An Interpolation Method that Minimizes an Energy Integral of Fractional Order

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The Problem

Given $N+1$ points (x_0, c_0), $(x_1, c_1), \ldots, (x_N, c_N)$ with

$$0 = x_0 < x_1 < \cdots < x_N = 1 \text{ and } c_0 = c_N = 0,$$

we are interested in finding a sufficiently smooth function u on $[0,1]$ that passes through the given points and minimizes the energy integral

$$E_\alpha(u) := \int_0^1 |u^{(\alpha)}(x)|^2 dx,$$

where $u^{(\alpha)}$ denotes the fractional derivative of u of order α.

For $\alpha = 1$, the energy integral

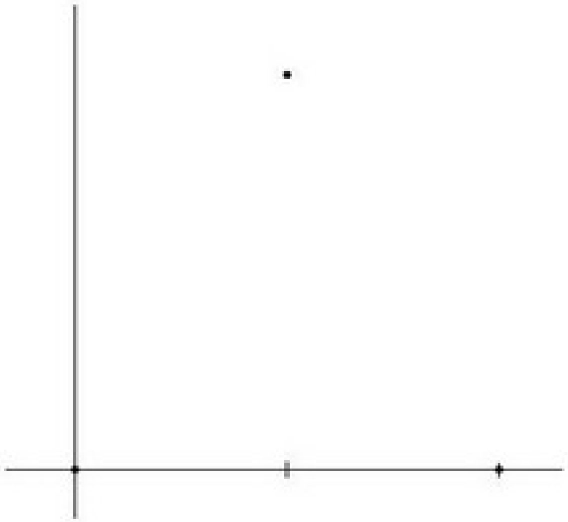
$$E_1(u) := \int_0^1 |u'(x)|^2 dx.$$

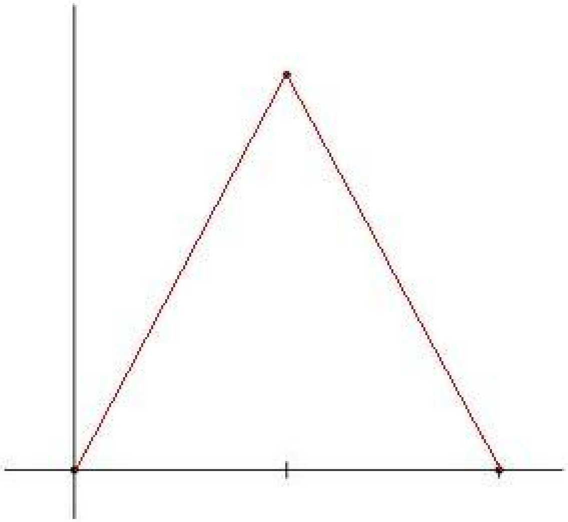
represents the tension (or the potential energy of axial load) of u.

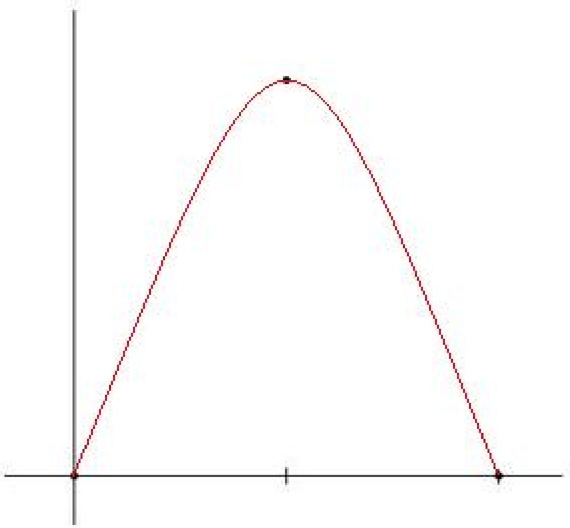
For $\alpha = 2$, the energy integral

$$E_2(u) := \int_0^1 |u''(x)|^2 dx.$$

represents the curvature (or the strain energy of bending) of u.







