ON MAXIMAL OPERATORS ASSOCIATED TO LAPLACE OPERATORS

H. GUNAWAN AND A. SIKORA

Abstract. Let F be a bounded Borel function on $\mathbf{R}^+$ and L be a second order uniformly elliptic operator in divergence form on $\mathbf{R}^d$. By using Mellin transform arguments and known estimates for the imaginary powers of L, we prove the boundedness of the maximal operator $M_{F,L}$, given by $M_{F,L}f = \sup_{t>0} |F(tL)f|$, on $L^p(\mathbf{R}^d)$ for some $p \in (1, \infty]$.

1. Introduction

Let L be a positive self-adjoint operator defined initially on $\mathcal{S}(\mathbf{R}^d)$ by the formula

$$Lf = -\sum \partial_j a_{jk} \partial_k f$$

where $a_{jk} \in C^\infty(\mathbf{R}^d)$ such that $a_{jk} = a_{kj}$ for $1 \leq j, k \leq d$. (A standard example of such an operator is the Laplace operator $\Delta = -\sum \partial_j^2$.) One basic property of L is that it has a spectral resolution

$$L = \int_{\mathbf{R}^+} \lambda \, dE_L(\lambda)$$

where $E_L(\lambda)$'s are the spectral projectors. This allows us to define the operator $F(L)$ by the formula

$$F(L) = \int_{\mathbf{R}^+} F(\lambda) \, dE_L(\lambda)$$

for any bounded Borel function $F : \mathbf{R}^+ \to \mathbf{C}$. See [3].

In particular, for every $u \in \mathbf{R}$, one may define L^{iu}, the imaginary power of L, by

$$L^{iu} = \int_{\mathbf{R}^+} \lambda^{iu} \, dE_L(\lambda).$$

By spectral theory, we have $\|L^{iu}\|_{L^2 \to L^2} = 1$, so that L^{iu} extends to an isometry on $L^2(\mathbf{R}^d)$. Moreover, L^{iu} is of weak-type (1,1) with

$$\|L^{iu}\|_{L^1 \to L^{1,\infty}} \leq C(1 + |u|)^{d/2}.$$

By interpolation and duality arguments, it follows that L^{iu} extends to a bounded operator on $L^p(\mathbf{R}^d)$ for $1 < p < \infty$, with

$$\|L^{iu}\|_{L^p \to L^p} \leq C_p(1 + |u|)^{d|1/p - 1/2|}.$$

This estimate is sharp in the sense that one cannot have

$$\|L^{iu}\|_{L^p \to L^p} \leq C_p(1 + |u|)^{d|1/p - 1/2| - \epsilon}$$

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for any $\epsilon > 0$. See [6].

The study of imaginary powers of operators plays an important role in the theory of spectral multipliers (see e.g. [2] and [5]). In this note, we shall show that, by using Mellin transform arguments, the above estimate for L^{iu} can actually lead us to the boundedness of a class of maximal operators associated to L on $L^p(\mathbf{R}^d)$ for some $p \in (1, \infty]$. See [1] and [4] for previous results.

Throughout this note, C, C_p or C_ϵ will always denote positive constants, which may vary from line to line.

2. Preliminary Results

Given a bounded Borel function $F : \mathbf{R}^+ \to \mathbf{C}$, define the maximal operator $M_{F,L}$ by the formula

$$M_{F,L}f = \sup_{t>0} |F(tL)f|, \quad f \in \mathcal{S}(\mathbf{R}^d).$$

Assuming that $F(0) = 0$, we may write (at least formally)

$$F(\lambda) = \int_{\mathbf{R}} A(u)\lambda^{iu} du, \quad \lambda \in \mathbf{R}^+.$$

By Mellin transform, this holds if and only if

$$A(u) = \frac{1}{2\pi} \int_{\mathbf{R}^+} F(\lambda)\lambda^{-1-iu} d\lambda, \quad u \in \mathbf{R}.$$

Lemma 1. *Define $\tilde{F}$ by the formula $\tilde{F}(r) = F(e^r)$, $r \in \mathbf{R}$. Then, for any $\epsilon > 0$,*

$$\int_{\mathbf{R}} |A(u)|(1 + |u|)^s du \leq C_\epsilon \|\tilde{F}\|_{H_{s+1/2+\epsilon}},$$

where $\|F\|_{H_s} = \|(I - \frac{d^2}{dx^2})^{\frac{s}{2}} F\|_{L^2}$ is the norm of F in the Sobolev space $H_s(\mathbf{R})$.

