WEIGHTED ESTIMATES FOR SOME SINGULAR INTEGRALS

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Abstract. Let K_u be the distribution on $\mathbf{R}^d,\ d \geq 1$, whose Fourier transform is $\widehat{K_u}(\xi) := |\xi|^{-iu},\ u \in \mathbf{R}$. We prove the unweighted $H^1 - L^1$ and weighted $L^p - L^p$ estimates for $K_u * f$ with bound of order $(1+|u|)^{d/2}$. These improve previous estimates obtained by [CM] and [G1]. Our method of proving the weighted estimate for $K_u * f$ also applies to singular integrals with oscillatory factors that are exponentials of some imaginary polynomials, at least in the one dimensional case.

1. Introduction and Main Results

In order to obtain estimates for the spherical maximal function $M_{\rm S}f := \sup_{r>0} |\phi_r * f|$ where ϕ_r denotes the normalised surface measure on the sphere of centre 0 and radius r in $\mathbf{R}^d$ (see [S2]), one may use the Mellin transform method to control $M_{\rm S}f$ by

$$M_{\rm S}f \leq M_{\rm P}f + C \int_{\mathbf{R}} (1+|u|)^{-d/2} |K_u * f(\cdot)|\, du$$

where $M_{\rm P}f$ denotes the maximal function related to Poisson kernels and K_u is the distribution whose Fourier transform is $\widehat{K_u}(\xi) = |\xi|^{-iu}$, and then try to get a good estimate for $K_u * f$ (see [CM] and [G1]).

For any small positive δ, it can be shown that the following estimate

$$\|K_u * f\|_p \leq C_\delta (1+|u|)^{d|1/p-1/2|+\delta} \|f\|_p, \quad f \in L^p(\mathbf{R}^d),$$

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holds for $1 < p < \infty$ (see [CM]). For the weighted case, [G1] shows that the following estimate

$$\|K_u * f\|_{p,w} \le C_\delta (1+|u|)^{d/2+\delta} \|f\|_{p,w}, \quad f \in L^p_w(\mathbf{R}^d),$$

holds whenever $w \in A_p$, $1 < p < \infty$ (see [GR] for discussion of A_p weights). In both estimates, however, the smaller the value of δ, the bigger the constant C_δ. In this note, we shall show that the appearance of δ is in fact unnecessary and accordingly its effect to the constant bound is removed. Precisely, we have the following theorems:

Theorem A. *The unweighted estimate*

$$\|K_u * f\|_1 \le C(1+|u|)^{d/2} \|f\|_{H^1}, \quad f \in H^1(\mathbf{R}^d), \tag{1.1}$$

holds, and consequently the unweighted estimate

$$\|K_u * f\|_p \le C(1+|u|)^{d|1/2-1/p|} \|f\|_p, \quad f \in L^p(\mathbf{R}^d), \tag{1.2}$$

holds for $1 < p < \infty$.

Theorem B. *The weighted estimate*

$$\|K_u * f\|_{p,w} \le C(1+|u|)^{d/2} \|f\|_{p,w}, \quad f \in L^p_w(\mathbf{R}^d), \tag{1.3}$$

holds whenever $w \in A_p$, $1 < p < \infty$.

Note that the bound in (1.1) and (1.3) is sharp in the sense that exponent of $(1+|u|)$ cannot be less than $d/2$ for otherwise the spherical maximal operator M_{S} would be bounded on $L^p(\mathbf{R}^d)$ for some $p < d/(d-1)$. (We also expect that the weighted weak $L^1 - L^1$ estimate

$$w(\{x : |K_u * f(x)| \ge t\}) \le Ct^{-1}(1+|u|)^{d/2} \|f\|_{1,w}, \quad f \in L^1_w(\mathbf{R}^d),\ t > 0,$$

holds whenever $w \in A_1$, but the proof can be quite involved. See [SW] for the unweighted case.)

