from $L^{p_1,\phi_1}(\mu) \times \cdots \times L^{p_m,\phi_m}(\mu)$ to $L^{p,\phi}(\mu)$, with*

$$\|M^\mu_m \mathbf{f}\|_{p,\phi,\mu} \le C \prod_{j=1}^m \|f_j\|_{p_j,\phi_j,\mu}.$$

Theorem 4.5 *Suppose that ϕ^p satisfies the integral condition (3.2) and the inequality (3.3). Then $I^\mu_{m,\rho}$ is a bounded operator from $L^{p_1,\phi_1}(\mu) \times \cdots \times L^{p_m,\phi_m}(\mu)$ to $L^{q,\phi^{p/q}}(\mu)$, with*

$$\|I^\mu_{m,\rho} \mathbf{f}\|_{q,\phi^{p/q},\mu} \le C \prod_{j=1}^m \|f_j\|_{p_j,\phi_j,\mu}.$$

Corollary 4.6 *With the same hypotheses as in Theorem 4.5, we have*

$$\|W \cdot I^\mu_{m,\rho} \mathbf{f}\|_{p,\phi,\mu} \le C\, \|W\|_{s,\phi^{p/s},\mu} \prod_{j=1}^m \|f_j\|_{p_j,\phi_j,\mu},$$

where $\frac{1}{s} := \frac{1}{p} - \frac{1}{q}$.

5. Concluding remarks

As in [20], another version of maximal operators may be defined on the non-homogeneous space by the formula

$$M^{\kappa,\mu} f(x) := \sup_{B \ni x} \frac{1}{\mu(\kappa B)} \int_B |f(y)|\, d\mu(y),$$

where $\kappa > 1$. This operator is bounded on the generalized Morrey space $L^{p,\phi}(\kappa,\mu)$ for $1 < p < \infty$, where the norm is given by

$$\|f\|_{p,\phi,\kappa,\mu} := \sup_{B=B(a,r)} \frac{1}{\phi(\mu(\kappa B))} \Big[\frac{1}{\mu(\kappa B)} \int_B |f(x)|^p d\mu(x)\Big]^{1/p}.$$

In [9], the boundedness of the corresponding fractional integral operator $I_\rho^{\kappa,\mu}$ on such spaces is also proved. Using the same strategy, we can obtain analogous results to Theorems 4.4–4.6 for the multisublinear maximal operator $M_m^{\kappa,\mu}$ and the multilinear fractional integral operator $I_{m,\rho}^{\kappa,\mu}$.

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