Generalized Fractional Integral Operators and Olsen Inequalities

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Outline

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Abstract

We review the boundedness of the classical fractional integral operator I_α on Morrey spaces as well as on generalized Morrey spaces.

We also discuss recent results for the generalized fractional integral operator T_ρ associated to a function $\rho : (0, \infty) \to (0, \infty)$.

Further, for a function W on $\mathbb{R}^n$, we shall be interested in the boundedness of the multiplication operator $f \mapsto W \cdot T_\rho f$ on generalized Morrey spaces.

For $\rho(t) = t^\alpha,\ 0 < \alpha < n$, our results for T_ρ and $W \cdot T_\rho$ reduce to Olsen's and Kurata-Nishigaki-Sugano's.

The Fractional Integral Operator

For $0 < \alpha < n$, let I_α denote the Riesz potential or the (classical) fractional integral operator, which is given by the formula

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

Formally, through its Fourier transform, the operator I_α can be recognized as a multiple of the Laplacian to the power of $-\frac{\alpha}{2}$, that is,

$$I_\alpha f = \kappa(n,\alpha).(-\Delta)^{-\frac{\alpha}{2}} f,$$

(see, for instance, [2, 20, 22]).

The Boundedness of I_α on Lebesgue Spaces

A well-known result for I_α is the Hardy-Littlewood-Sobolev inequality, which was proved by Hardy and Littlewood [8, 10] and Sobolev [21] around the 1930's.

Theorem 2.1 (Hardy-Littlewood; Sobolev)

For $1 < p < \frac{n}{\alpha}$, we have the inequality

$$\|I_\alpha f\|_q \le C_p \|f\|_p, \tag{1}$$

that is, I_α is bounded from L^p to L^q, provided that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

As an immediate consequence of this inequality, one has the following estimate for $(-\Delta)^{-1}$:

$$\|(-\Delta)^{-1}f\|_{np/(n-2)} \leq C_p\|f\|_p,$$

for $1 < p < \frac{n}{2},\ n \geq 3$.

Here $u := (-\Delta)^{-1}f$ is a solution of the Poisson equation $-\Delta u = f$.

From (1) we can also prove Sobolev's embedding theorems (see [22]).

The (Classical) Morrey Spaces

For $1 \leq p < \infty$ and $0 \leq \lambda \leq n$, the (classical) Morrey space $L^{p,\lambda} = L^{p,\lambda}(\mathbb{R}^n)$ is defined to be the space of all functions $f \in L^p_{\text{loc}}(\mathbb{R}^n)$ for which

$$\|f\|_{p,\lambda} := \sup_{B=B(a,r)} \left(\frac{1}{r^\lambda} \int_B |f(y)|^p \, dy \right)^{1/p} < \infty,$$

where $B(a,r)$ denotes the (open) ball centered at $a \in \mathbb{R}^n$ with radius $r > 0$ [13].

Here $\| \cdot \|_{p,\lambda}$ defines a semi-norm on $L^{p,\lambda}$. Note particularly that $L^{p,0} = L^p$ and $L^{p,n} = L^\infty$.

For the structure of Morrey spaces and their generalizations, see the works of S. Campanato [3], J. Peetre [19], C.T. Zorko [23], and the references therein.

The Boundedness of I_α on Morrey Spaces

As stated in [19], S. Spanne proved that I_α is bounded from $L^{p,\lambda}$ to $L^{q,\lambda q/p}$ for $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $0 \le \lambda < n$.

A stronger result was obtained by D.R. Adams [1] and reproved by F. Chiarenza and M. Frasca [4].

Theorem 2.2 (Adams; Chiarenza-Frasca)

For $1 < p < \frac{n}{\alpha}$ and $0 \le \lambda < n - \alpha p$, we have the inequality

$$\|I_\alpha f\|_{q,\lambda} \le C_{p,\lambda} \|f\|_{p,\lambda}$$

provided that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$.

The Hardy-Littlewood Maximal Operator

The proof usually involves the properties of the Hardy-Littlewood maximal operator M, defined by the formula

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy,$$

where $|B(x,r)| = c\, r^n$ is the Lebesgue measure of $B(x,r)$.

The operator M is known to be bounded on L^p for $1 < p \leq \infty$ [9].

Chiarenza and Frasca [4] proved that M is also bounded on Morrey spaces.

