G-orthogonality in n-inner product spaces

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Introduction

Several notions of orthogonality in a normed space have been developed and studied by many authors. For examples, see [Alonso], [Desbiens], [Diminnie], [Guijarro & Tomas], [Milicic], [Partington], [Serb]. Among them are the Pythagorean, isosceles, and the Birkhoff-James orthogonality, defined in a (real) normed space $(X, \|\cdot\|)$ as follows:

P-orthogonality: x is P-orthogonal to y (denoted by $x \perp_{\mathsf{P}} y$) if only if

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

I-orthogonality: x is I-orthogonal to y (denoted by $x \perp_{\mathrm{I}} y$) if only if

$$\|x+y\| = \|x-y\|.$$

BJ-orthogonality: x is BJ-orthogonal to y (denoted by $x \perp_{\mathsf{BJ}} y$) if only if

$$\|x+\alpha y\| \geq \|x\| \text{ for every } \alpha \in \mathbf{R}.$$

Note that if X is equipped with an inner product $\langle \cdot, \cdot \rangle$, then $x \perp_{\mathsf{P}} y$, $x \perp_{\mathsf{I}} y$, and $x \perp_{\mathsf{BJ}} y$ are all equivalent to the condition that $\langle x, y \rangle = 0$, where we have the usual orthogonality $x \perp y$.

The following examples show that in a normed space which is not an inner product space, one notion of orthogonality does not imply another.

Example 1.1. Let $X = \ell^1$, equipped with the norm $\|x\|_1 = \sum_{k=1}^{\infty} |\xi_k|$.

(a) Take $x = (3, 6, 0, \ldots)$ and $y = (8, -4, 0, \ldots)$. Then, $x \perp_{\mathsf{P}} y$ but $x \not\perp_{\mathsf{I}} y$ and $x \not\perp_{\mathsf{BJ}} y$.

(b) Take $x = (1, 1, 0, \ldots)$ and $y = (2, -1, 0, \ldots)$. Then, $x \perp_{\mathsf{I}} y$ but $x \not\perp_{\mathsf{P}} y$ and $x \not\perp_{\mathsf{BJ}} y$.

(c) Take $x = (1, 0, 0, \ldots)$ and $y = (-1, 1, 0, \ldots)$. Then, $x \perp_{\mathsf{BJ}} y$ but $x \not\perp_{\mathsf{P}} y$ and $x \not\perp_{\mathsf{I}} y$.

The three notions of orthogonality have been extended to 2-normed spaces by several researchers. For example, see [Cho & Kim] and [Khan].

A (real) 2-normed space is a (real) vector space X equipped with a *2-norm* $\|\cdot,\cdot\| : X \times X \to \mathbf{R}$ satisfying the following properties:

(2N.1) $\|x, y\| \geq 0$ for every $x, y \in X$; $\|x, y\| = 0$ if and only if x and y are linearly dependent;

(2N.2) $\|x, y\| = \|y, x\|$ for every $x, y \in X$;

(2N.3) $\|x, \alpha y\| = |\alpha|\|x, y\|$ for every $x, y \in X$ and $\alpha \in \mathbf{R}$;

(2N.4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for every $x, y, z \in X$.

As the notions of orthogonality in normed spaces are inspired by that in inner product spaces, the notions of orthogonality in 2-normed spaces are also connected to that in 2-inner product spaces. A (real) 2-inner product space is a (real) vector space X equipped with an *2-inner product* $\langle \cdot, \cdot | \cdot \rangle : X \times X \times X \to \mathbf{R}$ satisfying

(2I.1) $\langle x, x | z \rangle \geq 0$ for every $x, z \in X$; $\langle x, x | z \rangle = 0$ if and only if x and z are linearly dependent;

(2I.2) $\langle x, y | z \rangle = \langle y, x | z \rangle$ for every $x, y, z \in X$;

(2I.3) $\langle \alpha x, y | z \rangle = \alpha \langle x, y | z \rangle$ for every $x, y, z \in X$ and $\alpha \in \mathbf{R}$;

(2I.4) $\langle x_1 + x_2, y | z \rangle = \langle x_1, y | z \rangle + \langle x_2, y | z \rangle$ for every $x_1, x_2, y, z \in X$.

For historical background about 2-inner product spaces and 2-normed spaces, see [Diminnie, Gahler & White] and [Gahler], respectively.