Previous Work

The case where $\alpha = 2$ was studied by Alghofari [1], by using Fourier series approach.

From the literature we know the solution to the problem is a cubic spline (see [12]).

Our Work

We show that the problem of finding an interpolant u that minimizes the energy integral $E_\alpha(u)$ has a unique solution and that the solution is continuous on $[0, 1]$ if and only if $\alpha > \frac{1}{2}$.

We also present an iterative procedure to obtain the solution and discuss some examples.

Let $u : [0, 1] \to \mathbb{R}$ be a continuous function with $u(0) = u(1) = 0$.

If, for instance, u is piecewise smooth, then u may be expressed as a Fourier sine series

$$u(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x, \quad x \in [0, 1],$$

where

$$a_n = 2 \int_0^1 u(x) \sin n\pi x \, dx, \quad n = 1, 2, 3, \dots .$$

Parseval's identity

$$2\int_0^1 |u(x)|^2 dx = \sum_{n=1}^{\infty} a_n^2.$$

If u is of class $C^{(k-1)}$ and $u^{(k-1)}$ is piecewise smooth, then

$$\sum_{n=1}^{\infty} n^{2k} a_n^2 < \infty \tag{1}$$

(see, for instance, [4]).

Conversely, if the a_n's satisfy the condition (1), then $u, \ldots, u^{(k-1)}$ are absolutely continuous and $u^{(k)}$ is square integrable with

$$\|u^{(k)}\|_2^2 := 2\int_0^1 |u^{(k)}(x)|^2 dx = \pi^{2k} \sum_{n=1}^{\infty} n^{2k} a_n^2$$

(see, for instance, [13]). Thus $u^{(k)}$ may be identified with $(n^k a_n)$.

Here $n^k a_n$'s are the Fourier coefficients of $u^{(k)}$, from which we can recover $u^{(k)}$ almost everywhere through the formula

$$u^{(k)}(x) = \pi^k \sum_{n=1}^{\infty} n^k a_n \sin(n\pi x + k\tfrac{\pi}{2}).$$

Note that $\pi^k n^k \sin(n\pi x + k\frac{\pi}{2})$ is the k-th derivative of $\sin(n\pi x)$.

Inspired by the above facts, we may define the fractional derivative of u of order $\alpha \geq 0$, denoted by $u^{(\alpha)}$, almost everywhere by

$$u^{(\alpha)}(x) = \pi^\alpha \sum_{n=1}^{\infty} n^\alpha a_n \sin(n\pi x + \alpha\tfrac{\pi}{2}),$$

provided that $\sum_{n=1}^{\infty} n^{2\alpha} a_n^2 < \infty$.

Notice that $\pi^\alpha n^\alpha \sin(n\pi x + \alpha\frac{\pi}{2})$ is the fractional derivative of $\sin n\pi x$ of order α (see [11]).

Moreover, the family $\{\sin(n\pi x + \alpha\frac{\pi}{2})\} : n \in \mathbb{N}\}$ forms an orthogonal system and that

$$2\int_0^1 |u^{(\alpha)}(x)|^2 dx = \pi^{2\alpha} \sum_{n=1}^{\infty} n^{2\alpha} a_n^2.$$

Hence $u^{(\alpha)}$ is a square integrable function on $[0,1]$, which may be identified with $(n^\alpha a_n)$.

Our Problem Repeated

Given $0 = x_0 < x_1 < \cdots < x_N = 1$ and $c_0 = c_N = 0$,

$$\text{Minimize } E_\alpha(u), \tag{2}$$

subject to

$$u(x_i) = c_i, \quad i = 0, 1, \ldots, N.$$

To solve the problem, we consider the space $W = W_\alpha$ consisting of all functions u on $[0, 1]$ of the form

$$u(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x$$

with

$$\sum_{n=1}^{\infty} n^{2\alpha} a_n^2 < \infty.$$

On W, we define the inner product

$$\langle u, v \rangle := \sum_{n=1}^{\infty} n^{2\alpha} a_n b_n,$$

where a_n's and b_n's are the coefficients of u and v, respectively.