Our method of proving the weighted estimate (1.3) also applies to singular integrals with oscillatory factors that are exponentials of some imaginary polynomials, at least in the one dimensional case. In particular, we have:

Theorem C. *Let P be a real polynomial on $\mathbf{R}$ of degree $n \geq 2$. Define the operator T_P by*

$$T_P f(x) := \text{p.v.} \int_{\mathbf{R}} e^{iP(x-y)} \frac{1}{x-y} f(y)\, dy,$$

for any suitable function f on $\mathbf{R}$. Then the weighted estimate

$$\|T_P f\|_{p,w} \leq C\|f\|_{p,w}, \quad f \in L^p_w(\mathbf{R}), \tag{1.4}$$

holds for all $w \in A_p,\ 1 < p < \infty$. Here C may depend on the degree but not on the coefficents of P.

Theorem C may be extended to the d-dimensional case as follows. Let P be a real polynomial on $\mathbf{R}$ of degree $n \geq 2$ that has no linear terms and K be a standard Calderon-Zygmund kernel. Define the operator $T_{P,K}$ by

$$T_{P,K} f(x) := e^{iP(|\cdot|)} K * f(x),$$

for any suitable function f on $\mathbf{R}^d$. Then the weighted estimate

$$\|T_{P,K} f\|_{p,w} \leq C\|f\|_{p,w}, \quad f \in L^p_w(\mathbf{R}^d),$$

holds for all $w \in A_p,\ 1 < p < \infty$.

Unweighted estimates for more general operators of this type can be found in, e.g., [RS]. See also [CRW] for related work.

2. Proof of Theorems A and B

As it is known, K_u may be given explicitly by $K_u(x) := C(u)|x|^{-d+iu}$, where $C(u) := \pi^{-d/2+iu}\Gamma\big(\frac{d-iu}{2}\big)/\Gamma\big(\frac{iu}{2}\big)$ (see [S1, p. 117]). Using properties of Gamma functions (see, e.g., [T, p. 151]), one may show that $C(u) = \mathrm{O}((1+|u|)^{d/2})$. Further, for $|x| \geq 2|y| > 0$, we have the following estimates

$$|K_u(x-y) - K_u(x)| \leq C(1+|u|)^{d/2}|x|^{-d} \tag{2.1}$$

and

$$|K_u(x-y) - K_u(x)| \le C(1+|u|)^{d/2+1}|y||x|^{-d-1}. \tag{2.2}$$

Since we are mainly concerned with large magnitudes of u, let us assume hereafter that $|u| \ge 2$ (so that (2.1) and (2.2) hold for $|x| \ge |u||y|$ and $|C(u)| \le C|u|^{d/2}$). As usual, C will denote a constant, which may depend on d, p and w, but not on u, and its value may vary from line to line. Other constants will be denoted according to their dependence.

Proof of Theorem A. The idea is to split the integral at $|u||y|$ rather than at $2|y|$ and exploit the L^2 theory. By writing f as $f := \sum_j \lambda_j a_j$ for some $(\lambda_j) \in l^1$ and a sequence of atoms (a_j), where $\sum_j |\lambda_j| \le \|f\|_{H^1}$ and each a_j is supported in a cube Q_j such that $\|a_j\|_\infty \le \frac{1}{|Q_j|}$ and $\int_{\mathbf{R}^d} a_j(x)\,dx = 0$, it suffices to show

$$\|K_u * a\|_1 \le C\,|u|^{d/2}$$

for any atom a. Without loss of generality, we may assume that a is supported in $Q := [-R,R]^d$. Now,

$$\|K_u * a\|_1 = \int_{\mathbf{R}^d} |K_u * a(x)|\,dx = \int_{|x|\le|u|R} |K_u * a(x)|\,dx + \int_{|x|\ge|u|R} |K_u * a(x)|\,dx.$$

