ON FINDING THE FUNDAMENTAL DOMINANT WEIGHTS OF A ROOT SYSTEM

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Abstract. This note offers a formula which can be used as an alternative method of finding the fundamental dominant weights of a root system, as suggested in [3]. We explain how the formula actually works and verify the fundamental dominant weights of root systems of type $\mathsf{A} - \mathsf{G}$.

Introduction

Almost all symbols and terminology we use here are the same as in [4, Ch. III]. Let Φ be a root system of rank l. Fix $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ to be a base of Φ. Relative to Δ, denote by Φ^+ the set of positive roots, whose members are of the form $\alpha = \sum_{j=1}^{l} k_j(\alpha)\alpha_j$ where the $k_j(\alpha)$'s are all nonnegative integers. Denote by δ the special element, namely $\delta = \frac{1}{2} \sum_{\alpha \in \Phi^+} \alpha$. Now let $\{\lambda_1, \ldots, \lambda_l\}$ be the dual basis, for which $\langle \lambda_j, \alpha_k \rangle = \delta_{jk},\ j, k \in \{1, \ldots, l\}$. Then we know that $\delta = \sum_{j=1}^{l} \lambda_j$, showing that δ is a dominant weight.

Let us fix any $j_0 \in \{1, \ldots, l\}$ and put $\Phi_0^+ = \{\alpha \in \Phi^+ : k_{j_0}(\alpha) = 0\}$. Then $\Phi_0 = \Phi_0^+ \cup (-\Phi_0^+)$ forms a root system of rank $l - 1$. Φ_0^+ is obviously the corresponding set of positive roots relative to the base $\Delta_0 = \{\alpha_j : j \neq j_0\}$. Let $\delta_0 = \frac{1}{2} \sum_{\alpha \in \Phi_0^+} \alpha$ and set $\delta_1 = \delta - \delta_0$. (Note that δ_0 depends on the choice of j_0, and so does δ_1.) One may observe that δ_1 is actually the projection of δ on λ_{j_0}. Moreover, as announced in [3], we have the following result.

Theorem. $\lambda_{j_0} = \frac{\delta_1}{\langle \delta_1, \alpha_{j_0} \rangle}$.

The theorem offers a formula which can be used as an alternative method of finding the fundamental dominant weight λ_{j_0} for any given $j_0 \in \{1, \ldots, l\}$. Using this formula, we can

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find one particular fundamental dominant weight without worrying about the others. It is —in this case— less labor than employing the Cartan matrix (where we have to invert the matrix). Note, however, that the formula is useful only when we know δ and δ_0 (so we can find δ_1).

We give here a shorter proof of the theorem. We also explain how to find δ_1, and then compute and verify the fundamental dominant weights $\lambda_{j_0},\ j_0 \in \{1, \dots, l\}$, of root systems of type $\mathsf{A} - \mathsf{G}$.

1. Proof of the Theorem

The theorem follows from the lemma below, which we have stated before.

Lemma. $\delta_1 = \frac{1}{2}\langle \delta, \lambda_{j_0}\rangle \lambda_{j_0}$.

Proof. For each $j \in \{1, \dots, l\}$, set $\lambda_j^* = \lambda_j - \frac{1}{2}\langle \lambda_j, \lambda_{j_0}\rangle \lambda_{j_0}$. We observe that for $j, k \neq j_0$,

$$\langle \lambda_j^*, \alpha_k\rangle = \langle \lambda_j, \alpha_k\rangle - \frac{1}{2}\langle \lambda_j, \lambda_{j_0}\rangle\langle \lambda_{j_0}, \alpha_k\rangle = \langle \lambda_j, \alpha_k\rangle = \delta_{jk},$$

that is, $\{\lambda_j^* : j \neq j_0\}$ is the dual basis relative to Δ_0 (see [3] for more details). From this and the fact that $\lambda_{j_0}^* = 0$, we have $\delta_0 = \sum_{j=1}^{l} \lambda_j^*$, and accordingly

$$\begin{aligned}
\delta_1 &= \delta - \delta_0 \\
&= \sum_{j=1}^{l} \lambda_j - \sum_{j=1}^{l} \lambda_j^* \\
&= \frac{1}{2}\sum_{j=1}^{l} \langle \lambda_j, \lambda_{j_0}\rangle \lambda_{j_0} \\
&= \frac{1}{2}\langle \sum_{j=1}^{l} \lambda_j, \lambda_{j_0}\rangle \lambda_{j_0} \\
&= \frac{1}{2}\langle \delta, \lambda_{j_0}\rangle \lambda_{j_0},
\end{aligned}$$

proving the lemma. □

Now we prove the theorem.

Proof of the Theorem. Using the above lemma, we have

$$\langle \delta_1, \alpha_{j_0}\rangle = \frac{1}{2}\langle \delta, \lambda_{j_0}\rangle\langle \lambda_{j_0}, \alpha_{j_0}\rangle = \frac{1}{2}\langle \delta, \lambda_{j_0}\rangle.$$

Hence we obtain that

$$\delta_1 = \langle \delta_1, \alpha_{j_0} \rangle \lambda_{j_0}$$

or

$$\lambda_{j_0} = \frac{\delta_1}{\langle \delta_1, \alpha_{j_0} \rangle},$$

as stated. $\square$

2. Finding δ_1

We use the following technique to find δ_1. (In [2], the same technique is used for a different purpose.) Suppose, for example, Φ is a root system of type E_6, whose Dynkin diagram is as follows.

