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INNER PRODUCTS ON n-INNER PRODUCT SPACES

BY

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Abstract. In this note, we show that in any n-inner product space with $n \geq 2$ we can explicitly derive an inner product or, more generally, an $(n-k)$-inner product from the n-inner product, for each $k \in \{1, \ldots, n-1\}$. We also present some related results on n-normed spaces.

1. Introduction

Let n be a nonnegative integer and X be a real vector space of dimension $d \geq n$ (d may be infinite). A real-valued function $\langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle$ on X^{n+1} satisfying the following five properties:

(I1) $\langle x_1, x_1 | x_2, \ldots, x_n \rangle \geq 0$; $\langle x_1, x_1 | x_2, \ldots, x_n \rangle = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent;

(I2) $\langle x_1, x_1 | x_2, \ldots, x_n \rangle = \langle x_{i_1}, x_{i_1} | x_{i_2}, \ldots, x_{i_n} \rangle$ for every permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$;

(I3) $\langle x, y | x_2, \ldots, x_n \rangle = \langle y, x | x_2, \ldots, x_n \rangle$;

(I4) $\langle \alpha x, y | x_2, \ldots, x_n \rangle = \alpha \langle x, y | x_2, \ldots, x_n \rangle, \ \alpha \in \mathbf{R}$;

(I5) $\langle x + x', y | x_2, \ldots, x_n \rangle = \langle x, y | x_2, \ldots, x_n \rangle + \langle x', y | x_2, \ldots, x_n \rangle$,

is called an *n-inner product* on X, and the pair $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ is called an *n-inner product space*.

For $n = 1$, the expression $\langle x, y | x_2, \ldots, x_n \rangle$ is to be understood as $\langle x, y \rangle$, which denotes nothing but an inner product on X. The concept of 2-inner product spaces was first introduced by Diminnie, Gähler and White [2, 3, 7] in 1970's,

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while its generalization for $n \geq 2$ was developed by Misiak [12] in 1989. Note here that our definition of n-inner products is slightly simpler than, but equivalent to, that in [12].

On an n-inner product space $(X, \langle\cdot,\cdot|\cdot,\dots,\cdot\rangle)$, one may observe that the following function

$$\|x_1, x_2, \dots, x_n\| := \langle x_1, x_1|x_2, \dots, x_n\rangle^{1/2},$$

defines an n-norm, which enjoys the following four properties:

(N1) $\|x_1, \dots, x_n\| \geq 0; \|x_1, \dots, x_n\| = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent;

(N2) $\|x_1, \dots, x_n\|$ is invariant under permutation;

(N3) $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha|\,\|x_1, x_2, \dots, x_n\|,\ \alpha \in \mathbf{R}$;

(N4) $\|x+y, x_2, \dots, x_n\| \leq \|x, x_2, \dots, x_n\| + \|y, x_2, \dots, x_n\|$.

Just as in an inner product space, we have the Cauchy-Schwarz inequality:

$$|\langle x, y|x_2, \dots, x_n\rangle| \leq \|x, x_2, \dots, x_n\|\,\|y, x_2, \dots, x_n\|,$$

and the equality holds if and only if $x, y, x_2, \dots, x_n$ are linearly dependent (see [9]). Furthermore, we have the polarization identity:

$$\|x+y, x_2, \dots, x_n\|^2 - \|x-y, x_2, \dots, x_n\|^2 = 4\langle x, y|x_2, \dots, x_n\rangle,$$

and the parallelogram law:

$$\|x+y, x_2, \dots, x_n\|^2 + \|x-y, x_2, \dots, x_n\|^2 = 2(\|x, x_2, \dots, x_n\|^2 + \|y, x_2, \dots, x_n\|^2).$$

The latter gives a characterization of n-inner product spaces.

By the polarization identity and the property (I2), one may observe that

$$\langle x, y|x_2, \dots, x_n\rangle = \langle x, y|x_{i_2}, \dots, x_{i_n}\rangle,$$

for every permutation $(i_2, \dots, i_n)$ of $(2, \dots, n)$. Moreover, one can also show that

$$\langle x, y|x_2, \dots, x_n\rangle = 0,$$

when x or y is a linear combination of $x_2, \dots, x_n$, or when $x_2, \dots, x_n$ are linearly dependent.