Here minimizing the integral

$$\int_0^1 |u^{(\alpha)}(x)|^2 dx$$

is equivalent to minimizing the sum

$$\sum_{n=1}^{\infty} n^{2\alpha} a_n^2 =: \|u\|^2.$$

With respect to the above inner product, W is complete, that is, $(W, \langle \cdot, \cdot \rangle)$ is a Hilbert space. Further, we have:

Theorem 3.1

Suppose that $\|u_m - u\| \to 0$ as $m \to \infty$. If $\alpha > \frac{1}{2}$, then (u_m) converges uniformly to u on $[0, 1]$. More generally, $\left(u_m^{(\beta)}\right)$ converges uniformly to $u^{(\beta)}$ on $[0, 1]$ for $0 \leq \beta < \alpha - \frac{1}{2}$.

Theorem 3.2

Let $\alpha > \frac{1}{2}$. If $u \in W$, then $u^{(\beta)}$ is continuous for $0 \leq \beta < \alpha - \frac{1}{2}$. In particular, every function in W is continuous.

Proof. For each β with $0 \leq \beta < \alpha - \frac{1}{2}$, $u^{(\beta)}$ is a limit, and hence a uniform limit, of its partial sums. Now since the partial sums are continuous, $u^{(\beta)}$ too must be continuous. □

Because we are looking for a continuous solution, we assume from now on that $\alpha > \frac{1}{2}$.

This is not only a sufficient condition, but also a necessary condition to have a continuous solution.

Consider the subspace V of W consisting of all functions u that vanish at $x_i,\ i = 1, ..., N-1$; that is,

$$V := \{u \in W\ :\ u(x_i) = 0,\ i = 1, \ldots, N-1\}.$$

Meanwhile, let U be the subset of W given by

$$U := \{u \in W\ :\ u(x_i) = c_i,\ i = 1, \ldots, N-1\}.$$

Theorem 3.3

V is closed, while U is nonempty, closed and convex.

Theorem 3.4

The minimization problem (2) has a unique solution in W, and the solution is given by

$$u = u_0 - \mathrm{proj}_V(u_0),$$

where u_0 is an arbitrary element of U and $\mathrm{proj}_V(u_0)$ denotes the orthogonal projection of u_0 on V.

Note. To find an element in U is easy. What is rather difficult is to find an orthonormal basis for V. In the next section, we develop a procedure to find an initial element in U and an orthonormal basis for V, and to obtain the minimum solution iteratively through finite computations.

Given $N+1$ points (x_0, c_0), $(x_1, c_1), \ldots$, (x_N, c_N) with $0 = x_0 < x_1 < \cdots < x_N = 1$ and $c_0 = c_N = 0$, we can obtain the solution in W through the following steps.

Step 1

To obtain an initial element in U, we solve the system of equations

$$\sum_{j=1}^{N-1} b_j \sin j\pi x_i = c_i, \quad i = 1, \dots, N-1,$$

for the coefficients b_j's.

The $(N-1) \times (N-1)$ matrix $[\sin j\pi x_i]_{i,j}$ is always nonsingular (see [2]), and so the above system has a solution.

Having found b_j's, we put $u_0(x) = \sum_{j=1}^{N-1} b_j \sin j\pi x$.

Step 2

To obtain a basis for V, we consider the system of equations

$$\sum_{n=1}^{\infty} a_n \sin n\pi x_i = 0, \quad i = 1, \ldots, N-1,$$

each of which contains infinitely many unknowns a_n's.

From this system, $a_1, \ldots, a_{N-1}$ can be expressed in terms of $a_n,\ n \geq N$.