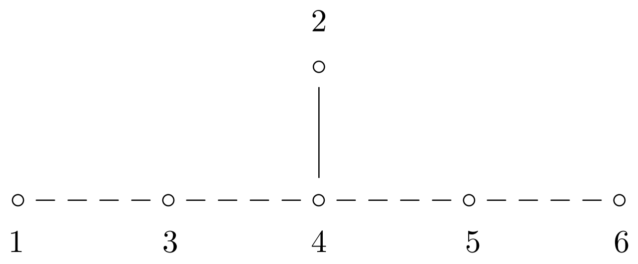

Figure 1. The Dynkin diagram of E_6

Remove the j_0-th vertex from the diagram to obtain a new diagram, which in general will be a union of Dynkin diagrams. For example, if we remove the 4th vertex, then we get a diagram for $\mathsf{A}_2 \cup \mathsf{A}_1 \cup \mathsf{A}_2$. The resulting diagram is the Dynkin diagram of the root system Φ_0. Keeping in mind the roots associated with the vertices, we determine the weight δ_0, using available formulae. We then easily compute $\delta_1 = \delta - \delta_0$. For our example, by removing the 4th vertex we get

$$\delta_0 = \alpha_1 + \frac{1}{2}\alpha_2 + \alpha_3 + \alpha_5 + \alpha_6.$$

We know that

$$\delta = 8\alpha_1 + 11\alpha_2 + 15\alpha_3 + 21\alpha_4 + 15\alpha_5 + 8\alpha_6,$$

and therefore

$$\delta_1 = 7\alpha_1 + \frac{21}{2}\alpha_2 + 14\alpha_3 + 21\alpha_4 + 14\alpha_5 + 7\alpha_6.$$

For root systems of type A – G, we use the (ordered) bases as in [4, pp. 64-65]. With respect to these bases, explicit formulae for the weight δ are obtainable (see, for example, [1]).

2.1 A_l $(l \geq 1)$. Here we have $\delta = \frac{1}{2}\sum_{j=1}^{l} j(l-j+1)\alpha_j$. In general, removing the j_0-th vertex from the Dynkin diagram will yield a diagram for $\mathsf{A}_{j_0-1} \cup \mathsf{A}_{l-j_0}$. Accordingly we get $\delta_0 = \frac{1}{2}\sum_{j=1}^{l} c_j(\delta_0)\alpha_j$ with

$$c_j(\delta_0) = \begin{cases} j(j_0-j), & \text{if } j < j_0; \\ 0, & \text{if } j = j_0; \\ (j-j_0)(l-j+1), & \text{if } j > j_0. \end{cases}$$

Subtracting δ_0 from δ gives $\delta_1 = \frac{1}{2}\sum_{j=1}^{l} c_j(\delta_1)\alpha_j$ with

$$c_j(\delta_1) = \begin{cases} j(l-j_0+1), & \text{if } j < j_0; \\ j_0(l-j+1), & \text{if } j \geq j_0. \end{cases}$$

2.2 B_l $(l \geq 2)$. Here $\delta = \frac{1}{2}\sum_{j=1}^{l} j(2l-j)\alpha_j$. In general, removing the j_0-th vertex from the Dynkin diagram will give a diagram for $\mathsf{A}_{j_0-1}\cup$ B_{l-j_0}. Thus we get $\delta_0 = \frac{1}{2}\sum_{j=1}^{l} c_j(\delta_0)\alpha_j$ with

$$c_j(\delta_0) = \begin{cases} j(j_0-j), & \text{if } j < j_0; \\ 0, & \text{if } j = j_0; \\ (j-j_0)(2l-j_0-j), & \text{if } j > j_0; \end{cases}$$

and hence we obtain $\delta_1 = \frac{1}{2}\sum_{j=1}^{l} c_j(\delta_1)\alpha_j$ with

$$c_j(\delta_1) = \begin{cases} j(2l-j_0), & \text{if } j < j_0; \\ j_0(2l-j_0), & \text{if } j \geq j_0. \end{cases}$$

2.3 C_l $(l \geq 3)$. Here we have $\delta = \frac{1}{2}\sum_{j=1}^{l-1} j(2l-j+1)\alpha_j + \frac{1}{4}l(l+1)\alpha_l$. In general, removing the j_0-th vertex from the Dynkin diagram will yield a diagram for $\mathsf{A}_{j_0-1}\cup$ C_{l-j_0}. So we get $\delta_0 = \frac{1}{2}\sum_{j=1}^{l} c_j(\delta_0)\alpha_j$ with

$$c_j(\delta_0) = \begin{cases} j(j_0-j), & \text{if } j < j_0; \\ 0, & \text{if } j = j_0; \\ (j-j_0)(2l-j_0-j+1), & \text{if } j_0 < j < l; \\ \frac{1}{2}(l-j_0)(l-j_0+1), & \text{if } j = l. \end{cases}$$