Now, for example, if $(X, \langle\cdot,\cdot\rangle)$ is an inner product space, then the following function

$$\langle x, y|x_2, \ldots, x_n\rangle := \begin{vmatrix} \langle x,y\rangle & \langle x,x_2\rangle & \cdots & \langle x,x_n\rangle \\ \langle x_2,y\rangle & \langle x_2,x_2\rangle & \cdots & \langle x_2,x_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_n,y\rangle & \langle x_n,x_2\rangle & \cdots & \langle x_n,x_n\rangle \end{vmatrix},$$

defines an n-inner product, called the standard (or simple) n-inner product on X. Its induced n-norm $\|x_1, x_2, \ldots, x_n\|$ represents the volume of the n-dimensional parallelepiped spanned by $x_1, x_2, \ldots, x_n$.

Historically, the concept of n-norms were introduced earlier by Gähler in order to generalize the notion of length, area and volume in a real vector space (see [4, 5, 6]). The objects studied here are n-dimensional parallelepipeds. The concept of n-inner products is thus useful when one talks about the angle between two n-dimensional parallelepipeds having the same $(n-1)$-dimensional base.

In this note, we shall show that in any n-inner product space with $n \geq 2$ we can derive an $(n-k)$-inner product from the n-inner product for each $k \in \{1, \ldots, n-1\}$. In particular, in any n-inner product space, we can derive an inner product from the n-inner product, so that one can talk about, for instance, the angle between two vectors, as one might like to.

In addition, we shall present some related results on n-normed spaces. See [5] and [11] for previous results on these spaces.

2. Main Results

To avoid confusion, we shall sometimes denote an n-inner product by $\langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle_n$ and an n-norm by $\|\cdot,\ldots,\cdot\|_n$.

Let $(X, \langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle_n)$ be an n-inner product space with $n \geq 2$. Fix a linearly independent set $\{a_1, \ldots, a_n\}$ in X. With respect to $\{a_1, \ldots, a_n\}$, define for each $k \in \{1, \ldots, n-1\}$ the function $\langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle_{n-k}$ on X^{n-k+1} by

$$\langle x, y|x_2, \ldots, x_{n-k}\rangle := \sum_{\{i_1,\ldots,i_k\}\subseteq\{1,\ldots,n\}} \langle x, y|x_2, \ldots, x_{n-k}, a_{i_1}, \ldots, a_{i_k}\rangle.$$

Then we have the following fact:

Fact 2.1. *For every* $k \in \{1, \dots, n-1\}$*, the function* $\langle \cdot, \cdot | \cdot, \dots, \cdot \rangle_{n-k}$ *defines an* $(n-k)$*-inner product on* X*. In particular, when* $k = n-1$*,*

$$\langle x, y \rangle := \sum_{\{i_2, \dots, i_n\} \subseteq \{1, \dots, n\}} \langle x, y | a_{i_2}, \dots, a_{i_n} \rangle, \tag{1}$$

defines an inner product on X*.*

Proof. It is not hard to see that the function $\langle \cdot, \cdot | \cdot, \dots, \cdot \rangle_{n-k}$ satisfies the five properties (I1)–(I5) of an $(n-k)$-inner product, except perhaps to establish the second part of (I1). To verify this property, suppose that $x_1, \dots, x_{n-k}$ are linearly dependent. Then $\langle x_1, x_1 | x_2, \dots, x_{n-k}, a_{i_1}, \dots, a_{i_k} \rangle = 0$ for every $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$, and hence $\langle x_1, x_1 | x_2, \dots, x_{n-k} \rangle = 0$. Conversely, suppose that $\langle x_1, x_1 | x_2, \dots, x_{n-k} \rangle = 0$. Then $\langle x_1, x_1 | x_2, \dots, x_{n-k}, a_{i_1}, \dots, a_{i_k} \rangle = 0$ for every $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$. Hence $\{x_1, \dots, x_{n-k}, a_{i_1}, \dots, a_{i_k}\}$ are linearly dependent for every $\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}$. By elementary linear algebra, this can only happen if $x_1, \dots, x_{n-k}$ are linearly dependent (or if $x_1 = 0$ when $k = n-1$).

Corollary 2.2. *Any* n*-inner product space is an* $(n-k)$*-inner product space for every* $k = 1, \dots, n-1$*. In particular, an* n*-inner product space is an inner product space.*

Corollary 2.3. *Let* $\|\cdot, \dots, \cdot\|_n$ *be the induced* n*-norm on* X*. Then, for each* $k \in \{1, \dots, n-1\}$*, the following function*

$$\|x_1, \dots, x_{n-k}\| := \Bigg(\sum_{\{i_1, \dots, i_k\} \subseteq \{1, \dots, n\}} \|x_1, \dots, x_{n-k}, a_{i_1}, \dots, a_{i_k}\|^2 \Bigg)^{1/2},$$

defines an $(n-k)$*-norm that corresponds to* $\langle \cdot, \cdot | \cdot, \dots, \cdot \rangle_{n-k}$ *on* X*. In particular,*

$$\|x\| := \Bigg(\sum_{\{i_2, \dots, i_n\} \subseteq \{1, \dots, n\}} \|x, a_{i_2}, \dots, a_{i_n}\|^2 \Bigg)^{1/2},$$

defines a norm that corresponds to the derived inner product $\langle \cdot, \cdot \rangle$ *on* X*.*

