A WEIGHTED INEQUALITY FOR STEIN'S MAXIMAL FUNCTION

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Abstract. This note presents a weighted inequality for Stein's maximal function which is better than the one in [5] and comparable to that in [6].

1. Introduction

Let ϕ_r be the normalised surface measure on the sphere in $\mathbf{R}^n$ $(n \geq 3)$ with centre 0 and radius r. We consider Stein's maximal function $M_{\mathrm{S}}f$, defined by

$$M_{\mathrm{S}}f(x) = \sup_{r>0} |\phi_r * f(x)|, \quad x \in \mathbf{R}^n,$$

for any nice function f on $\mathbf{R}^n$ (see [8]). We know, for example, that the weighted estimate

$$\|M_{\mathrm{S}}f\|_{p,w} \leq C\|f\|_{p,w}$$

holds for $w \in A_p^{1-p'/n}$ (see [3] and [5]) and for $w \in A_1^{(n-p')/(n-1)}$, where $n/(n-1) < p \leq 2$ (see [3]). In [6], it is shown that the estimate also holds for $w \in A_p^{(n-2)/(n+p-2)}$, $2 \leq p \leq \infty$ and for $w \in A_{p(n-2)/(n-p)}^{1-p/n}$, $n/(n-1) < p < n$. Its proof relies on known estimates for the analytic family of operators $\{M_\alpha\}$ which was studied by Stein in [8], in which he proves the unweighted estimate for $M_{\mathrm{S}}f$.

In this note, we shall prove that the estimate holds for a different subclass of A_p weights. Our technique is based on the use of Mellin transform but we shall exploit more our knowledge on the estimates for $K_u * f$, where $K_u(x) = \pi^{-n/2+iu}\Gamma(\frac{n-iu}{2})/\Gamma(\frac{iu}{2})$ $|x|^{-n+iu}$, $u \in \mathbf{R}$. (K_u may be thought of as the distribution on $\mathbf{R}^n$ whose Fourier transform is $\widehat{K_u}(\xi) = |\xi|^{-iu}$ (see [7], p. 117)). As explained in [5] (see also [2]), $M_{\mathrm{S}}f$

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is controlled by

$$\|M_{\rm S}f\|_{p,w} \le C\int_{\mathbf{R}}(1+|u|)^{-n/2}\|K_u * f\|_{p,w}du$$

and so our task is to get such an estimate for $K_u * f$, namely

$$\|K_u * f\|_{p,w} \le C(1+|u|)^s\|f\|_{p,w}$$

with $s - n/2 < -1$.

For discussion of A_p weights, we refer the reader to [4], Chapter IV. For Mellin transform approach, see [2] or [5]. For related work, see e.g. [1].

2. Main Results

Our main results are the following:

Theorem 1. *For any $\delta \in (0,1)$, the weighted estimate*

$$\|K_u * f\|_{p,w} \le C(1+|u|)^{(n/p-n/2)(1-\theta)+(n/2)\theta+\delta}\|f\|_{p,w}$$

holds for all $w = v_1^{(2-p)(1-\theta)}v_2^{\theta}$ where $v_1 \in A_1,\ v_2 \in A_p,\ 0 \le \theta \le 1$, and $1 < p \le 2$.

Consequently, we have:

Theorem 2. *The weighted estimate*

$$\|M_{\rm S}f\|_{p,w} \le C\|f\|_{p,w}$$

holds for all $w = v_1^{(2-p)(1-\theta)}v_2^{\theta}$ where $v_1 \in A_1,\ v_2 \in A_p,\ 0 \le \theta < 1 - p'/n$, and $n/(n-1) < p \le 2$.

