

FRACTIONAL INTEGRALS AND GENERALIZED OLSEN INEQUALITIES

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ABSTRACT. Let T_ρ be the generalized fractional integral operator associated to a function $\rho : (0, \infty) \rightarrow (0, \infty)$, as defined in [15]. For a function W on \mathbb{R}^n , we shall be interested in the boundedness of the multiplication operator $f \mapsto W \cdot T_\rho f$ on generalized Morrey spaces. Under some assumptions on ρ , we obtain an inequality for $W \cdot T_\rho$, which can be viewed as an extension of Olsen's and Kurata-Nishigaki-Sugano's results.

1. INTRODUCTION

For $0 < \alpha < n$, let I_α denote the Riesz potential or the (classical) fractional integral operator, which is given by the formula

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

Formally, through its Fourier transform, the operator I_α can be recognized as a multiple of the Laplacian to the power of $-\frac{\alpha}{2}$, that is,

$$I_\alpha f = \kappa(-\Delta)^{-\frac{\alpha}{2}} f,$$

where $\kappa = \kappa(n, \alpha)$ (see, for instance, [2, 21, 23]).

A well-known result for I_α is the Hardy-Littlewood-Sobolev inequality, which was proved by Hardy and Littlewood [8, 10] and Sobolev [22] around the 1930's.

Theorem 1.1 (Hardy-Littlewood; Sobolev) *For $1 < p < \frac{n}{\alpha}$, we have the inequality*

$$(1.1) \quad \|I_\alpha f\|_q \leq C_p \|f\|_p,$$

that is, I_α is bounded from $L^p(\mathbb{R}^n)$ to $L^q(\mathbb{R}^n)$, provided that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

As an immediate consequence of this inequality, one has the following estimate for $(-\Delta)^{-1}$:

$$\|(-\Delta)^{-1} f\|_{np/(n-2)} \leq C_p \|f\|_p,$$

for $1 < p < \frac{n}{2}$, $n \geq 3$. Here $u := (-\Delta)^{-1} f$ is a solution of the Poisson equation $-\Delta u = f$. From (1.1) one can also prove Sobolev's embedding theorems (see [23]).

Decades later, the inequality has been extended from Lebesgue spaces to Morrey spaces. For $1 \leq p < \infty$ and $0 \leq \lambda \leq n$, the (classical) Morrey space $L^{p,\lambda} = L^{p,\lambda}(\mathbb{R}^n)$ is defined to be the space of all functions $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{p,\lambda} := \sup_{B=B(a,r)} \left(\frac{1}{r^\lambda} \int_B |f(y)|^p dy \right)^{1/p} < \infty,$$

where $B(a, r)$ denotes the (open) ball centered at $a \in \mathbb{R}^n$ with radius $r > 0$ [13]. Here $\|\cdot\|_{p,\lambda}$ defines a semi-norm on $L^{p,\lambda}$. Note particularly that $L^{p,0} = L^p$ and $L^{p,n} = L^\infty$. For

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the structure of Morrey spaces and their generalizations, see the works of S. Campanato [3], J. Peetre [20], C.T. Zorko [25], and the references therein.

In the 1960's, S. Spanne proved that I_α is bounded from $L^{p,\lambda}$ to $L^{q,\lambda q/p}$ for $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $0 \leq \lambda < n$, as stated in [20]. A stronger result was obtained by D.R. Adams [1] and reproved by F. Chiarenza and M. Frasca [4].

Theorem 1.2 (Adams; Chiarenza-Frasca) *For $1 < p < \frac{n}{\alpha}$ and $0 \leq \lambda < n - \alpha p$, we have the inequality*

$$\|I_\alpha f\|_{q,\lambda} \leq C_{p,\lambda} \|f\|_{p,\lambda}$$

provided that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$.

The proof usually involves the properties of the Hardy-Littlewood maximal operator M , defined by the formula

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| dy,$$

where $|B(x,r)| = cr^n$ is the Lebesgue measure of $B(x,r)$. The operator M is known to be bounded on L^p for $1 < p \leq \infty$ [9]. Chiarenza and Frasca [4] proved that M is also bounded on Morrey spaces.

Theorem 1.3 (Chiarenza-Frasca) *The inequality*

$$\|Mf\|_{p,\lambda} \leq C_{p,\lambda} \|f\|_{p,\lambda}$$

holds for $p > 1$ and $0 \leq \lambda < n$.