Note that by using a derived inner product, one can develop the notion of orthogonality and the Fourier series theory in an n-inner product space, just like in an inner product space (see [3] and [13] for previous results in this direction). With respect to the derived inner product $\langle\cdot,\cdot\rangle$ defined by (1), one may observe that the set $\{a_1,\dots,a_n\}$ is orthogonal and that $\|a_i\| = \|a_1,\dots,a_n\|$ for every $i = 1,\dots,n$ (see [8]). In particular, if X is n-dimensional, then $\{a_1,\dots,a_n\}$ forms an orthogonal basis for X and each $x \in X$ can be written as $x = \|a_1,\dots,a_n\|^{-2}\sum_{i=1}^{n}\langle x, a_i\rangle a_i$.

Unlike in [13], we can now have an orthogonal set of m vectors with $1 \le m < n$. In general, by using a derived inner product, we have a more relaxed condition for orthogonality than that in [3] or [13].

Furthermore, one may also use the derived inner products and their induced norm to study the convergence of sequences of vectors in an n-inner product space. See some recent results in [10].

2.1. Related results on n-normed spaces

Suppose now that $(X, \|\cdot,\dots,\cdot\|_n)$ is an n-normed space and, as before, $\{a_1,\dots,a_n\}$ is a linearly independent set in X. Then one may check that for each $k \in \{1,\dots,n-1\}$

$$\|x_1,\dots,x_{n-k}\| := \Bigg(\sum_{\{i_1,\dots,i_k\}\subseteq\{1,\dots,n\}} \|x_1,\dots,x_{n-k},a_{i_1},\dots,a_{i_k}\|^2\Bigg)^{1/2}, \qquad (2)$$

defines an $(n-k)$-norm on X (see [5] and [11] for similar results). In particular, the triangle inequality can be verified as follows:

$$\begin{aligned}
&\|x+y,x_2,\dots,x_{n-k}\| \\
&= \Bigg(\sum_{\{i_1,\dots,i_k\}\subseteq\{1,\dots,n\}} \|x+y,x_2,\dots,x_{n-k},a_{i_1},\dots,a_{i_k}\|^2\Bigg)^{1/2} \\
&\le \Bigg(\sum_{\{i_1,\dots,i_k\}\subseteq\{1,\dots,n\}} \Big(\|x,x_2,\dots,x_{n-k},a_{i_1},\dots,a_{i_k}\| \\
&\qquad\qquad + \|y,x_2,\dots,x_{n-k},a_{i_1},\dots,a_{i_k}\|\Big)^2\Bigg)^{1/2} \\
&\le \Bigg(\sum_{\{i_1,\dots,i_k\}\subseteq\{1,\dots,n\}} \|x,x_2,\dots,x_{n-k},a_{i_1},\dots,a_{i_k}\|^2\Bigg)^{1/2}
\end{aligned}$$

$$+\Big(\sum_{\{i_1,\dots,i_k\}\subseteq\{1,\dots,n\}}\|y,x_2,\dots,x_{n-k},a_{i_1},\dots,a_{i_k}\|^2\Big)^{1/2}$$
$$=\|x,x_2,\dots,x_{n-k}\|+\|y,x_2,\dots,x_{n-k}\|.$$

The first inequality follows from the triangle inequality for the n-norm, while the second one follows from the triangle inequality for the l^2-type norm.

In general, for $1\le p\le\infty$, one may observe that

$$\|x_1,\dots,x_{n-k}\|:=\Big(\sum_{\{i_1,\dots,i_k\}\subseteq\{1,\dots,n\}}\|x_1,\dots,x_{n-k},a_{i_1},\dots,a_{i_k}\|^p\Big)^{1/p},$$

also defines an $(n-k)$-norm on X. Among these derived $(n-k)$-norms, however, the case $p=2$ is special in the following sense.