Now if $(a_1, \dots, a_{N-1}, a_N, a_{N+1}, \dots)$ stands for $\sum_{n=1}^{\infty} a_n \sin n\pi x$, then by expressing $a_1, \dots, a_{N-1}$ in terms of a_n with $n \geq N$, every element in V can be expressed as

$$a_N(*, \dots, *, 1, 0, 0, \dots) + a_{N+1}(*, \dots, *, 0, 1, 0, \dots) + a_{N+2}(*, \dots, *, 0, 0, 1, \dots) + \cdots ,$$

where the first $N - 1$ terms marked by asterisks come from $a_1, \dots, a_{N-1}$.

The following sequence

$$\begin{aligned} v_1 &:= (*, \ldots, *, 1, 0, 0, \ldots), \\ v_2 &:= (*, \ldots, *, 0, 1, 0, \ldots), \\ v_3 &:= (*, \ldots, *, 0, 0, 1, \ldots), \\ &\vdots \end{aligned}$$

form a basis for V.

Step 3

The minimum solution u is given by $u = u_0 - \mathrm{proj}_V(u_0)$. To find (or approximate) it, we compute the orthogonal projection of u_0 on the subspace $V_m := \mathrm{span}\{v_1, \dots, v_m\}$ for $m = 1, 2, 3, \dots$ iteratively.

If $u_m := u_0 - \mathrm{proj}_{V_m}(u_0)$, then the sequence (u_m) approximates the minimum solution u. Indeed, $\|u_m\|$ gets smaller and $\|u_m - u\| \to 0$ as $m \to \infty$.

In practice, we may stop the iteration process at u_M basically when

$$\|u_M - u_{M-1}\| < \epsilon$$

for a given value of ϵ. Note that the larger the value of α the faster the convergence of (u_m).

Example 1

Suppose that we wish to find a continuous, piecewise smooth function u on $[0,1]$ that minimizes the integral

$$E_1(u) := \int_0^1 |u'(x)|^2 \, dx, \tag{3}$$

subject to the condition that

$$u(0) = u(1) = 0 \text{ and } u\left(\tfrac{1}{2}\right) = 1.$$

For this, consider the following subspace V of W

$$V := \{u \in W \, : \, u\left(\tfrac{1}{2}\right) = 0\},$$

and the subset U of W given by

$$U := \{u \in W \, : \, u\left(\tfrac{1}{2}\right) = 1\}.$$

Our initial approximation is $u_0(x) = \sin n\pi x$. Next, if

$$v(x) := \sum_{n=1}^{\infty} a_n \sin n\pi x$$

is in V, then $v\left(\frac{1}{2}\right) = 0$ is equivalent to

$$a_1 - a_3 + a_5 - a_7 + - \cdots = 0,$$

for which we get

$$a_1 = a_3 - a_5 + a_7 - a_9 + - \cdots .$$

Hence, every element $(a_1, a_2, a_3, a_4, a_5, \dots)$ in V can be expressed as

$$a_2(0, 1, 0, 0, 0, \dots) + a_3(1, 0, 1, 0, 0, \dots) +$$
$$a_4(0, 0, 0, 1, 0, \dots) + a_5(-1, 0, 0, 0, 1, \dots) + \cdots .$$

From this we get the following basis for V:

$$\begin{aligned} v_1 &:= (0, 1, 0, 0, 0, \dots), \\ v_2 &:= (1, 0, 1, 0, 0, \dots), \\ v_3 &:= (0, 0, 0, 1, 0, \dots), \\ v_4 &:= (-1, 0, 0, 0, 1, \dots), \\ &\vdots \end{aligned}$$

Now if one carries out Step 3 as prescribed, one will get

$$u_1 = (1,0,0,0,0,\dots),\ u_2 = u_3 = \frac{9}{10}(1,0,-\frac{1}{3^2},0,0,\dots),$$

and so on. The limiting solution is

$$u = \frac{8}{\pi^2}(1,0,-\frac{1}{3^2},0,\frac{1}{5^2},\dots),$$

that is,

$$u(x) = \frac{8}{\pi^2}\Big(\sin \pi x - \frac{1}{3^2}\sin 3\pi x + \frac{1}{5^2}\sin 5\pi x - \frac{1}{7^2}\sin 7\pi x + - \cdots\Big).$$

As one would expect, this is nothing but the Fourier sine series of the piecewise linear function f given by

$$f(x) = \begin{cases} 2x, & 0 \leq x \leq \frac{1}{2} \\ 2(1-x), & \frac{1}{2} < x \leq 1. \end{cases}$$

The difference between the sequence (u_m) and the Fourier partial sums is that each u_m passes through the point $\left(\frac{1}{2}, 1\right)$ while the Fourier partial sums do not.