For the first integral, we use Cauchy-Schwarz inequality and the L^2 theory to obtain

$$\begin{aligned}\int_{|x|\le|u|R} |K_u * a(x)|\,dx &\le (|u|R)^{d/2}\|K_u * a\|_2 = |u|^{d/2}R^{d/2}\|a\|_2 \\ &\le |u|^{d/2}R^{d/2}R^{-d}R^{d/2} = |u|^{d/2}.\end{aligned}$$

Meanwhile, for the second integral, we use (2.2) to get

$$\begin{aligned}\int_{|x|\ge|u|R} |K_u * a(x)|\,dx &= C(u)\int_{|x|\ge|u|R}\left|\int_{|y|\le R} a(y)|x-y|^{-d+iu}dy\right|dx \\ &= C(u)\int_{|x|\ge|u|R}\left|\int_{|y|\le R} a(y)\left[|x-y|^{-d+iu} - |x|^{-d+iu}\right]dy\right|dx \\ &\le C|u|^{d/2}\int_{|y|\le R}|a(y)|\int_{|x|\ge|u|R}\left||x-y|^{-d+iu} - |x|^{-d+iu}\right|dx\,dy \\ &\le C|u|^{d/2+1}\int_{|y|\le R}|a(y)||y|\int_{|x|\ge|u|R}|x|^{-d-1}\,dx\,dy \\ &\le C|u|^{d/2+1}R|u|^{-1}R^{-1} \\ &= C|u|^{d/2}.\end{aligned}$$

This verifies (1.1). By interpolation with the trivial $L^2 - L^2$ estimate and duality arguments, (1.2) follows. □

Proof of Theorem B. As argued in [KW] (see also [GR, p. 411]), in order to verify (1.3), we only need to show that for every $r \in (1,2)$ we have the pointwise estimate

$$(K_u * f)^{\#}(x) \leq C_r |u|^{d/2} M_r f(x),$$

where $M_r f := (M|f|^r)^{1/r}$. (Here $^{\#}$ denotes the sharp maximal operator and M is the Hardy-Littlewood maximal operator). By translation invariance, it suffices to prove it at the origin.

Let $Q := [-R, R]^d$. If we can find a constant c_Q depending on Q such that

$$\frac{1}{|Q|} \int_Q |K_u * f(x) - c_Q| \, dx \leq C_r |u|^{d/2} M_r f(0),$$

then we are done. Choose any nonnegative function $\psi \in C_c^\infty(\mathbf{R}^d)$ such that $\mathrm{supp}(\psi) \subseteq \{y : \frac{1}{2} \leq |y| \leq 2\} =: I$ and $\sum_{j\in\mathbf{Z}} \psi(2^{-j}y) \equiv 1$ on $\mathbf{R}^d \setminus \{0\}$. Write

$$f(y) = \sum_{j\in\mathbf{Z}} \psi(2^{-j}y) f(y) =: \sum_{j\in\mathbf{Z}} f_j(y).$$

Then c_Q will be of the form $c_Q = \sum_{j\in\mathbf{Z}} c_j$ where c_j is either 0 or $K_u * f_j(0)$, and

$$\frac{1}{|Q|} \int_Q |K_u * f(x) - c_Q| \, dx \leq \sum_{j\in\mathbf{Z}} \frac{1}{|Q|} \int_Q |K_u * f_j(x) - c_j| \, dx.$$

To estimate the sum, we consider the following three cases.

Case 1. For $j \in \mathbf{Z}$ such that $2^j \leq 8R$, take $c_j := 0$. Then we have

$$\begin{aligned}
\sum_{2^j \leq 8R} \frac{1}{|Q|} \int_Q |K_u * f_j(x)| \, dx &\leq \sum_{2^j \leq 8R} |Q|^{-1/r} \Big[\int_Q |K_u * f_j(x)|^r dx \Big]^{1/r} \\
&\leq \sum_{2^j \leq 8R} C_r |u|^{d/2} |Q|^{-1/r} \|f_j\|_r \quad \text{(by Theorem A)} \\
&\leq \sum_{2^j \leq 8R} C_r |u|^{d/2} R^{-d/r} \Big[\int_{|x| \leq 2^{j+1}} |f(x)|^r dx \Big]^{1/r} \\
&\leq C_r |u|^{d/2} M_r f(0) \sum_{2^j \leq 8R} \big(\frac{2^j}{R}\big)^{d/r} \\
&\leq C_r |u|^{d/2} M_r f(0).
\end{aligned}$$