Fact 2.4. *If the n-norm satisfies the parallelogram law*

$$\|x+y,x_2,\dots,x_n\|^2+\|x-y,x_2,\dots,x_n\|^2=2(\|x,x_2,\dots,x_n\|^2+\|y,x_2,\dots,x_n\|^2),$$

then the derived $(n-k)$-norm given by (2) *satisfies*

$$\|x+y,x_2,\dots,x_{n-k}\|^2+\|x-y,x_2,\dots,x_{n-k}\|^2$$
$$=2(\|x,x_2,\dots,x_{n-k}\|^2+\|y,x_2,\dots,x_{n-k}\|^2).$$

In particular, the derived norm satisfies

$$\|x+y\|^2+\|x-y\|^2=2(\|x\|^2+\|y\|^2).$$

Proof. There are two ways to prove it. The first one is by establishing the parallelogram law directly. Indeed, by definition and hypothesis, we have

$$\|x+y,x_2,\dots,x_{n-k}\|^2+\|x-y,x_2,\dots,x_{n-k}\|^2$$
$$=\sum_{\{i_1,\dots,i_k\}\subseteq\{1,\dots,n\}}\Big(\|x+y,x_2,\dots,x_{n-k},a_{i_1},\dots,a_{i_k}\|^2$$
$$+\|x-y,x_2,\dots,x_{n-k},a_{i_1},\dots,a_{i_k}\|^2\Big)$$
$$=\sum_{\{i_1,\dots,i_n\}\subseteq\{1,\dots,n\}}2\Big(\|x,x_2,\dots,x_{n-k},a_{i_1},\dots,a_{i_k}\|^2+\|y,x_2,\dots,x_{n-k},a_{i_1},\dots,a_{i_k}\|^2\Big)$$
$$=2(\|x,x_2,\dots,x_{n-k}\|^2+\|y,x_2,\dots,x_{n-k}\|^2),$$

as desired.

The second way to prove it is by defining an n-inner product $\langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle_n$ on X by

$$\langle x,y|x_2,\ldots,x_n\rangle := \frac{1}{4}(\|x+y,x_2,\ldots,x_n\|^2 - \|x-y,x_2,\ldots,x_n\|^2),$$

and deriving an $(n-k)$-inner product from it with respect to $\{a_1,\ldots,a_n\}$. One will then realize that the derived $(n-k)$-norm is the induced $(n-k)$-norm from the derived $(n-k)$-inner product, and hence the parallelogram law follows.

2.2. Finite-dimensional case

Suppose here that $(X,\langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle_n)$ is an n-inner product space of finite-dimension $d \geq n$. Then one can derive an $(n-k)$-inner product from the n-inner product in a slightly different way. To be precise, take a linearly independent set $\{a_1,\ldots,a_m\}$ in X, with $n \leq m \leq d$. With respect to $\{a_1,\ldots,a_m\}$, define for each $k \in \{1,\ldots,n-1\}$ the function $\langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle_{n-k}$ on X^{n-k+1} by

$$\langle x,y|x_2,\ldots,x_{n-k}\rangle := \sum_{\{i_1,\ldots,i_k\}\subseteq\{1,\ldots,m\}} \langle x,y|x_2,\ldots,x_{n-k},a_{i_1},\ldots,a_{i_k}\rangle.$$

Then we have:

Fact 2.5. *The function $\langle\cdot,\cdot|\cdot,\ldots,\cdot\rangle_{n-k}$ defines an $(n-k)$-inner product on X.*

Proof. Similar to the proof of Fact 2.1.

As we shall see in the next section, we may obtain an interesting inner product from the n-inner product by using a set of d, rather than just n, linearly independent vectors in X (that is, by using a basis for X).

3. Examples

We shall here present some examples showing us what sort of inner products can be derived through (1) when the n-inner product is simple, and how they are related to the original inner product.

Example 3.1. Let $X = \mathbf{R}^n$ be equipped with the standard n-inner product

$$\langle x, y|x_2, \dots, x_n\rangle := \begin{vmatrix} x\cdot y & x\cdot x_2 & \cdots & x\cdot x_n \\ x_2\cdot y & x_2\cdot x_2 & \cdots & x_2\cdot x_n \\ \vdots & \vdots & \ddots & \vdots \\ x_n\cdot y & x_n\cdot x_2 & \cdots & x_n\cdot x_n \end{vmatrix}, \tag{3}$$

where $x \cdot y$ is the usual inner product on $\mathbf{R}^n$. Then one may observe that the derived $(n-k)$-inner product with respect to an orthonormal basis $\{b_1, \dots, b_n\}$ coincides with the standard $(n-k)$-inner product on $\mathbf{R}^n$, that is,

$$\langle x, y|x_2, \dots, x_{n-k}\rangle = \begin{vmatrix} x\cdot y & x\cdot x_2 & \cdots & x\cdot x_{n-k} \\ x_2\cdot y & x_2\cdot x_2 & \cdots & x_2\cdot x_{n-k} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-k}\cdot y & x_{n-k}\cdot x_2 & \cdots & x_{n-k}\cdot x_{n-k} \end{vmatrix}.$$

In particular, the derived inner product $\langle x, y\rangle$ with respect to $\{b_1, \dots, b_n\}$, which is given by

$$\langle x, y\rangle = \langle x, y|b_2, b_3, \dots, b_n\rangle + \langle x, y|b_1, b_3, \dots, b_n\rangle + \cdots + \langle x, y|b_1, b_2, \dots, b_{n-1}\rangle, \tag{4}$$

is precisely the usual inner product $x \cdot y$.