Theorem 2.3 (Chiarenza-Frasca)

The inequality

$$\|Mf\|_{p,\lambda} \le C_{p,\lambda}\|f\|_{p,\lambda}$$

holds for $p > 1$ and $0 \le \lambda < n$.

The Proof of Theorem 1.2

For each $x \in \mathbb{R}^n$, write $I_\alpha f(x) = I_1(x) + I_2(x)$ where

$$I_1(x) := \int_{|x-y|<R} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy; \quad I_2(x) := \int_{|x-y|\geq R} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy,$$

with R being an arbitrary positive number.

For I_1, we have the following estimate:

$$\begin{aligned}
|I_1(x)| &\le \sum_{k=-\infty}^{-1} \int_{2^k R \le |x-y| < 2^{k+1} R} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy \\
&\le \sum_{k=-\infty}^{-1} (2^k R)^{\alpha-n} \int_{B(x,2^{k+1}R)} |f(y)|\, dy \\
&\le C\, Mf(x) \sum_{k=-\infty}^{-1} (2^k R)^{\alpha} \\
&\le C\, R^{\alpha} Mf(x).
\end{aligned}$$

To estimate I_2, we proceed as follows:

$$\begin{aligned}
|I_2(x)| &\le \sum_{k=0}^{\infty} \int_{2^k R \le |x-y| < 2^{k+1} R} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy \\
&\le \sum_{k=0}^{\infty} (2^k R)^{\alpha-n} \int_{B(x,2^{k+1}R)} |f(y)|\, dy \\
&\le C \sum_{k=0}^{\infty} (2^k R)^{\alpha-\frac{n}{p}} \left(\int_{B(x,2^{k+1}R)} |f(y)|^p dy \right)^{1/p} \\
&\le C \sum_{k=0}^{\infty} (2^k R)^{\alpha+\frac{\lambda-n}{p}} \|f\|_{p,\lambda} \\
&\le C\, R^{\alpha+\frac{\lambda-n}{p}} \|f\|_{p,\lambda}.
\end{aligned}$$

The last inequality holds because $\sum_{k=0}^{\infty} 2^{k(\alpha+\frac{\lambda-n}{p})}$ converges, for we have $\alpha + \frac{\lambda-n}{p} < 0$.

Combining the two estimates, we get

$$|I_\alpha f(x)| \le C\, R^\alpha [Mf(x) + R^{\frac{\lambda-n}{p}} \|f\|_{p,\lambda}].$$

Assuming that $f \ne 0$ and Mf is finite everywhere, we choose $R = \left(\frac{Mf(x)}{\|f\|_{p,\lambda}}\right)^{p/(\lambda-n)}$. Then, we have

$$\begin{aligned} |I_\alpha f(x)| &\le C\, [Mf(x)]^{1-\alpha p/(n-\lambda)} \|f\|_{p,\lambda}^{\alpha p/(n-\lambda)} \\ &= C\, [Mf(x)]^{p/q} \|f\|_{p,\lambda}^{1-p/q}. \end{aligned}$$

The desired inequality then follows from this and the boundedness of M on L^p. □

The Boundedness of I_α on Generalized Morrey Spaces

For $1 \le p < \infty$ and a suitable function $\phi : (0,\infty) \to (0,\infty)$, we define the generalized Morrey space $\mathcal{M}_{p,\phi} = \mathcal{M}_{p,\phi}(\mathbf{R}^n)$ to be the space of all functions $f \in L^p_{\rm loc}(\mathbf{R}^n)$ for which

$$\|f\|_{p,\phi} := \sup_{B=B(a,r)} \frac{1}{\phi(r)} \left(\frac{1}{|B|} \int_B |f(y)|^p dy \right)^{1/p} < \infty.$$

Note: If $\phi(t) = t^{(\lambda-n)/p}$, $0 \le \lambda \le n$, we have $\mathcal{M}_{p,\phi} = L^{p,\lambda}$ — the classical Morrey space.

Unless otherwise stated, ϕ satisfies the following two conditions:

(2.1) $\frac{1}{2} \le \frac{r}{s} \le 2 \Rightarrow \frac{1}{C_1} \le \frac{\phi(r)}{\phi(s)} \le C_1$ (*the doubling condition*);

(2.2) $\displaystyle\int_r^\infty \frac{\phi^p(t)}{t}\,dt \le C_2\,\phi^p(r)$ for $1 < p < \infty$.