In [Gunawan *et al.*], it is shown that the 'standard' definition of orthogonality in a 2-inner product space $(X, \langle \cdot, \cdot | \cdot \rangle)$, where $\dim(X) \geq 3$, is the following:

Definition 1.2. (G-orthogonality in 2-inner product spaces)

x is *G-orthogonal to* y, denoted by $x \perp_G y$, if and only if there exists a subspace V of X with $\text{codim}(V) = 1$ such that $\langle x, y | z \rangle = 0$ for all $z \in V$.

We say that the above definition is 'standard' because when X is a standard 2-inner product space, that is, when X is actually equipped with an inner product $\langle\cdot,\cdot\rangle$ together with the 2-inner product

$$\langle x, y|z\rangle := \begin{vmatrix} \langle x, y\rangle & \langle x, z\rangle \\ \langle z, y\rangle & \langle z, z\rangle \end{vmatrix},$$

then we have $x \perp_G y$ if and only if $x \perp y$ (see [Gunawan *et al.*]).

Accordingly, we may define P-, I-, and BJ-orthogonality in a 2-normed space $(X, \|\cdot,\cdot\|)$ of dimension 3 or higher as follows:

Definition 1.2 (P-, I-, and BJ-orthogonality in 2-normed spaces)

(a) $x \perp_{\mathsf{P}} y$ if only if there exists a subspace V of X with codim$(V) = 1$ such that

$$\|x+y,z\|^2 = \|x,z\|^2 + \|y,z\|^2 \quad \text{for every } z \in V;$$

(b) $x \perp_{\mathsf{I}} y$ if only if there exists a subspace V of X with codim$(V) = 1$ such that

$$\|x+y,z\| = \|x-y,z\| \quad \text{for every } z \in V;$$

(c) $x \perp_{\mathsf{BJ}} y$ if only if there exists a subspace V of X with codim$(V) = 1$ such that

$$\|x+\alpha y,z\| \geq \|x,z\| \quad \text{for every } z \in V \text{and } \alpha \in \mathbf{R}.$$

In this talk, we shall discuss the notion of G-orthogonality in n-inner product spaces. We show that in the standard case, our notions of orthogonality also coincide with the usual one. This research is joint with E. Kikianty, Mashadi, S. Gemawati, and I. Sihwaningrum.

Throughout this paper, X will always denote a real vector space, unless otherwise stated.

Main Results

Let $n \geq 2$ be a nonnegative integer and X be a vector space of dimension n or higher. A real-valued function $\|\cdot, \dots, \cdot\|$ on X^n satisfying the following four properties

(2.1) $\|x_1, \dots, x_n\| = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent;

(2.2) $\|x_1, \dots, x_n\|$ is invariant under permutation;

(2.3) $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \, \|x_1, x_2, \dots, x_n\|$ for any $\alpha \in \mathbf{R}$;

(2.4) $\|x + x', x_2, \dots, x_n\| \leq \|x, x_2, \dots, x_n\| + \|x', x_2, \dots, x_n\|$,

is called an n-*norm* on X.

The pair $(X, \|\cdot, \dots, \cdot\|)$ is called an n-normed space.

Next, a real-valued function $\langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle$ on X^{n+1} satisfying the following five properties

(2.5) $\langle z_1, z_1 | z_2, \ldots, z_n \rangle \geq 0$; $\langle z_1, z_1 | z_2, \ldots, z_n \rangle = 0$ if and only if $z_1, z_2, \ldots, z_n$ are linearly dependent;

(2.6) $\langle z_1, z_1 | z_2, \ldots, z_n \rangle = \langle z_{i_1}, z_{i_1} | z_{i_2}, \ldots, z_{i_n} \rangle$ for every permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$;

(2.7) $\langle x, y | z_2, \ldots, z_n \rangle = \langle y, x | z_2, \ldots, z_n \rangle$;

(2.8) $\langle \alpha x, y | z_2, \ldots, z_n \rangle = \alpha \langle x, y | z_2, \ldots, z_n \rangle$, $\alpha \in \mathbf{R}$;

(2.9) $\langle x + x', y | z_2, \ldots, z_n \rangle = \langle x, y | z_2, \ldots, z_n \rangle + \langle x', y | z_2, \ldots, z_n \rangle$,

is called an n-*inner product* on X.

The pair $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ is called an n-inner product space.

