WEIGHTED ESTIMATES FOR IMAGINARY POWERS OF THE LAPLACE OPERATOR*

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Abstract. In this paper we review some estimates for singular integral operators that are imaginary powers of the Laplace operator in $\mathbf{R}^n$. These estimates lead to some estimates for a family of maximal operators, some of which are closely related to the (weak) solution of the wave equation. We shall also discuss recent weighted estimates for these operators, including some weighted $H^p - L^p$ estimates with $0 < p \leq 1$.

1. Introduction

Let Δ be the standard Laplace operator in $\mathbf{R}^n$, given by

$$\Delta = -\sum_{j=1}^{n} \partial_j^2.$$

If $\widehat{f}(\xi) = \int_{\mathbf{R}^n} e^{2\pi i x\cdot\xi} f(x)\, dx$, then

$$(\Delta f)\widehat{\ }(\xi) = (2\pi|\xi|)^2 \widehat{f}(\xi).$$

Inspired by this relation, one may then define $\Delta^{\beta/2}$ for any exponent β by

$$(\Delta^{\beta/2} f)\widehat{\ }(\xi) = (2\pi|\xi|)^\beta \widehat{f}(\xi).$$

In particular, for each $0 < \alpha < n$, the operator

$$I_\alpha : f \mapsto \Delta^{-\alpha/2} f$$

is known as the Riesz potential. Here I_α may be expressed as

$$I_\alpha f = K_\alpha * f$$

where $K_\alpha(x) = \pi^{-n/2} 2^{-\alpha} \Gamma(\frac{n-\alpha}{2})/\Gamma(\frac{\alpha}{2})\, |x|^{-n+\alpha}$, and this tells us that I_α is an integral operator (see [8], p. 117).

In this paper, we shall consider the operator I_{iu}, $u \in \mathbf{R} \setminus \{0\}$, given by

$$I_{iu} f = K_{iu} * f,$$

which makes sense via

$$(I_{iu} f)\widehat{\ }(\xi) = (2\pi|\xi|)^{-iu} \widehat{f}(\xi),$$

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that is, $I_{iu} = \Delta^{-iu/2}$, an imaginary power of Δ. This operator was studied by B. Muckenhoupt [7] in 1960 and used by Cowling and Mauceri [1] in 1978 to prove E.M. Stein's theorem on the spherical maximal function [9].

Note that $|\widehat{K_{iu}}(\xi)| = |(2\pi|\xi|)^{-iu}| = 1$, so that by Plancherel's theorem we have

$$\|I_{iu}f\|_2 = \|f\|_2.$$

Hence I_{iu}, which is initially defined on $S(\mathbf{R}^n)$, extends to an isometry (hence a bounded operator) on $L^2(\mathbf{R}^n)$. By using further properties of the kernel K_{iu} (particularly the fact that it is locally integrable away from the origin and satisfies $|K_{iu}(x)| \leq C(1+|u|)^{n/2}|x|^{-n}$ and $|\nabla K_{iu}(x)| \leq C(1+|u|)^{n/2+1}|x|^{-n-1}$ for $x \neq 0$), one may observe that I_{iu} also extends to a bounded operator from the Hardy space $H^1(\mathbf{R}^n)$ to $L^1(\mathbf{R}^n)$. Indeed, one may show that

$$\|I_{iu}f\|_1 \leq C(1+|u|)^{n/2}\|f\|_{H^1}.$$

By interpolation and duality arguments, we obtain

$$\|I_{iu}f\|_p \leq C_p(1+|u|)^{|n/p-n/2|}\|f\|_p$$

for $1 < p < \infty$, that is, I_{iu} extends to a bounded operator on $L^p(\mathbf{R}^n)$ (see [5]).