This example tells us that, on $\mathbf{R}^n$, we can define the standard n-inner product by using the usual inner product as in (3) and, conversely, derive the usual inner product from the standard n-inner product via (4).

Example 3.2. Let $X = \mathbf{R}^n$ be equipped with the standard n-inner product as in the preceeding example. Then one may verify that the derived inner product with respect to an arbitrary linearly independent set $\{a_1, \dots, a_n\}$ in X is given by

$$\langle x, y\rangle = \|a_1, \dots, a_n\|^2\,(A^{-1}x)\cdot(A^{-1}y),$$

where A is the $n \times n$ matrix whose i-th column is the vector a_i. Note that, for every $i, j \in \{1, \dots, n\}$, we have

$$\langle a_i, a_j\rangle = \|a_1, \dots, a_n\|^2\, b_i \cdot b_j,$$

where $\{b_1, \dots, b_n\}$ is the standard basis for $\mathbf{R}^n$. This means that $\{a_1, \dots, a_n\}$ is an orthogonal basis for $(X, \langle\cdot,\cdot\rangle)$, as remarked previously in §2.

Remark. By invoking Parseval's identity (see, e.g., [1], p. 354), Examples 3.1 and 3.2 extend to any n-dimensional inner product space X.

Example 3.3. Suppose that X is an inner product space of dimension $d \geq n$ and $\{e_1, \ldots, e_n\}$ is an orthonormal set in X. Equip X with the standard n-inner product as in (3), with $x \cdot y$ being the inner product on X. Then one may observe that the derived inner product with respect to $\{e_1, \ldots, e_n\}$ is given by

$$\langle x, y\rangle = Px \cdot Py + n(Qx \cdot Qy),$$

where P denotes the orthogonal projection on the subspace spanned by $\{e_1, \ldots, e_n\}$ and $Q = I - P$ is its complementary projection. Notice here that its induced norm is equivalent to the original norm.

Although a little bit messy, it is also possible to obtain the expression for the derived $(n-k)$-inner product for each $k \in \{1, \ldots, n-1\}$. For example, the derived $(n-1)$-inner product with respect to $\{e_1, \ldots, e_n\}$ is given by

$$\begin{aligned}
&\langle x, y|x_2, \ldots, x_{n-1}\rangle \\
&= \begin{vmatrix} x \cdot y & x \cdot x_2 & \cdots & x \cdot x_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_i \cdot y & x_i \cdot x_2 & \cdots & x_i \cdot x_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1} \cdot y & x_{n-1} \cdot x_2 & \cdots & x_{n-1} \cdot x_{n-1} \end{vmatrix} + \sum_{i=1}^{n-1} \begin{vmatrix} x \cdot y & x \cdot x_2 & \cdots & x \cdot x_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ Qx_i \cdot Qy & Qx_i \cdot Qx_2 & \cdots & Qx_i \cdot Qx_{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n-1} \cdot y & x_{n-1} \cdot x_2 & \cdots & x_{n-1} \cdot x_{n-1} \end{vmatrix},
\end{aligned}$$

with x_1 being identified as x.

Example 3.4. Let $X = \mathbf{R}^d$ be equipped with the standard n-inner product as in (3), with $x \cdot y$ being the usual inner product on $\mathbf{R}^d$. Then one may particularly observe that the derived inner product with respect to an orthonormal basis $\{b_1, \ldots, b_d\}$ is given by

$$\langle x, y\rangle = \sum_{\{i_2, \ldots, i_n\} \subseteq \{1, \ldots, d\}} \langle x, y|b_{i_2}, \ldots, b_{i_n}\rangle = C_{n-1}^{d-1}\, x \cdot y,$$

where $C_{n-1}^{d-1} = \frac{(d-1)!}{(d-n)!(n-1)!}$. This derived inner product is better than the previous one in the sense that it is only a multiple of the usual inner product.

Remark. By invoking Parseval's identity, Example 3.4 may also be extended to any finite d-dimensional inner product space X.

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