For any function ψ that satisfies the doubling condition, we have

$$\int_{2^k r}^{2^{k+1} r} \frac{\psi(t)}{t}\,dt \sim \psi(2^k r)$$

for every integer k and $r > 0$.

In [14], E. Nakai proved the boundedness of the Hardy-Littlewood maximal operator on generalized Morrey spaces.

Theorem 2.4 (Nakai)

The inequality

$$\|Mf\|_{p,\phi} \leq C_{p,\phi}\|f\|_{p,\phi}$$

holds for $1 < p < \infty$.

Nakai also obtained the boundedness of I_α on generalized Morrey spaces, which can be viewed as an extension of Spanne's result.

The following theorem can be considered as an extension of Adams-Chiarenza-Frasca's result.

Theorem 2.5

Suppose that, in addition to the condition (2.1) and (2.2), ϕ satisfies the inequality $\phi(t) \le Ct^{\beta}$ for $-\frac{n}{p} \le \beta < -\alpha,\ 1 < p < \frac{n}{\alpha}$. Then, we have

$$\|I_\alpha f\|_{q,\phi^{p/q}} \le C_{p,\beta}\|f\|_{p,\phi}$$

where $q = \frac{\beta p}{\alpha+\beta}$.

Remark. Observe that when $\phi(t) = t^{(\lambda-n)/p}$, $0 \le \lambda < n - \alpha p$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$, Theorem 2.5 reduces to Theorem 2.2.

The Generalized Fractional Integral Operator T_ρ

For a given function $\rho : (0, \infty) \to (0, \infty)$, we define the (generalized) fractional integral operator T_ρ by

$$T_\rho f(x) := \int_{\mathbf{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y)\, dy$$

For $\rho(t) = t^\alpha,\ 0 < \alpha < n$, we have $T_\rho = I_\alpha$ — the classical fractional integral operator.

The operator T_ρ was first studied by Nakai [15]. Recent results on T_ρ can be found in [5, 6, 7, 16, 17].

The Boundedness of T_ρ on Generalized Morrey spaces

A slight modification of Theorem 2.5 may be formulated for T_ρ as follows.

Theorem 3.1

Suppose that $\rho(t) \le C_1 t^\alpha$ for some $0 < \alpha < n$, and, in addition to the condition (2.1) and (2.2), ϕ satisfies the inequality $\phi(t) \le C_2 t^\beta$ for $-\frac{n}{p} \le \beta < -\alpha,\ 1 < p < \frac{n}{\alpha}$. Then, for $q = \frac{\beta p}{\alpha+\beta}$, we have

$$\|T_\rho f\|_{q,\phi^{p/q}} \le C_{p,\beta}\|f\|_{p,\phi},$$

that is, T_ρ is bounded from $\mathcal{M}_{p,\phi}$ to $\mathcal{M}_{q,\phi^{p/q}}$.

A further generalization of Theorem 2.2 can be found in [7].

Theorem 3.2 (G)

Suppose that, in addition to the condition (2.1) and (2.2), ϕ is surjective. If ρ satisfies the doubling condition and

$$\int_0^r \frac{\rho(t)}{t}\,dt \le C\phi(r)^{(p-q)/q} \quad \text{and} \quad \int_r^\infty \frac{\rho(t)\phi(t)}{t}dt \le C\phi(r)^{p/q},$$

for $1 < p < q < \infty$, then we have

$$\|T_\rho f\|_{q,\phi^{p/q}} \le C_{p,\phi}\|f\|_{p,\phi}.$$

Olsen Inequality

In studying a Schrödinger equation

$$(-\Delta + V(x) + W(x))u(x) = f(x)$$

with perturbed potentials W on $\mathbb{R}^n$ (particularly for $n = 3$), P.A. Olsen [18] proved the following result.

Theorem 4.1 (Olsen)

For $1 < p < \frac{n}{\alpha}$ and $0 \leq \lambda < n - \alpha p$, we have

$$\|W \cdot I_\alpha f\|_{p,\lambda} \leq C_{p,\lambda} \|W\|_{(n-\lambda)/\alpha,\lambda} \|f\|_{p,\lambda},$$

that is, $W \cdot I_\alpha$ is bounded on $L^{p,\lambda}$, provided that $W \in L^{(n-\lambda)/\alpha,\lambda}$.