Proof. First observe that

$$A(u) = \frac{1}{2\pi} \int_{\mathbf{R}} \tilde{F}(r)e^{-iru} dr = \frac{1}{2\pi} \widehat{\tilde{F}}(u), \quad u \in \mathbf{R}.$$

Now, for any $\epsilon > 0$, we have

$$\begin{aligned}\int_{\mathbf{R}} |A(u)|(1 + |u|)^s du &\leq \Big[\int_{\mathbf{R}} |A(u)|^2(1 + |u|)^{2s+1+2\epsilon} du\Big]^{1/2} \Big[\int_{\mathbf{R}} (1 + |u|)^{-1-2\epsilon} du\Big]^{1/2} \\ &\leq C_\epsilon \|\tilde{F}\|_{H_{s+1/2+\epsilon}}\end{aligned}$$

as stated. □

Theorem 2. *If, for some $1 < p < \infty$, we have*

$$\int_{\mathbf{R}} |A(u)| \|L^{iu}\|_{L^p \to L^p} du \leq C_p,$$

then

$$\|M_{F,L}\|_{L^p \to L^p} \leq C_p,$$

that is, $M_{F,L}$ is bounded on $L^p(\mathbf{R}^d)$.

Proof. For each $t > 0$, we have

$$F(tL) = \int_{\mathbf{R}} A(u) t^{iu} L^{iu} du.$$

Since $|t^{iu}| = 1$ for every $t > 0$ and $u \in \mathbf{R}$, it follows that

$$\sup_{t>0} |F(tL)f| \leq \int_{\mathbf{R}} |A(u)||L^{iu} f| du, \quad f \in \mathcal{S}(\mathbf{R}^d).$$

Hence, by Minkowski's inequality, we obtain

$$\|M_{F,L}\|_{L^p \to L^p} \leq \int_{\mathbf{R}} |A(u)| \|L^{iu}\|_{L^p \to L^p} du \leq C_p,$$

that is, $M_{F,L}$ is bounded on $L^p(\mathbf{R}^d)$. □

`Remark`. Unlike on $L^1(\mathbf{R}^d)$, the boundedness of $M_{F,L}$ on $L^\infty(\mathbf{R}^d)$ can usually be verified separately (trivial).

Following Lemma 1 and Theorem 2, together with the estimate for L^{iu}, we have

Corollary 3. *If $\tilde{F} \in H_{d|1/p-1/2|+1/2+\epsilon}$ for some $1 < p < \infty$ and $\epsilon > 0$ and L is a self-adjoint operator as in the introduction, then $M_{F,L}$ is bounded on $L^p(\mathbf{R}^d)$.*

3. Examples

Example A. For $\mathrm{Re}(\alpha) > 0$, let

$$I_\alpha(\lambda) = \int_{-1}^{1} (1 - x^2)^{\alpha - 1} e^{i\lambda x} dx, \quad \lambda \in \mathbf{R}^+.$$

Note that $I_0(\lambda) = \cos\lambda$ and $I_1(\lambda) = 2\frac{\sin\lambda}{\lambda}$ are connected with the solution of the wave equation corresponding to the operator L. Also, $I_{n+1} = -\frac{2}{\lambda} I_n'$ for $n > 0$, and so I_α's are some version of Bessel functions. In fact, $I_\alpha(\lambda) = c_\alpha \lambda^{-\alpha+\frac{1}{2}} J_{\alpha-\frac{1}{2}}(\lambda)$ for some constant c_α depending on α.

Now observe that $I_\alpha(0) = \frac{\Gamma(\alpha)\Gamma(\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})}$. Next put

$$I_\alpha^*(\lambda) = I_\alpha(\lambda) - I_\alpha(0) e^{-\lambda^2},$$

so that $I_\alpha^*(0) = 0$, and write

$$I_\alpha^*(\lambda) = \int_{\mathbf{R}} A_\alpha(u) \lambda^{iu} du, \quad \lambda \in \mathbf{R}^+.$$

By Mellin transform, this holds if and only if

$$A_\alpha(u) = \frac{1}{2\pi} \int_{\mathbf{R}^+} I_\alpha^*(\lambda) \lambda^{-1-iu} d\lambda, \quad u \in \mathbf{R}.$$

Using the fact that the Fourier transform of $|\cdot|^{-1-iu}$, in the distribution sense, is a multiple of $|\cdot|^{iu}$ (see [7], p. 117), together with the definition and basic properties of gamma functions (see [9], pp. 55-58), one may compute that

$$A_\alpha(u) = \frac{\Gamma(\alpha)\Gamma(-\frac{iu}{2})}{4\pi^{1/2}} \Big[\frac{2^{-iu}}{\Gamma(\alpha + \frac{1}{2} + \frac{iu}{2})} - \frac{1}{\Gamma(\alpha + \frac{1}{2})} \Big].$$

From this formula we see that $A_\alpha(u)$ behaves like 1 near 0 and like $|u|^{-\mathrm{Re}(\alpha)-\frac{1}{2}}$ at infinity, so that $A_\alpha(u) = \mathrm{O}((1 + |u|)^{-\mathrm{Re}(\alpha)-\frac{1}{2}})$.