Example 2

In general, given $N + 1$ points (x_0, c_0), $(x_1, c_1), \ldots,$ (x_N, c_N) with $0 = x_0 < x_1 < \cdots < x_N = 1$ and $c_0 = c_N = 0$, the solution to our minimization problem for the case where $\alpha = 1$ is the Fourier sine series of the piecewise linear function f for which $f(x_i) = c_i$ and f is linear on each subinterval $[x_{i-1}, x_i]$.

For example, let

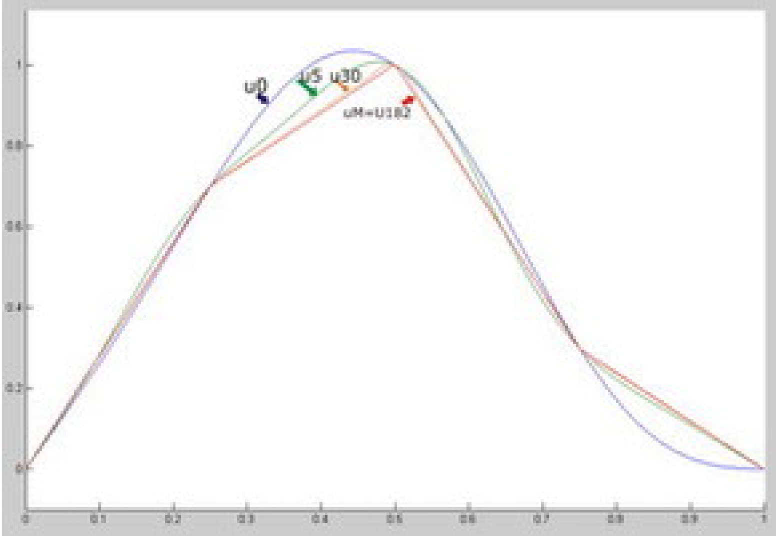
$$x_i = \frac{i}{4},\ i = 0, \ldots, 4,$$

$$c_0 = 0,\ c_1 = \frac{7}{10},\ c_2 = 1, c_3 = \frac{3}{10},\ c_4 = 0.$$

With a computer program, we apply our procedure and get a sequence (u_m) that approximates the solution in W.

For $\epsilon = 0.01$, the iterations stop at u_{182}.

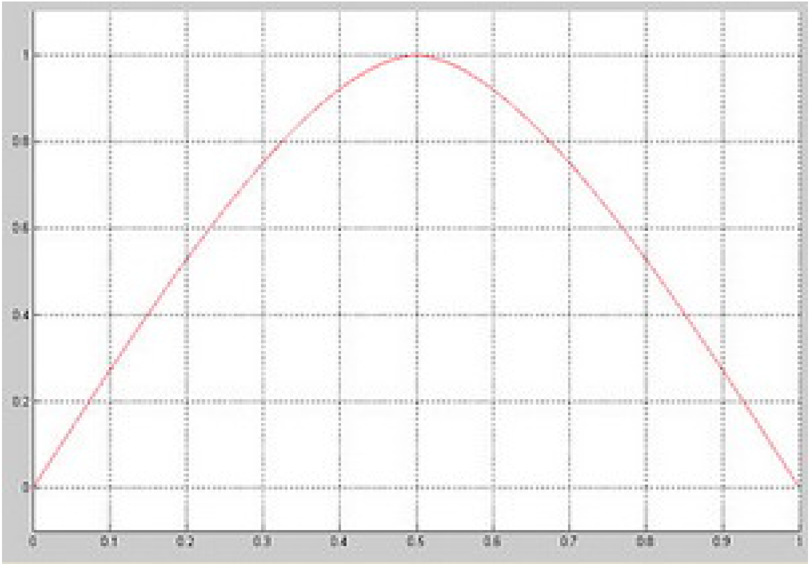
The following figure shows the graphs of u_0, u_5, u_{30}, and $u_M = u_{182}$. Note that the limiting series is the piecewise linear function passing through the points $(x_i, c_i),\ i = 0, \ldots, 4$.