Case 2. For $j \in \mathbf{Z}$ such that $2^j \geq 8|u|R$, take $c_j := K_u * f_j(0)$. Then, by (2.2), we have

$$\begin{aligned}
\sum_{2^j \geq 8|u|R} & \frac{1}{|Q|} \int_Q |K_u * f_j(x) - K_u * f_j(0)| \, dx \\
&\leq \sum_{2^j \geq 8|u|R} \frac{1}{|Q|} \int_Q \Big| \int_{\mathbf{R}^d} \big(K_u(x-y) - K_u(-y)\big) f_j(y) dy \Big| \, dx \\
&\leq \sum_{2^j \geq 8|u|R} C|u|^{d/2+1} \frac{1}{|Q|} \int_Q \int_{2^{j-1} \leq |y| \leq 2^{j+1}} |x||y|^{-d-1} |f(y)| \, dy \, dx \\
&\leq C|u|^{d/2} Mf(0) \sum_{2^j \geq 8|u|R} \frac{|u|R}{2^j} \\
&\leq C|u|^{d/2} Mf(0) \leq C|u|^{d/2} M_r f(0).
\end{aligned}$$

Case 3. For $j \in \mathbf{Z}$ such that $8R < 2^j < 8|u|R$, take $c_j := 0$. Observe that for every $x \in Q$

$$\begin{aligned}
K_u * f_j(x) &= C(u) \int_{\mathbf{R}^d} e^{iu \log |x-y|} |x-y|^{-d} f_j(y) \, dy \\
&= C(u) \int_{\mathbf{R}^d} e^{iu \log |x-y|} |y|^{-d} f_j(y) \, dy \\
&\quad + C(u) \int_{\mathbf{R}^d} e^{iu \log |x-y|} (|x-y|^{-d} - |y|^{-d}) f_j(y) \, dy.
\end{aligned}$$

The error term is fine since it is bounded by $C|u|^{d/2} R \, 2^{-j} Mf(0)$ and $\sum_{8R < 2^j < 8|u|R} \frac{R}{2^j} \leq C$. Next consider

$$\begin{aligned}
\int_{\mathbf{R}^d} & e^{iu \log |x-y|} |y|^{-d} f_j(y) \, dy \\
&= \int_{2^{j-1} \leq |y| \leq 2^{j+1}} e^{i(u/2) \log |x-y|^2} \psi(2^{-j} y) f(y) |y|^{-d} dy \\
&= \int_{1/2 \leq |y| \leq 2} e^{i(u/2) \log |x - 2^j y|^2} \psi(y) f(2^j y) |y|^{-d} dy \\
&=: \int_{\mathbf{R}^d} e^{i\lambda_u \Phi(x,y)} \psi(y) \tilde{f}(y) \, dy \\
&=: I_{\lambda_u}(\tilde{f})
\end{aligned}$$

where

$$\begin{aligned}
\lambda_u &:= 2^{-j} u \\
\Phi(x,y) &:= 2^{j-1} \log |x - 2^j y|^2 \\
\tilde{f}(y) &:= f(2^j y) |y|^{-d} \chi_I(y).
\end{aligned}$$

Here we have

Claim 1. $\left|\det\left(\frac{\partial^2\Phi}{\partial x_k \partial y_l}\right)\right| > 3^{-2d}$ on $Q \times I$.