For $1 \leq p < \infty$ and a suitable function $\phi : (0, \infty) \rightarrow (0, \infty)$, we define the (generalized) Morrey space $\mathcal{M}_{p,\phi} = \mathcal{M}_{p,\phi}(\mathbb{R}^n)$ to be the space of all functions $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{p,\phi} := \sup_{B=B(a,r)} \frac{1}{\phi(r)} \left(\frac{1}{|B|} \int_B |f(y)|^p dy \right)^{1/p} < \infty.$$

Note that for $\phi(t) = t^{(\lambda-n)/p}$, $0 \leq \lambda \leq n$, we have $\mathcal{M}_{p,\phi} = L^{p,\lambda}$ — the classical Morrey space. Unless stated otherwise, we assume hereafter that the function ϕ satisfies the following two conditions:

$$(1.1) \quad \frac{1}{2} \leq \frac{r}{s} \leq 2 \Rightarrow \frac{1}{C_1} \leq \frac{\phi(r)}{\phi(s)} \leq C_1.$$

$$(1.2) \quad \int_r^\infty \frac{\phi^p(t)}{t} dt \leq C_2 \phi^p(r) \text{ for } 1 < p < \infty.$$

The condition (1.1) is known as *the doubling condition* (with a doubling constant C_1). Note that for any function ψ that satisfies the doubling condition, we have

$$\int_{2^k r}^{2^{k+1} r} \frac{\psi(t)}{t} dt \sim \psi(2^k r)$$

for every integer k and $r > 0$.

Now, for a given function $\rho : (0, \infty) \rightarrow (0, \infty)$, we define the (generalized) fractional integral operator T_ρ by

$$T_\rho f(x) := \int_{\mathbb{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y) dy.$$

For $\rho(t) = t^\alpha$, $0 < \alpha < n$, we have $T_\rho = I_\alpha$ — the classical fractional integral operator. The boundedness of the operator T_ρ on the generalized Morrey space $\mathcal{M}_{p,\phi}$ was first studied by Nakai [15]. Recent results on T_ρ can be found in [5, 6, 7, 16, 17, 18].

In this paper, we shall be interested in the boundedness of the multiplication operators $f \mapsto W \cdot I_\alpha f$ and $f \mapsto W \cdot T_\rho f$ on generalized Morrey spaces. In both cases, W is just a function on \mathbb{R}^n . We prove an inequality for $W \cdot I_\alpha$ [Theorem 3.3] and, under some assumptions on ρ , we also obtain an inequality for $W \cdot T_\rho$ [Theorem 3.5]. Our results can be viewed as an extension of Olsen's and Kurata-Nishigaki-Sugano's results. Indeed, for $\rho(t) = t^\alpha$, $0 < \alpha < n$, the inequalities for $W \cdot T_\rho$ reduce to those for the classical fractional integral operator $W \cdot I_\alpha$.

2. INEQUALITIES FOR I_α AND T_ρ

In [14], E. Nakai proved the boundedness of the Hardy-Littlewood maximal operator on generalized Morrey spaces.

Theorem 2.1 (Nakai) *The inequality*

$$\|Mf\|_{p,\phi} \leq C_{p,\phi} \|f\|_{p,\phi}$$

holds for $1 < p < \infty$.

Nakai also obtained the boundedness of I_α on generalized Morrey spaces, which can be viewed as an extension of Spanne's result. A similar result was also obtained by Sugano-Tanaka [24]. The following theorem can be considered as an extension of Adams-Chiarenza-Frasca's result.

Theorem 2.2 *Suppose that, in addition to the condition (1.1) and (1.2), ϕ satisfies the inequality $\phi(t) \leq Ct^\beta$ for $-\frac{n}{p} \leq \beta < -\alpha$, $1 < p < \frac{n}{\alpha}$. Then, for $q = \frac{\beta p}{\alpha + \beta}$, we have*

$$\|I_\alpha f\|_{q,\phi^{p/q}} \leq C_{p,\beta} \|f\|_{p,\phi}.$$

Proof. As before, we assume that $f \neq 0$ and Mf is finite everywhere. For each $x \in \mathbb{R}^n$, write $I_\alpha f(x) = I_1(x) + I_2(x)$ where