u0
u5
u30
uM=u182

Example 3

Suppose that $\alpha = 1.5$ and we wish to find a smooth function u on $[0, 1]$ that minimizes the integral $E_\alpha(u)$ subject to the condition that $u(0) = u(1) = 0$ and $u\left(\frac{1}{2}\right) = 1$.

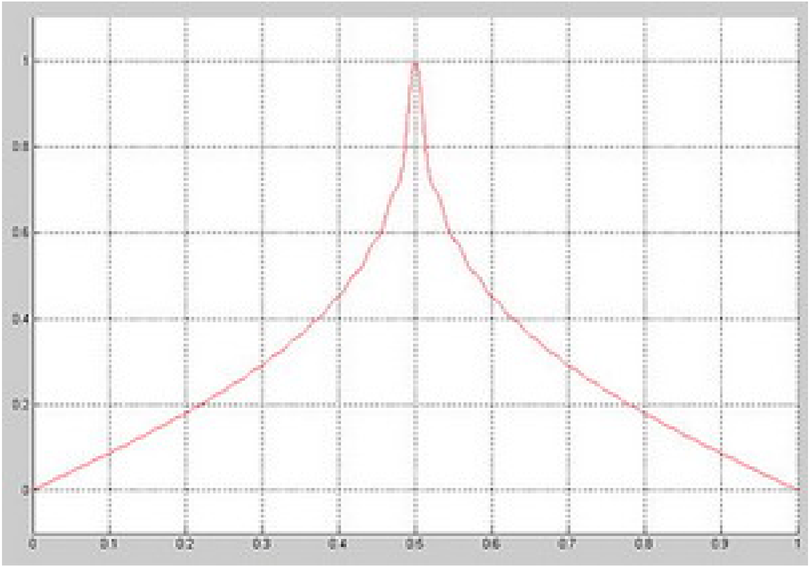
The following figure shows the graph of the approximate solution (for $\epsilon = 0.01$).


1
0.8
0.6
0.4
0.2
0
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

Example 4

Suppose that $\alpha = 0.6$ and we wish to find a continuous function u on $[0, 1]$ that minimizes the integral $E_\alpha(u)$ subject to the condition that $u(0) = u(1) = 0$ and $u\left(\frac{1}{2}\right) = 1$.

The following figure shows the graph of the approximate solution (for $\epsilon = 0.05$).

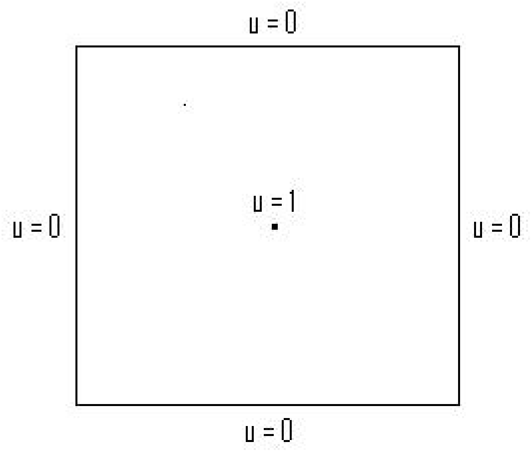


Remark

Our procedure also works for an energy functional which is a linear combination of several E_α's with at least one of $\alpha's$ is greater than $\frac{1}{2}$.

We have also been successful in extending our method to solve the 2-dimensional problem.

2-dimensional analog



Acknowledgement

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