So Hörmander's theorem (see [S3, p. 377]) applies to $I_{\lambda_u}(\tilde{f})$ and we have

$$\|I_{\lambda_u}(\tilde{f})\|_{r'} \le C_r \lambda_u^{-d/r'} \|\tilde{f}\|_r.$$

But $\lambda_u = 2^{-j}u$ and

$$\begin{aligned}
\|\tilde{f}\|_r^r &= \int_{1/2 \le |y| \le 2} |f(2^j y)|^r |y|^{-dr} dy \\
&\le C \int_{1/2 \le |y| \le 2} |f(2^j y)|^r dy \\
&= C 2^{-jd} \int_{2^{j-1} \le |y| \le 2^{j+1}} |f(y)|^r dy \\
&\le C\big(M_r f(0)\big)^r.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{1}{|Q|}\int_Q |K_u * f_j(x)|\, dx &\le C|u|^{d/2} \frac{1}{|Q|}\int_Q |I_{\lambda_u}(\tilde{f})|\, dx + C|u|^{d/2} R\, 2^{-j} M f(0) \\
&\le C|u|^{d/2} |Q|^{-1/r'} \|I_{\lambda_u}(\tilde{f})\|_{r'} + C|u|^{d/2} R\, 2^{-j} M_r f(0) \\
&\le C|u|^{d/2 - d/r'} R^{-d/r'} 2^{jd/r'} M_r f(0) + C|u|^{d/2} R\, 2^{-j} M_r f(0).
\end{aligned}$$

Summing over $j \in \mathbf{Z}$ for which $8R < 2^j < 8|u|R$, we obtain

$$\begin{aligned}
&\sum_{8R < 2^j < 8|u|R} \frac{1}{|Q|}\int_Q |K_u * f_j(x)|\, dx \\
&\qquad \le C|u|^{d/2} M_r f(0) \Big[\sum_{2^j < 8|u|R} \big(\frac{2^j}{|u|R}\big)^{d/r} + \sum_{8R < 2^j} \frac{R}{2^j} \Big] \\
&\qquad \le C|u|^{d/2} M_r f(0).
\end{aligned}$$

as desired, and this completes the proof of Theorem B. □

3. Proof of Theorem C

The idea of the proof will be similar to that of Theorem B, except that now we need to tackle the oscillatory factors more carefully. Suppose $P(x) := \sum_{m=0}^n a_m x^m$, $a_m \in \mathbf{R}$. Since

$$T_P f(x) = e^{i(a_0 + a_1 x)} T_Q(e^{-ia_1 y} f(y))(x)$$

with $Q(x) := \sum_{m=2}^{n} a_m x^m$, we may assume that $a_0 = a_1 = 0$. As before, in order to verify (1.4), we only need to show that for every $r \in (1, 2)$ we have

$$(T_P f)^{\#}(0) \leq C_r M_r f(0),$$

where C_r may depend on the degree but not on the coefficients of P.

To do so, let $Q := [-R, R]$. Our task then is to find a constant c_Q depending on Q such that

$$\frac{1}{|Q|} \int_Q |T_P f(x) - c_Q| \, dx \leq C_r M_r f(0).$$

Again, choose any nonnegative function $\psi \in C_c^\infty(\mathbf{R})$ such that $\mathrm{supp}(\psi) \subseteq \{y : \frac{1}{2} \leq |y| \leq 2\} =: I$ and $\sum_{j \in \mathbf{Z}} \psi(2^{-j} y) \equiv 1$ on $\mathbf{R} \setminus \{0\}$, and write

$$f(y) = \sum_{j \in \mathbf{Z}} \psi(2^{-j} y) f(y) =: \sum_{j \in \mathbf{Z}} f_j(y).$$

Then c_Q will be of the form $c_Q = \sum_{j \in \mathbf{Z}} c_j$ where c_j is either 0 or $-\int_{\mathbf{R}} e^{iP(-y)} f_j(y) y^{-1} dy$, and

$$\frac{1}{|Q|} \int_Q |T_P f(x) - c_Q| \, dx \leq \sum_{j \in \mathbf{Z}} \frac{1}{|Q|} \int_Q |T_P f_j(x) - c_j| \, dx.$$

To estimate the sum, we consider several cases.