$$I_1(x) := \int_{|x-y| < R} \frac{f(y)}{|x-y|^{n-\alpha}} dy \quad \text{and} \quad I_2(x) := \int_{|x-y| \geq R} \frac{f(y)}{|x-y|^{n-\alpha}} dy,$$

with R being an arbitrary positive number. Then, $|I_1(x)| \leq C R^\alpha Mf(x)$, while for I_2 we have

$$\begin{aligned} |I_2(x)| &\leq \sum_{k=0}^{\infty} \int_{2^k R \leq |x-y| < 2^{k+1} R} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \\ &\leq \sum_{k=0}^{\infty} (2^k R)^{\alpha-n} \int_{B(x, 2^{k+1} R)} |f(y)| dy \\ &\leq C \sum_{k=0}^{\infty} (2^k R)^{\alpha-\frac{n}{p}} \left(\int_{B(x, 2^{k+1} R)} |f(y)|^p dy \right)^{1/p} \\ &\leq C \sum_{k=0}^{\infty} (2^k R)^\alpha \phi(2^k R) \|f\|_{p,\phi} \\ &\leq C \|f\|_{p,\phi} \sum_{k=0}^{\infty} (2^k R)^{\alpha+\beta} \\ &\leq C R^{\alpha+\beta} \|f\|_{p,\phi}. \end{aligned}$$

Now choose $R = \left(\frac{Mf(x)}{\|f\|_{p,\phi}}\right)^{1/\beta}$ to get

$$|I_\alpha f(x)| \leq |I_1(x)| + |I_2(x)| \leq C [Mf(x)]^{(\alpha+\beta)/\beta} \|f\|_{p,\phi}^{-\alpha/\beta} = C [Mf(x)]^{p/q} \|f\|_{p,\phi}^{1-p/q}.$$

The inequality then follows from this and Theorem 2.1. \square

Remark. Observe that when $\phi(t) = t^{(\lambda-n)/p}$, $0 \leq \lambda < n - \alpha p$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$, Theorem 2.2 reduces to Theorem 1.2.

A slight modification of Theorem 2.2 may be formulated for T_ρ as follows. We leave its proof to the reader.

Theorem 2.3 *Suppose that $\rho(t) \leq C_1 t^\alpha$ for some $0 < \alpha < n$, and, in addition to the condition (1.1) and (1.2), $\phi(t) \leq C_2 t^\beta$ for $-\frac{n}{p} \leq \beta < -\alpha$, $1 < p < \frac{n}{\alpha}$. Then, for $q = \frac{\beta p}{\alpha + \beta}$, we have*

$$\|T_\rho f\|_{q,\phi^{p/q}} \leq C_{p,\beta} \|f\|_{p,\phi}.$$

The following result of H. Gunawan [7] gives a further generalization of Theorem 1.2.

Theorem 2.4 (Gunawan) *Suppose that, in addition to the condition (1.1) and (1.2), ϕ is surjective. If ρ satisfies the doubling condition and*

$$\int_0^r \frac{\rho(t)}{t} dt \leq C \phi(r)^{(p-q)/q} \quad \text{and} \quad \int_r^\infty \frac{\rho(t)\phi(t)}{t} dt \leq C \phi(r)^{p/q},$$

for $1 < p < q < \infty$, then we have

$$\|T_\rho f\|_{q,\phi^{p/q}} \leq C_{p,\phi} \|f\|_{p,\phi},$$

that is, T_ρ is bounded from $\mathcal{M}_{p,\phi}$ to $\mathcal{M}_{q,\phi^{p/q}}$.

3. INEQUALITIES FOR $W \cdot I_\alpha$ AND $W \cdot T_\rho$

In studying a Schrödinger equation with perturbed potentials W on \mathbb{R}^n (particularly for $n = 3$), P.A. Olsen [19] proved the following result.

Theorem 3.1 (Olsen) *For $1 < p < \frac{n}{\alpha}$ and $0 \leq \lambda < n - \alpha p$, we have*

$$\|W \cdot I_\alpha f\|_{p,\lambda} \leq C_{p,\lambda} \|W\|_{(n-\lambda)/\alpha,\lambda} \|f\|_{p,\lambda},$$

that is, $W \cdot I_\alpha$ is bounded on $L^{p,\lambda}$, provided that $W \in L^{(n-\lambda)/\alpha,\lambda}$.