As a consequence of Theorem 4.1, we see that for $1 < p < \frac{n}{2},\ n \geq 3$, the estimate

$$\|W \cdot (-\Delta)^{-1} f\|_{p,\lambda} \leq C_{p,\lambda}\, \|W\|_{(n-\lambda)/2,\lambda} \|f\|_{p,\lambda},$$

holds provided that $W \in L^{(n-\lambda)/2,\lambda},\ 0 \leq \lambda < n - 2p$.

In particular, when $\lambda = 0$, one has

$$\|W \cdot (-\Delta)^{-1} f\|_{p} \leq C_{p}\, \|W\|_{n/2} \|f\|_{p}$$

provided that $W \in L^{n/2}$.

K. Kurata *et al.* [12] extended Olsen's result by proving that, for some $p > 1$ and a function ϕ satisfying several conditions (including the doubling condition), the operator $W \cdot I_\alpha$ is bounded on generalized Morrey spaces $\mathcal{M}_{p,\phi}$, provided that $W \in \mathcal{M}_{s_1,\phi} \cap \mathcal{M}_{s_2,\phi}$ for some indices s_1 and s_2.

Their estimate, however, is rather complicated. We shall here present simpler estimates for $W \cdot I_\alpha$ on generalized Morrey spaces.

Our Results: First Estimate for $W \cdot I_\alpha$

Theorem 4.2

Suppose that, in addition to the condition (2.1) and (2.2), ϕ satisfies the inequality $\phi(t) \leq Ct^\beta$ for $-\frac{n}{p} \leq \beta < -\alpha,\ 1 < p < \frac{n}{\alpha}$. Then, we have

$$\|W \cdot I_\alpha f\|_{p,\phi} \leq C_{p,\beta}\|W\|_{s,\phi^{p/s}}\|f\|_{p,\phi}$$

provided that $W \in \mathcal{M}_{s,\phi^{p/s}}$ where $s = -\frac{\beta p}{\alpha}$.

Proof. Use Hölder's inequality and Theorem 2.5. □

Our Results: Second Estimate for $W \cdot I_\alpha$

Theorem 4.3

Suppose that ϕ satisfies the doubling condition and the inequality

$$\int_r^\infty t^{\alpha-1}\phi(t)\,dt \le C\,r^\alpha \phi(r).$$

Then, for $1 < p < \frac{n}{\alpha}$, we have

$$\|W \cdot I_\alpha f\|_{p,\phi} \le C_{p,\phi}\|W\|_{n/\alpha}\|f\|_{p,\phi},$$

provided that $W \in L^{n/\alpha}$.

Proof. For $a \in \mathbf{R}^n$ and $r > 0$, let $B = B(a, r)$, $\tilde{B} = B(a, 2r)$, and write $f = f_1 + f_2 := f\chi_{\tilde{B}} + f\chi_{\tilde{B}^c}$. We observe that $f_1 \in L^p$ with

$$\|f_1\|_p = \Big(\int_{\mathbf{R}^n} |f_1(y)|^p dy\Big)^{1/p} = \Big(\int_{\tilde{B}} |f(y)|^p dy\Big)^{1/p} \le C\, r^{n/p}\phi(r)\|f\|_{p,\phi}.$$

Hence, by applying Theorem 4.1 for $\lambda = 0$, we get

$$\begin{aligned}\Big(\int_B |W \cdot I_\alpha f_1(x)|^p dx\Big)^{1/p} &\le \|W \cdot I_\alpha f_1\|_p \le C\,\|W\|_{n/\alpha}\|f_1\|_p \\ &\le C\, r^{n/p}\phi(r)\|W\|_{n/\alpha}\|f\|_{p,\phi},\end{aligned}$$

whence

$$\frac{1}{\phi(r)}\Big(\frac{1}{|B|}\int_B |W \cdot I_\alpha f_1(x)|^p dx\Big)^{1/p} \le C\,\|W\|_{n/\alpha}\|f\|_{p,\phi}.$$

Next, for $x \in B$, we have

$$|I_\alpha f_2(x)| \le \int_{\tilde{B}^c} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \le \int_{|x-y|\ge r} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy$$