Case 1. For $j \in \mathbf{Z}$ such that $2^j \leq 8R$, take $c_j := 0$. Then, using the fact that T_P is bounded on $L^r(\mathbf{R})$ (see [RS]), we have

$$\begin{aligned}
\sum_{2^j \leq 8R} \frac{1}{|Q|} \int_Q |T_P f_j(x)| \, dx &\leq \sum_{2^j \leq 8R} |Q|^{-1/r} \Big[\int_{\mathbf{R}} |T_P f_j(x)|^r dx \Big]^{1/r} \\
&\leq C_r \sum_{2^j \leq 8R} |Q|^{-1/r} \|f_j\|_r \\
&\leq C_r \sum_{2^j \leq 8R} R^{-1/r} \Big[\int_{|x| \leq 2^{j+1}} |f(x)|^r dx \Big]^{1/r} \\
&\leq C_r M_r f(0) \sum_{2^j \leq 8R} \big(\frac{2^j}{R} \big)^{1/r} \\
&\leq C_r M_r f(0).
\end{aligned}$$

Case 2. For $j \in \mathbf{Z}$ such that $2^j > 8R$, define

$$S_P f_j(x) := -\int_{\mathbf{R}} e^{iP(x-y)} f_j(y) y^{-1} dy.$$

Observe that for $x \in Q$, we have

$$|T_P f_j(x) - S_P f_j(x)| \le |x| \int_{\mathbf{R}} |f_j(y)||y|^{-2} dy \le R\, 2^{-j} Mf(0).$$

But $\sum_{2^j > 8R} \frac{R}{2^j} \le C$, and so we obtain

$$\begin{aligned}
&\sum_{2^j > 8R} \frac{1}{|Q|} \int_Q |T_P f_j(x) - c_j|\, dx \\
&\le \sum_{2^j > 8R} \Big[\frac{1}{|Q|} \int_Q |T_P f_j(x) - S_P f_j(x)|\, dx + \frac{1}{|Q|} \int_Q |S_P f_j(x) - c_j|\, dx \Big] \\
&\le C\, Mf(0) + \sum_{2^j > 8R} \frac{1}{|Q|} \int_Q |S_P f_j(x) - c_j|\, dx.
\end{aligned}$$

It now remains to estimate the last sum. Here we shall apply an extension of Lemma 2.5 of [CRW] to $P(x) = \sum_{m=2}^n a_m x^m$ to obtain a constant $A = A_n$ and a decomposition of $\mathbf{R}$ into a family of disjoint symmetric intervals

$$\mathbf{R} = \bigcup_{k=1}^{K_1} I_k \;\cup \bigcup_{k=1}^{K_2} J_k$$

where $K_1 \le n$ and $K_2 \le 8n^2$. Each J_k is a dyadic interval of the form $[-A^{l_k+1}, -A^{l_k}] \cup [A^{l_k}, A^{l_k+1}]$ and the I_k's (which are called "gaps") have the following property: for each I_k there exist constants $0 < \epsilon_n < C_n$, an integer m_k where $2 \le m_k \le n$, and a constant D_k (depending on the coefficients of P) such that for every $x \in I_k$,

(i) $\epsilon_n D_k |x|^{m_k} \le |P(x)| \le C_n D_k |x|^{m_k}$,

(ii) $\frac{\epsilon_n}{|x|} \le \left| \frac{P'(x)}{P(x)} \right| \le \frac{C_n}{|x|}$,

(iii) $\frac{\epsilon_n}{x^2} \le \left| \frac{P''(x)}{P(x)} \right| \le \frac{C_n}{x^2}$.

Note that (i) and the first inequality in (ii) are contained in the original lemma. For the proof of the remaining inequalities, especially that of the first inequality in (iii) (which we will refer to as Claim 2), see Appendix.