As a consequence of Theorem 3.1, we see that for $1 < p < \frac{n}{2}$, $n \geq 3$, the estimate

$$\|W \cdot (-\Delta)^{-1} f\|_{p,\lambda} \leq C_{p,\lambda} \|W\|_{(n-\lambda)/2,\lambda} \|f\|_{p,\lambda},$$

holds provided that $W \in L^{(n-\lambda)/2,\lambda}$, $0 \leq \lambda < n - 2p$. In particular, when $\lambda = 0$, one has

$$\|W \cdot (-\Delta)^{-1} f\|_p \leq C_p \|W\|_{n/2} \|f\|_p$$

provided that $W \in L^{n/2}$.

K. Kurata *et al.* [12] extended Olsen's result by proving that, for some $p > 1$ and a function ϕ satisfying several conditions (including the doubling condition), the operator $W \cdot I_\alpha$ is bounded on generalized Morrey spaces $\mathcal{M}_{p,\phi}$, provided that $W \in \mathcal{M}_{s_1,\phi} \cap \mathcal{M}_{s_2,\phi}$ for some indices s_1 and s_2 . Their estimate, however, is rather complicated. We shall here present simpler estimates for $W \cdot I_\alpha$ on generalized Morrey spaces.

The first estimate below is a consequence of Theorem 2.2, while the second one is obtained directly without using Theorem 2.2.

Theorem 3.2 *Suppose that, in addition to the condition (1.1) and (1.2), ϕ satisfies the inequality $\phi(t) \leq Ct^\beta$ for $-\frac{n}{p} \leq \beta < -\alpha$, $1 < p < \frac{n}{\alpha}$. Then, we have*

$$\|W \cdot I_\alpha f\|_{p,\phi} \leq C_{p,\beta} \|W\|_{s,\phi^{p/s}} \|f\|_{p,\phi}$$

provided that $W \in \mathcal{M}_{s,\phi^{p/s}}$ where $s = -\frac{\beta p}{\alpha}$.

Proof. Use Hölder's inequality and Theorem 2.2. □

Theorem 3.3 *Suppose that ϕ satisfies the doubling condition and the inequality*

$$\int_r^\infty t^{\alpha-1} \phi(t) dt \leq C r^\alpha \phi(r).$$

Then, for $1 < p < \frac{n}{\alpha}$, we have

$$\|W \cdot I_\alpha f\|_{p,\phi} \leq C_{p,\phi} \|W\|_{n/\alpha} \|f\|_{p,\phi},$$

provided that $W \in L^{n/\alpha}$.

Proof. For $a \in \mathbb{R}^n$ and $r > 0$, let $B = B(a, r)$, $\tilde{B} = B(a, 2r)$, and write $f = f_1 + f_2 := f\chi_{\tilde{B}} + f\chi_{\tilde{B}^c}$. We observe that $f_1 \in L^p$ with

$$\|f_1\|_p = \left(\int_{\mathbb{R}^n} |f_1(y)|^p dy \right)^{1/p} = \left(\int_{\tilde{B}} |f(y)|^p dy \right)^{1/p} \leq C r^{n/p} \phi(r) \|f\|_{p,\phi}.$$

Hence, by applying Theorem 3.1 for $\lambda = 0$, we get

$$\left(\int_B |W \cdot I_\alpha f_1(x)|^p dx \right)^{1/p} \leq \|W \cdot I_\alpha f_1\|_p \leq C \|W\|_{n/\alpha} \|f_1\|_p \leq C r^{n/p} \phi(r) \|W\|_{n/\alpha} \|f\|_{p,\phi},$$

whence

$$\frac{1}{\phi(r)} \left(\frac{1}{|B|} \int_B |W \cdot I_\alpha f_1(x)|^p dx \right)^{1/p} \leq C \|W\|_{n/\alpha} \|f\|_{p,\phi}.$$

Next, for $x \in B$, we have

$$|I_\alpha f_2(x)| \leq \int_{\tilde{B}^c} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \leq \int_{|x-y| \geq r} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy$$