Hence we have the following result:

Theorem 4. *Let* $L = \Delta^{\frac{1}{2}}$. *Then* $M_{I_\alpha,L}$ *is bounded on* $L^p(\mathbf{R}^d)$ *for*
(a) $\mathrm{Re}(\alpha) > \frac{d}{p} - \frac{d}{2} + \frac{1}{2}$, $1 < p \le 2$, *and*
(b) $\mathrm{Re}(\alpha) > \frac{d}{2} - \frac{d}{p} + \frac{2}{p} - \frac{1}{2}$, $2 \le p \le \infty$.

Proof. For $1 < p \le 2$, it follows from Theorem 2 that $M_{I^*_\alpha,L}$ is bounded on $L^p(\mathbf{R}^d)$ provided that $\frac{d}{p} - \frac{d}{2} - \mathrm{Re}(\alpha) - \frac{1}{2} < -1$ or $\mathrm{Re}(\alpha) > \frac{d}{p} - \frac{d}{2} + \frac{1}{2}$. Since the maximal operator $f \mapsto \sup_{t>0} |e^{-t^2L^2} f|$ is bounded on each $L^p(\mathbf{R}^d)$, the boundedness of $M_{I_\alpha,L}$ follows immediately from that of $M_{I^*_\alpha,L}$, and thus we have proved the first part of the theorem.

To prove the second part, we observe that $I_\alpha(L)$ is actually a convolution operator with kernel $c_\alpha(1 - |\cdot|^2)_+^{\alpha-1-\frac{d-1}{2}}$ for some constant c_α depending on α, and so $M_{I_\alpha,L}$ is bounded on $L^\infty(\mathbf{R}^d)$ for $\mathrm{Re}(\alpha) > \frac{d-1}{2}$. Interpolating the L^2 and L^∞ estimates, we obtain the desired range of α's in (b). □

Remark. Notice that if $\alpha = \frac{d+1}{2}$, then $M_{I_\alpha,L}$ is bounded on $L^p(\mathbf{R}^d)$ for $1 < p \le \infty$. Also, if $\alpha = \frac{d-1}{2}$ and $d \ge 3$, then $M_{I_\alpha,L}$ is bounded on $L^p(\mathbf{R}^d)$ for $\frac{d}{d-1} < p \le \infty$. The former corresponds to the Hardy-Littlewood maximal operator, while the latter corresponds to Stein's spherical maximal operator (see e.g. [8] for discussion on these two special maximal operators). Finally, if $\alpha = 1$, then $M_{I_\alpha,L}$ is bounded on $L^p(\mathbf{R}^d)$ for $\frac{2d}{d+1} < p \le \infty$ (if $d \le 3$) and for $\frac{2d}{d+1} < p < \frac{2(d-2)}{d-3}$ (if $d \ge 4$). This latest operator is connected with the solution of the wave equation corresponding to the operator L (see [8], p. 519).

Example B. For $\mathrm{Re}(\alpha) > 0$, let

$$F_\alpha(\lambda) = \frac{e^{i\lambda}}{(1+\lambda)^\alpha}, \quad \lambda \in \mathbf{R}^+.$$

Then one can show that $A(u) = \frac{1}{2\pi} \int_{\mathbf{R}^+} F_\alpha(\lambda)\lambda^{-1-iu} d\lambda \sim |u|^{-\mathrm{Re}(\alpha)-\frac{1}{2}}$ as $|u| >> 1$. Hence we have the following result:

Theorem 5. *Let* L *be a self-adjoint operator as in the introduction. Then* $M_{F_\alpha,L}$ *is bounded on* $L^p(\mathbf{R}^d)$ *for* $1 < p < \infty$ *provided that* $\mathrm{Re}(\alpha) > |\frac{d}{p} - \frac{d}{2}| + \frac{1}{2}$.

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Department of Mathematics, Bandung Institute of Technology, Bandung 40132, Indonesia

School of Mathematics, Australian National University, Canberra ACT 0200, Australia