For each k, $1 \le k \le K_1$, define

$$E_k := \{ j \in \mathbf{Z} \,:\, 2^j > 8R \text{ and } [-2^{j+2}, -2^{j-2}] \cup [2^{j-2}, 2^{j+2}] \subseteq I_k \}.$$

Now, for each $j \in \mathbf{Z}$ such that $2^j > 8R$ and $j \notin \bigcup_{k=1}^{K_1} E_k$, take $c_j := 0$. Since the number of such j's is bounded by a constant depending only on n and since $|S_P f_j(x)| \le Mf(0)$

for every $x \in Q$, we see that

$$\sum_{2^j > 8R} \frac{1}{|Q|} \int_Q |T_P f_j(x) - c_j| \, dx \le C\, Mf(0) + \sum_{k=1}^{K_1} \sum_{j \in E_k} \frac{1}{|Q|} \int_Q |S_P f_j(x) - c_j| \, dx.$$

Hence our task reduces to showing that

$$\sum_{j \in E_k} \frac{1}{|Q|} \int_Q |S_P f_j(x) - c_j| \, dx \le C_r M_r f(0).$$

for each k, $1 \le k \le K_1$. To do so, split $E_k := F_k \cup G_k$ where $F_k := \{j \in E_k : 2^j < (D_k R)^{-1/(m_k-1)}\}$ and $G_k := E_k \setminus F_k$.

Case 2.a. For each $j \in F_k$, take $c_j := -\int_{\mathbf{R}} e^{iP(-y)} f_j(y) y^{-1} dy$. Then, for every $x \in Q$, we have

$$\begin{aligned} |S_P f_j(x) - c_j| &\le \int_{\mathbf{R}} |P(x-y) - P(-y)| \, |f_j(y)||y|^{-1} dy \\ &\le \int_{2^{j-1} \le |y| \le 2^{j+1}} \int_{-y}^{x-y} |P'(z)| \, dz \, |f(y)||y|^{-1} dy \\ &\le C D_k R \int_{|y| \le 2^{j+1}} |f(y)||y|^{m_k-2} dy \quad (\text{since } z \in I_k) \\ &\le C D_k R \, 2^{j(m_k-1)} Mf(0). \end{aligned}$$

Hence

$$\sum_{j \in F_k} \frac{1}{|Q|} \int_Q |S_P f_j(x) - c_j| \, dx \le C\, Mf(0) \sum_{2^j < (D_k R)^{-1/(m_k-1)}} \frac{2^{j(m_k-1)}}{(D_k R)^{-1}} \le C\, Mf(0).$$

Case 2.b. For each $j \in G_k$, take $c_j := 0$. Write

$$\begin{aligned} S_P f_j(x) &= \int_{\mathbf{R}} e^{iP(x-y)} \psi(2^{-j} y) f(y) y^{-1} dy \\ &= \int_{\mathbf{R}} e^{iP(x-2^j y)} \psi(y) f(2^j y) y^{-1} dy \\ &=: \int_{\mathbf{R}} e^{i\lambda \phi(x,y)} \psi(y) \tilde{f}(y) dy \end{aligned}$$

where

$$\begin{aligned} \lambda &:= D_k 2^{j(m_k-1)} \\ \phi(x,y) &:= D_k^{-1} 2^{-j(m_k-1)} P(x - 2^j y) \\ \tilde{f}(y) &:= f(2^j y) y^{-1} \chi_I(y). \end{aligned}$$

Then, for $x \in Q$ and $y \in I$, we have

$$\frac{1}{B_n} \leq \left|\det\Big(\frac{\partial^2 \phi}{\partial x \partial y}(x,y)\Big)\right| \leq B_n$$

for some constant $B_n > 0$ depending only on n. Thus, by Hörmander's theorem, we have