Then, as in the proof of Theorem 2.4, we shall obtain

$$|I_\alpha f_2(x)| \leq C \|f\|_{p,\phi} \int_r^\infty t^{\alpha-1} \phi(t) dt \leq C r^\alpha \phi(r) \|f\|_{p,\phi}.$$

Hence

$$\begin{aligned} \left(\frac{1}{|B|} \int_B |W \cdot I_\alpha f_2(x)|^p dx \right)^{1/p} &\leq C r^\alpha \phi(r) \|f\|_{p,\phi} \left(\frac{1}{|B|} \int_B |W(x)|^p dx \right)^{1/p} \\ &\leq C r^\alpha \phi(r) \|f\|_{p,\phi} \left(\frac{1}{|B|} \int_B |W(x)|^{n/\alpha} dx \right)^{\alpha/n} \\ &\leq C \phi(r) \|W\|_{n/\alpha} \|f\|_{p,\phi}, \end{aligned}$$

and so

$$\frac{1}{\phi(r)} \left(\frac{1}{|B|} \int_B |W \cdot I_\alpha f_2(x)|^p dx \right)^{1/p} \leq C \|W\|_{n/\alpha} \|f\|_{p,\phi}.$$

The desired estimate follows from the two estimates via Minkowski inequality. □

The following two theorems provide estimates for $W \cdot T_\rho$ on generalized Morrey spaces. The first is a consequence of Theorem 2.3, while the second follows from Theorem 2.4. We leave the proof of the former to the reader.

Theorem 3.4 *Suppose that $\rho(t) \leq C_1 t^\alpha$ for some $0 < \alpha < n$, and, in addition to the condition (1.1) and (1.2), $\phi(t) \leq C_2 t^\beta$ for $-\frac{n}{p} \leq \beta < -\alpha$, $1 < p < \frac{n}{\alpha}$. Then, we have*

$$\|W \cdot T_\rho f\|_{p,\phi} \leq C_{p,\beta} \|W\|_{s,\phi^{p/s}} \|f\|_{p,\phi}$$

provided that $W \in \mathcal{M}_{s,\phi^{p/s}}$ where $s = -\frac{\beta p}{\alpha}$.

Theorem 3.5 *Suppose that, in addition to the condition (1.1) and (1.2), ϕ is surjective. If ρ satisfies the doubling condition and*

$$\int_0^r \frac{\rho(t)}{t} dt \leq C \phi(r)^{(p-q)/q} \quad \text{and} \quad \int_r^\infty \frac{\rho(t)\phi(t)}{t} dt \leq C \phi(r)^{p/q},$$

for $1 < p < q < \infty$, then we have

$$\|W \cdot T_\rho f\|_{p,\phi} \leq C_{p,\phi} \|W\|_{s,\phi^{p/s}} \|f\|_{p,\phi},$$

provided that $W \in \mathcal{M}_{s,\phi^{p/s}}$ where $\frac{1}{s} = \frac{1}{p} - \frac{1}{q}$.

Proof. Let $B = B(a, r)$ be an arbitrary ball in \mathbb{R}^n . By Hölder's inequality, we have

$$\frac{1}{|B|} \int_B |W \cdot T_\rho f(x)|^p dx \leq \left(\frac{1}{|B|} \int_B |W(x)|^s dx \right)^{p/s} \left(\frac{1}{|B|} \int_B |T_\rho f(x)|^q dx \right)^{p/q},$$

with $\frac{p}{s} + \frac{p}{q} = 1$. Now take the p -th roots and then divide both sides by $\phi(r)$ to get

$$\begin{aligned} \frac{1}{\phi(r)} \left(\frac{1}{|B|} \int_B |W \cdot T_\rho f(x)|^p dx \right)^{1/p} &\leq \frac{1}{\phi(r)^{p/s}} \left(\frac{1}{|B|} \int_B |W(x)|^s dx \right)^{1/s} \\ &\quad \times \frac{1}{\phi(r)^{p/q}} \left(\frac{1}{|B|} \int_B |T_\rho f(x)|^q dx \right)^{1/q} \\ &\leq C \|W\|_{s,\phi^{p/s}} \|T_\rho f\|_{q,\phi^{p/q}}. \end{aligned}$$

The desired inequality is obtained by taking the supremum over all balls B and using the fact that T_ρ is bounded from $\mathcal{M}_{p,\phi}$ to $\mathcal{M}_{q,\phi^{p/q}}$. \square