Then, we shall obtain

$$|I_\alpha f_2(x)| \le C\, \|f\|_{p,\phi} \int_r^\infty t^{\alpha-1}\phi(t)\, dt \le C\, r^\alpha \phi(r) \|f\|_{p,\phi}.$$

Hence

$$\Big(\frac{1}{|B|}\int_B |W.I_\alpha f_2(x)|^p dx\Big)^{\frac{1}{p}} \le C\, r^\alpha \phi(r)\|f\|_{p,\phi}\Big(\frac{1}{|B|}\int_B |W(x)|^p\, dx\Big)^{1/p}$$
$$\le C\, r^\alpha \phi(r)\|f\|_{p,\phi}\Big(\frac{1}{|B|}\int_B |W(x)|^{n/\alpha} dx\Big)^{\frac{\alpha}{n}}$$
$$\le C\, \phi(r)\|W\|_{n/\alpha}\|f\|_{p,\phi},$$

and so

$$\frac{1}{\phi(r)}\Big(\frac{1}{|B|}\int_B |W\cdot I_\alpha f_2(x)|^p dx\Big)^{1/p} \le C\,\|W\|_{n/\alpha}\|f\|_{p,\phi}.$$

The desired estimate follows from the two estimates via Minkowski inequality. □

Our Results: First Estimate for $W \cdot T_\rho$

Theorem 4.4

Suppose that $\rho(t) \le C_1 t^\alpha$ for some $0 < \alpha < n$, and, in addition to the condition (2.1) and (2.2), $\phi(t) \le C_2 t^\beta$ for $-\frac{n}{p} \le \beta < -\alpha$, $1 < p < \frac{n}{\alpha}$. Then, we have

$$\|W \cdot T_\rho f\|_{p,\phi} \le C_{p,\beta} \|W\|_{s,\phi^{p/s}} \|f\|_{p,\phi}$$

provided that $W \in \mathcal{M}_{s,\phi^{p/s}}$ where $s = -\frac{\beta p}{\alpha}$.

Proof. Use Hölder's inequality and Theorem 3.1. □

Our Results: Second Estimate for $W \cdot T_\rho$

Theorem 4.5

Suppose that, in addition to the condition (2.1) and (2.2), ϕ is surjective. If ρ satisfies the doubling condition and

$$\int_0^r \frac{\rho(t)}{t}\,dt \le C\phi(r)^{(p-q)/q} \quad \text{and} \quad \int_r^\infty \frac{\rho(t)\phi(t)}{t}\,dt \le C\phi(r)^{p/q},$$

for $1 < p < q < \infty$, then we have

$$\|W \cdot T_\rho f\|_{p,\phi} \le C_{p,\phi}\|W\|_{s,\phi^{p/s}}\|f\|_{p,\phi},$$

provided that $W \in \mathcal{M}_{s,\phi^{p/s}}$ where $\frac{1}{s} = \frac{1}{p} - \frac{1}{q}$.

Proof. Let $B = B(a, r)$ be an arbitrary ball in $\mathbf{R}^n$. By Hölder's inequality, we have

$$\frac{1}{|B|}\int_B |W\cdot T_\rho f(x)|^p dx \le \Big(\frac{1}{|B|}\int_B |W(x)|^s dx\Big)^{\frac{p}{s}} \Big(\frac{1}{|B|}\int_B |T_\rho f(x)|^q dx\Big)^{\frac{p}{q}}$$

with $\frac{p}{s}+\frac{p}{q}=1$. Now take the p-th roots and then divide both sides by $\phi(r)$ to get

$$\begin{aligned}\frac{1}{\phi(r)}\Big(\frac{1}{|B|}\int_B |W\cdot T_\rho f(x)|^p dx\Big)^{\frac{1}{p}} &\le \frac{1}{\phi(r)^{p/s}}\Big(\frac{1}{|B|}\int_B |W(x)|^s dx\Big)^{1/s} \\ &\quad\times \frac{1}{\phi(r)^{p/q}}\Big(\frac{1}{|B|}\int_B |T_\rho f(x)|^q dx\Big)^{\frac{1}{q}} \\ &\le C\,\|W\|_{s,\phi^{p/s}}\|T_\rho f\|_{q,\phi^{p/q}}.\end{aligned}$$

The desired inequality is obtained by taking the supremum over all balls B and using Theorem 3.2. □

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