Note that any inner product space $(X, \langle\cdot,\cdot\rangle)$ can be equipped with the standard n-norm

$$\|x_1, \dots, x_n\| := \sqrt{\det(\langle x_i, x_j\rangle)},$$

and the standard n-inner product

$$\langle x, y|z_2, \dots, z_n\rangle := \begin{vmatrix} \langle x, y\rangle & \langle x, z_2\rangle & \dots & \langle x, z_n\rangle \\ \langle z_2, y\rangle & \langle z_2, z_2\rangle & \dots & \langle z_2, z_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle z_n, y\rangle & \langle z_n, z_2\rangle & \dots & \langle z_n, z_n\rangle \end{vmatrix}.$$

The entity $\det(\langle x_i, x_j\rangle)$ is known as the Gramian of $x_1, \dots, x_n$. Geometrically, $\|x_1, \dots, x_n\|$ represents the volume of the n-dimensional parallelepiped spanned by $x_1, \dots, x_n$ in X. The standard n-inner product is also known as the *simple* n-inner product [Misiak1].

Hereafter, by a standard n-normed space (or a standard n-inner product space) we mean an inner product space equipped with the standard n-norm (or the standard n-inner product). Note that, in a standard n-normed space,

$$\langle x, y|z_2, \dots, z_n\rangle :=$$

$$\frac{1}{4}(\|x + y, z_2, \dots, z_n\|^2 - \|x - y, z_2, \dots, z_n\|^2)$$

is the standard n-inner product; and in a standard n-inner product space,

$$\|z_1, z_2, \dots, z_n\| := \sqrt{\langle z_1, z_1|z_2, \dots, z_n\rangle}$$

is the standard n-norm. Thus, a standard n-normed space is a standard n-inner product space, and vice versa. For further discussion about n-normed spaces and n-inner product spaces, see [Gunawan], [Gunawan & Mashadi], [Misiak1], [Misiak2].

The following result tells us that, in an n-inner product space $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$, we cannot define G-orthogonality between x and y by the condition that

$$\langle x, y | z_2, \dots, z_n \rangle = 0$$

for all $z_2, \dots, z_n \notin \text{span}\{x, y\}$, as suggested by [Cho & Kim] and [Godini].

Theorem 2.1. *Let $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$ be a standard n-inner product space of dimension $n+1$ or higher. Then, the condition that*

$$\langle x, y | z_2, \dots, z_n \rangle = 0$$

for all $z_2, \dots, z_n \notin \text{span}\{x, y\}$ is satisfied only by $x = 0$ or $y = 0$.

As in 2-inner product spaces and 2-normed spaces, we define the notions of G-orthogonality in n-inner product spaces and P-, I-, and BJ-orthogonality in n-normed spaces as follows.

Definition 2.2. Let $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$ be an n-inner product space of dimension $n+1$ or higher. For $x, y \in X$, we say that x is *G-orthogonal to* y and write $x \perp_G y$ if and only if there exists a subspace V of X with $\mathrm{codim}(V) = 1$ such that $\langle x, y | z_2, \dots, z_n \rangle = 0$ for every $z_2, \dots, z_n \in V$.

Definition 2.3. (P-, I-, and BJ-orthogonality) Let $(X, \|\cdot, \dots, \cdot\|)$ be an n-normed space of dimension $n+1$ or higher. For $x, y \in X$, we define

(a) $x \perp_{\mathsf{P}} y$ if only if there exists a subspace V of X with codim$(V) = 1$ such that

$$\|x+y, z_2, \dots, z_n\|^2 =$$

$$\|x, z_2, \dots, z_n\|^2 + \|y, z_2, \dots, z_n\|^2$$

for every $z_2, \dots, z_n \in V$;

(b) $x \perp_{\mathsf{I}} y$ if only if there exists a subspace V of X with codim$(V) = 1$ such that

$$\|x+y, z_2, \dots, z_n\| = \|x-y, z_2, \dots, z_n\|$$

for every $z_2, \dots, z_n \in V$;

(c) $x \perp_{\mathsf{BJ}} y$ if only if there exists a subspace V of X with codim$(V) = 1$ such that

$$\|x+\alpha y, z_2, \dots, z_n\| \geq \|x, z_2, \dots, z_n\|$$

for every $z_2, \dots, z_n \in V$ and $\alpha \in \mathbf{R}$.

Note that if X is an n-inner product space, then P-, I-, and BJ-orthogonality are equivalent to G-orthogonality. The following theorem states that in a standard n-inner product space, G-orthogonality is equivalent to the usual orthogonality (with respect to the inner product).

Theorem 2.4 *Let* $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ *be a standard* n*-inner product space of dimension* $n+1$ *or higher. Then,* $x \perp_G y$ *if and only if* $x \perp y$*.*

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