In the following sections, we shall indicate that I_{iu} also extends to a bounded operator from weighted Hardy spaces $H^p_w(\mathbf{R}^n)$ to weighted Lebesgue spaces $L^p_w(\mathbf{R}^n)$ for $0 < p \leq 1$ and w in the Muckenhoupt's class A_p. But first we shall demonstrate, via Mellin transform arguments, how the above estimates for I_{iu} can be used to obtain estimates for a family of maximal operators, some of which are closely related to the (weak) solution of the wave equation.

2. An application

Let L be a positive self-adjoint operator given by

$$Lf = -\sum_{j,k=1}^{n} \partial_j a_{jk} \partial_k f$$

where $a_{jk} \in C^\infty(\mathbf{R}^n)$, $a_{jk} = a_{kj}$ for $1 \leq j,k \leq n$. Then L has a spectral resolution

$$L = \int_{\mathbf{R}^+} \lambda \, dE_L(\lambda)$$

where $E_L(\lambda)$'s are the spectral projectors.

For any bounded Borel function $F : \mathbf{R}^+ \to \mathbf{C}$, define the operator $F(L)$ by

$$F(L) = \int_{\mathbf{R}^+} F(\lambda) \, dE_L(\lambda).$$

As in [6], we define the maximal operator $M_{F,L}$ by

$$M_{F,L}f = \sup_{t>0} |F(tL)f|.$$

Assuming $F(0)=0$, we may write

$$F(\lambda)=\int_{\mathbf{R}} A(u)\lambda^{iu}du.$$

By Mellin transform arguments, this holds if and only if

$$A(u)=\frac{1}{2\pi}\int_{\mathbf{R}^+} F(\lambda)\lambda^{-1-iu}d\lambda.$$

If, for some $1<p<\infty$, we have

$$\int_{\mathbf{R}} |A(u)|\,\|L^{iu}\|_{L^p\to L^p}du\le C_p,$$

then

$$\|F(tL)\|_{L^p\to L^p}\le C_p$$

for all $t>0$, and hence $M_{F,L}$ is bounded on $L^p(\mathbf{R}^n)$.

For instance, for $\mathrm{Re}(\alpha)>0$, let

$$F_\alpha(\lambda)=\int_{-1}^{1}(1-x^2)^{\alpha-1}e^{i\lambda x}dx.$$

Note that $F_0(\lambda)=\cos\lambda$ and $F_1(\lambda)=\frac{2\sin\lambda}{\lambda}$ are connected with the solution of the wave equation. Moreover, one may observe that

$$F_\alpha(\lambda)=c_\alpha\lambda^{-\alpha+1/2}J_{\alpha-1/2}(\lambda)$$

for some constant c_α.

Theorem 1. [6] *If $L=\sqrt{\Delta}$, then $M_{F_\alpha,L}$ is bounded on $L^p(\mathbf{R}^n)$ for*
(a) $\mathrm{Re}(\alpha)>\frac{n}{p}-\frac{n}{2}+\frac{1}{2},\ 1<p\le 2$*; and*
(b) $\mathrm{Re}(\alpha)>\frac{n}{2}-\frac{n}{p}+\frac{2}{p}-\frac{1}{2},\ 2\le p\le\infty$.

`Remark`. For $\alpha=\frac{n+1}{2}$, $M_{F_\alpha,\sqrt{\Delta}}$ is the Hardy-Littlewood maximal operator, while for $\alpha=\frac{n-1}{2}$, it is the Stein's spherical maximal operator (see [9]). For $\alpha=1$, $M_{F_1,\sqrt{\Delta}}$ is connected with the solution of the wave equation. Indeed, for an appropriate constant c_n, one may verify that $u(x,t)=c_ntF_1(t\sqrt{\Delta})f(x)$ is the weak solution of the wave equation

$$\partial_t^2u=\Delta u$$

subject to the initial data

$$u(x,0)=0,\ \partial_tu(x,0)=f(x)$$

(see [10], p. 519). From the above theorem, we see that $M_{F_1,\sqrt{\Delta}}$ is bounded for $\frac{2n}{n+1}<p\le\infty$ (if $n\le 3$) and for $\frac{2n}{n+1}<p<\frac{2(n-2)}{n-3}$ (if $n\ge 4$). Consequently, we have

$$\left\|\sup_{t>0}\left|\frac{u(\cdot,t)}{t}\right|\right\|_p\le C_p\|f\|_p$$

for the above values of p's.