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REFERENCES

- [1] D.R. ADAMS, "A note on Riesz potentials", *Duke Math. J.* **42** (1975), 765–778.
- [2] D.R. ADAMS AND L.I. HEDBERG, *Functions Spaces and Potential Theory*, Springer-Verlag, Berlin, 1996.
- [3] S. CAMPANATO, "Proprietà di una famiglia di spazi funzionali", *Ann. Scuola Norm. Sup. Pisa* **18** (1964), 137–160.
- [4] F. CHIARENZA AND M. FRASCA, "Morrey spaces and Hardy-Littlewood maximal function", *Rend. Mat.* **7** (1987), 273–279.
- [5] ERIDANI, "On the boundedness of a generalized fractional integral on generalized Morrey spaces", *Tamkang J. Math.* **33** (2002), 335–340.
- [6] ERIDANI, H. GUNAWAN, AND E. NAKAI, "On generalized fractional integral operators", *Sci. Math. Jpn.* **60** (2004), 539–550.
- [7] H. GUNAWAN, "A note on the generalized fractional integral operators", *J. Indones. Math. Soc. (MIHMI)* **9** (2003), 39–43.

- [8] G.H. HARDY AND J.E. LITTLEWOOD, “Some properties of fractional integrals. I”, *Math. Zeit.* **27** (1927), 565–606.
- [9] G.H. HARDY AND J.E. LITTLEWOOD, “A maximal theorem with function-theoretic applications”, *Acta Math.* **54** (1930), 81–116.
- [10] G.H. HARDY AND J.E. LITTLEWOOD, “Some properties of fractional integrals. II”, *Math. Zeit.* **34** (1932), 403–439.
- [11] L.I. HEDBERG, “On certain convolution inequalities”, *Proc. Amer. Math. Soc.* **36** (1972), 505–510.
- [12] K. KURATA, S. NISHIGAKI, AND S. SUGANO, “Boundedness of integral operators on generalized Morrey spaces and its application to Schrödinger operators”, *Proc. Amer. Math. Soc.* **128** (2002), 1125–1134.
- [13] C.B. MORREY, “Functions of several variables and absolute continuity”, *Duke Math. J.* **6** (1940), 187–215.
- [14] E. NAKAI, “Hardy-Littlewood maximal operator, singular integral operators, and the Riesz potentials on generalized Morrey spaces”, *Math. Nachr.* **166** (1994), 95–103.
- [15] E. NAKAI, “On generalized fractional integrals”, *Taiwanese J. Math.* **5** (2001), 587–602.
- [16] E. NAKAI, “On generalized fractional integrals on the weak Orlicz spaces, BMO_ϕ , the Morrey spaces and the Campanato spaces”, in *Function Spaces, Interpolation Theory and Related Topics (Lund, 2000)*, 95–103, de Gruyter, Berlin, 2002.
- [17] E. NAKAI, “Recent topics on fractional integral operators” (Japanese), *Sūgaku* **56** (2004), 260–280.
- [18] E. NAKAI, “Generalized fractional integrals on Orlicz-Morrey spaces”, in *Banach and Function Spaces*, 323–333, Yokohama Publ., Yokohama, 2004.
- [19] P.A. OLSEN, “Fractional integration, Morrey spaces and a Schrödinger equation”, *Comm. Partial Differential Equations* **20** (1995), 2005–2055.
- [20] J. PEETRE, “On the theory of $\mathcal{L}_{p,\lambda}$ spaces”, *J. Funct. Anal.* **4** (1969), 71–87.
- [21] B. RUBIN, *Fractional Integrals and Potentials*, Addison-Wesley, Essex, 1996.
- [22] S.L. SOBOLEV, “On a theorem in functional analysis” (Russian), *Mat. Sob.* **46** (1938), 471–497 [English translation in *Amer. Math. Soc. Transl. ser. 2* **34** (1963), 39–68].
- [23] E.M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, 1970.
- [24] S. SUGANO AND H. TANAKA, “Boundedness of fractional integral operators on generalized Morrey spaces”, *Sci. Math. Jpn.* **58** (2003), 531–540.
- [25] C.T. ZORKO, “Morrey space”, *Proc. Amer. Math. Soc.* **98** (1986), 586–592.

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