$$\begin{aligned}
\|S_P f_j\|_{r'} &\leq C_r \lambda^{-1/r'} \|\tilde{f}\|_r \\
&= C_r (D_k 2^{j(m_k-1)})^{-1/r'} \Big[\int_{1/2 \leq |y| \leq 2} |f(2^j y)|^r |y|^{-r} \, dy\Big]^{1/r} \\
&\leq C_r (D_k 2^{j(m_k-1)})^{-1/r'} M_r f(0).
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{j \in G_k} \frac{1}{|Q|} \int_Q |S_P f_j(x) - c_j| \, dx &\leq |Q|^{-1/r'} \sum_{2^j \geq (D_k R)^{-1/(m_k-1)}} \|S_P f_j\|_{r'} \\
&\leq C_r M_r f(0) \sum_{2^j \geq (D_k R)^{-1/(m_k-1)}} \Big[\frac{(D_k R)^{-1}}{2^{j(m_k-1)}}\Big]^{1/r'} \\
&\leq C_r M_r f(0).
\end{aligned}$$

This completes the proof of Theorem C. □

Appendix: Proof of Claims

Proof of Claim 1. Note that $\Phi(x,y) = 2^{j-1} \log(|x|^2 - 2^{j+1} x \cdot y + 2^{2j} |y|^2)$. Computing its mixed second derivatives, we get

$$\frac{\partial^2 \Phi}{\partial x_k \partial y_l} = -\frac{2^{2j}}{|x - 2^j y|^2} \delta_{kl} + \frac{2^{2j+1}}{|x - 2^j y|^4} (x_k - 2^j y_k)(x_l - 2^j y_l),$$

and so the mixed second derivatives matrix of Φ may be written as $M := a\mathbf{I} + b(z_k z_l)$ where $a := -\frac{2^{2j}}{|x-2^j y|^2}$, $b := \frac{2^{2j+1}}{|x-2^j y|^4}$, and $z_k := x_k - 2^j y_k$. Hence the Hessian of Φ is

$$\det(M) = a^d + a^{d-1} b |x - 2^j y|^2 = (-1)^{d-1} \Big[\frac{2^j}{|x - 2^j y|}\Big]^{2d}$$

(see [WWZ, §7]). Since $|x - 2^j y| \leq |x| + 2^j |y| < 3.2^j$ on $Q \times I$, the claim is clear. □

Proof of Claim 2. Leaving the proof of the second inequalities in (ii) and (iii) to the reader, we shall here prove the first inequality in (iii). With notations as in [CRW], we have

$$\frac{P''}{P}(x) = \Big[\frac{P'}{P}\Big]^2(x) + \Big[\frac{P'}{P}\Big]'(x) = \Big[\sum_{m=1}^{n} \frac{1}{x - x_m}\Big]^2 - \sum_{m=1}^{n} \frac{1}{(x - x_m)^2},$$

and since $\sum_{m=1}^{n} \frac{1}{(x-x_m)^2} \leq \frac{k}{(1-A^{-1})^2}\frac{1}{x^2} + \frac{n-k}{A-1}\frac{1}{x^2}$ we obtain

$$\begin{aligned}
\Big|\frac{P''}{P}(x)\Big| &\geq \Big[\Big|\sum_{m=1}^{k} \frac{1}{x-x_m}\Big| - \sum_{m=k+1}^{n} \frac{1}{|x-x_m|}\Big]^2 - \sum_{m=1}^{n} \frac{1}{|x-x_m|^2} \\
&\geq \Big[\frac{k(1-A^{-1})}{(1+A^{-1})^2}\frac{1}{|x|} - \frac{n-k}{A-1}\frac{1}{|x|}\Big]^2 - \sum_{m=1}^{n} \frac{1}{|x-x_m|^2} \\
&\geq \Big[\frac{k^2(1-A^{-1})^2}{(1+A^{-1})^4} - \frac{k}{(1-A^{-1})^2} - \frac{2k(n-k)(1-A^{-1})}{(A-1)(1+A^{-1})^2} + \frac{(n-k)(n-k-1)}{(A-1)^2}\Big]\frac{1}{x^2} \\
&\geq \frac{\epsilon_n}{x^2},
\end{aligned}$$

if A is large enough (since $k \geq 2$). □

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