The proof of Theorem 1 goes as follows. First note that, by Plancherel's theorem, we have

$$\|K_u * f\|_2 = \|f\|_2.$$

Next, for $|x| \ge 2|y| > 0$, we have the following kernel estimates

$$|K_u(x-y) - K_u(x)| \le C(1+|u|)^{n/2}|x|^{-n}$$

and

$$|K_u(x-y) - K_u(x)| \le C(1+|u|)^{n/2+1}|x|^{-n-1}|y|.$$

Interpolating these estimates, we get

$$|K_u(x-y) - K_u(x)| \le C(1+|u|)^{n/2+\delta}|x|^{-n-\delta}|y|^{\delta},$$

for any $\delta \in (0,1)$. Using this estimate, one may observe that K_u satisfies the L^r-Hörmander condition: for $R > 2|y| > 0$,

$$\sum_{j=1}^{\infty}(2^jR)^{n/r'}\Big(\int_{2^jR<|x|<2^{j+1}R}|K_u(x-y) - K_u(x)|^r dx\Big)^{1/r} \le C_\delta(1+|u|)^{n/2+\delta}$$

for every $r > 1$ (see [5]), and so Theorem 2(C) and (E) of [9] tells us that

$$\|K_u * f\|_{p,v} \le C(1+|u|)^{n/2+\delta}\|f\|_{p,v},$$

holds for all $v \in A_p, 1 < p < \infty$, and at the same time

$$w(\{x : |K_u * f(x)| \ge \lambda\}) \le \frac{C}{\lambda}(1+|u|)^{n/2+\delta}\|f\|_{1,w}$$

holds for all $w \in A_1$.

Interpolating the above weighted weak L^1-L^1 and the unweighted L^2-L^2 estimates, we get

$$\|K_u * f\|_{p,w^{2-p}} \le C(1+|u|)^{n/p-n/2+\delta(2/p-1)}\|f\|_{p,w^{2-p}} \le C(1+|u|)^{n/p-n/2+\delta}\|f\|_{p,w^{2-p}}$$

for all $w \in A_1,\ 1 < p \le 2$. From this and the previous weighted $L^p - L^p$ estimate, we obtain the desired estimate by interpolation.

We now come to the proof of Theorem 2. Recall that $M_{\mathrm{S}}f$ is controlled by

$$\|M_{\mathrm{S}}f\|_{p,w} \le C\int_{\mathbf{R}}(1+|u|)^{-n/2}\|K_u * f\|_{p,w}du.$$

Now if $\theta < 1 - p'/n$, then

$$(n/p - n/2)(1-\theta) + (n/2)\theta - n/2 = (n/p - n)(1-\theta) < -1.$$

So, if we take $\delta \in (0,1)$ sufficiently small such that $(n/p-n)(1-\theta)+\delta < -1$, then we have

$$\int_{\mathbf{R}}(1+|u|)^{(n/p-n)(1-\theta)+\delta}du < \infty$$

and therefore the weighted estimate for Stein's maximal function follows.

3. Concluding Remarks

Our estimate is clearly better than the one previously obtained in [5]. To illustrate our result, let us consider power weights $w(x) = |x|^a$. (Note that $|x|^b \in A_1$ if and only if $-n < b \leq 0$, and $|x|^b \in A_p$, $1 < p < \infty$, if and only if $-n < b < n(p-1)$.) We want to know how small/large a can be. By taking $v_1(x) = v_2(x) = |x|^b$ with b arbitrarily close to $-n$ (from right) and θ arbitrarily close to $1 - p'/n$ (from left), we see that a can be arbitrarily close to $p - n$ (from right). By taking $v_1(x) = 1$ and $v_2(x) = |x|^c$ with c arbitrarily close to $n(p-1)$ (from left) and θ arbitrarily close to $1 - p'/n$ (from left), we see that a can be arbitrarily close to $np - n - p$ (from left). Hence the weighted estimate for $M_{\mathrm{S}}f$ holds for power weights $|x|^a$ with $p - n < a < np - n - p$, $n/(n-1) < p \leq 2$. This agrees with the result in [6]. We admit, however, that this is not the optimal result for power weights since we know that it holds for $|x|^a$ with $1 - n < a < np - n - p$ (see [3]).

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