3. Recent estimates

The $H^1 - L^1$ estimates for I_{iu} can be extended to $H^p - L^p$ estimates for every $0 < p \leq 1$. To be precise, we have the following result.

Theorem 2. [4] *The $H^p - L^p$ inequality*

$$\|I_{iu}f\|_p \leq C_p(1 + |u|)^{n/p-n/2}\|f\|_{H^p}$$

holds for $0 < p \leq 1$.

There are also some weighted estimates for I_{iu}. Recall that a nonnegative locally integrable function w on $\mathbf{R}^n$ is said to belong to the *Muckenhoupt's class A_1* if

$$\frac{1}{|Q|}\int_Q w(x)dx \leq Cw(y), \quad \text{a.e. } y \in Q,$$

for all cubes Q in $\mathbf{R}^n$. Moreover, $w \in A_p,\ 1 < p < \infty$, if

$$\Big[\frac{1}{|Q|}\int_Q w(x)dx\Big]\Big[\frac{1}{|Q|}\int_Q w(x)^{-1/(p-1)}dx\Big]^{p-1} \leq C$$

for all cubes Q in $\mathbf{R}^n$. For example, $|\cdot|^a \in A_1$ if and only if $-n < a \leq 0$, and $|\cdot|^a \in A_p,\ 1 < p < \infty$, if and only if $-n < a < n(p-1)$.

Note that if $1 < p < q < \infty$, then $A_1 \subseteq A_p \subseteq A_q$. Also, if $w \in A_p,\ 1 < p < \infty$, then there exists $\epsilon > 0$ and $1 < q < p$ such that $w^{1+\epsilon} \in A_q$. Further, for every $1 < p < \infty$, we have $w \in A_p$ if and only if $w = w_1 w_2^{1-p}$ for some $w_1, w_2 \in A_1$. For further discussion on A_p weights, see Chapter IV of [3].

Let $\|f\|_{p,w} = \left[\int_{\mathbf{R}^n}|f(x)|^p w(x)dx\right]^{1/p}$ and $L^p_w(\mathbf{R}^n) = \{f : \mathbf{R}^n \to \mathbf{C} \mid \|f\|_{p,w} < \infty\}$. Denote also by $H^p_w(\mathbf{R}^n)$ the weighted Hardy spaces (see [2]). Then, we have the following results.

Theorem 3. [5] *For $1 < p < \infty$, the inequality*

$$\|I_{iu}f\|_{p,w} \leq C_{p,w}(1 + |u|)^{n/2}\|f\|_{p,w}$$

holds whenever $w \in A_p$.

Theorem 4. [4] *Let $0 < p \leq 1$. Suppose that $n/(n+k) < p \leq n/(n+k-1)$ for some $k \in \mathbf{N}$ and let $0 < \epsilon < n + k - n/p$. Then, the inequality*

$$\|I_{iu}f\|_p \leq C_p(1 + |u|)^{n/p-n/2+\epsilon}\|f\|_{H^p_w}$$

holds whenever $w \in A_{1+\epsilon p/n}$.

`Remark`. Notice that as ϵ tends to 0, the exponent of $(1 + |u|)$ tends to $n/p - n/2$ and the set of weights for which the inequality holds tends to the Muckenhoupt's class A_1. Up to now, however, we still do not know whether the inequality

$$\|I_{iu}f\|_{p,w} \leq C_{p,w}(1 + |u|)^{n/p-n/2}\|f\|_{H^p_w}$$

holds for $0 < p \leq 1$ and $w \in A_1$.

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