Therefore, $\delta_1 = \frac{1}{2}\sum_{j=1}^{l} c_j(\delta_1)\alpha_j$ with

$$c_j(\delta_1) = \begin{cases} j(2l - j_0 + 1), & \text{if } j < j_0; \\ j_0(2l - j_0 + 1), & \text{if } j_0 \le j < l; \\ \frac{1}{2}j_0(2l - j_0 + 1), & \text{if } j = l. \end{cases}$$

2.4 D_l $(l \ge 4)$. Here $\delta = \frac{1}{2}\sum_{j=1}^{l-2} j(2l - j - 1)\alpha_j + \frac{1}{4}l(l-1)(\alpha_{l-1} + \alpha_l)$. Removing the j_0-th vertex will give a diagram for $\mathsf{A}_{j_0-1} \cup \mathsf{D}_{l-j_0}$ when $j_0 \le l-2$, or A_{l-1} when $j_0 = l-1$ or l. Thus we have $\delta_0 = \frac{1}{2}\sum_{j=1}^{l} c_j(\delta_0)\alpha_j$ with

$$c_j(\delta_0) = \begin{cases} j(j_0 - j), & \text{if } j < j_0; \\ 0, & \text{if } j = j_0; \\ (j - j_0)(2l - j_0 - j - 1), & \text{if } j_0 < j < l-1; \\ \frac{1}{2}(l - j_0)(l - j_0 - 1), & \text{if } j \ge l-1, \end{cases}$$

when $j_0 \le l-2$, or

$$c_j(\delta_0) = \begin{cases} j(l-j), & \text{if } j < l-1; \\ 0, & \text{if } j = l-1; \\ l-1, & \text{if } j = l, \end{cases}$$

when $j_0 = l-1$, or

$$c_j(\delta_0) = \begin{cases} j(l-j), & \text{if } j < l; \\ 0, & \text{if } j = l. \end{cases}$$

when $j_0 = l$. Calculating $\delta - \delta_0$, we get $\delta_1 = \frac{1}{2}\sum_{j=1}^{l} c_j(\delta_1)\alpha_j$ with

$$c_j(\delta_1) = \begin{cases} j(2l - j_0 - 1), & \text{if } j < j_0; \\ j_0(2l - j_0 - 1), & \text{if } j_0 \le j < l-1; \\ \frac{1}{2}j_0(2l - j_0 - 1), & \text{if } j \ge l-1, \end{cases}$$

when $j_0 \le l-2$, or

$$c_j(\delta_1) = \begin{cases} j(l-1), & \text{if } j < l-1; \\ \frac{1}{2}l(l-1), & \text{if } j = l-1; \\ \frac{1}{2}(l-1)(l-2), & \text{if } j = l, \end{cases}$$

when $j_0 = l-1$, or

$$c_j(\delta_1) = \begin{cases} j(l-1), & \text{if } j < l-1; \\ \frac{1}{2}(l-1)(l-2), & \text{if } j = l-1; \\ \frac{1}{2}l(l-1), & \text{if } j = l, \end{cases}$$

when $j_0 = l$.

For convenience, we abbreviate $\sum_{j=1}^{l} k_j\alpha_j$ by $(k_1, \ldots, k_l)$.

2.5 E_6, E_7, and E_8. In type E_6, we have $\delta = (8, 11, 15, 21, 15, 8)$, as mentioned before. Below are the δ_0's and δ_1's obtained by removing the j_0-th vertex from the Dynkin diagram.

j_0	δ_0	δ_1
1	$(0, 5, 5, 9, 7, 4)$	$(8, 6, 10, 12, 8, 4)$
2	$(\frac{5}{2}, 0, 4, \frac{9}{2}, 4, \frac{5}{2})$	$(\frac{11}{2}, 11, 11, \frac{33}{2}, 11, \frac{11}{2})$
3	$(\frac{1}{2}, 2, 0, 3, 3, 2)$	$(\frac{15}{2}, 9, 15, 18, 12, 6)$
4	$(1, \frac{1}{2}, 1, 0, 1, 1)$	$(7, \frac{21}{2}, 14, 21, 14, 7)$
5	$(2, 2, 3, 3, 0, \frac{1}{2})$	$(6, 9, 12, 18, 15, \frac{15}{2})$
6	$(4, 5, 7, 9, 5, 0)$	$(4, 6, 8, 12, 10, 8)$

In type E_7, the special element is $\delta = (17, \frac{49}{2}, 33, 48, \frac{75}{2}, 26, \frac{27}{2})$. The δ_0's and δ_1's obtained by removing the j_0-th vertex from the Dynkin diagram are listed below.

j_0	δ_0	δ_1
1	$(0, \frac{15}{2}, \frac{15}{2}, 14, 12, 9, 5)$	$(17, 17, \frac{51}{2}, 34, \frac{51}{2}, 17, \frac{17}{2})$
2	$(3, 0, 5, 6, 6, 5, 3)$	$(14, \frac{49}{2}, 28, 42, \frac{63}{2}, 21, \frac{21}{2})$
3	$(\frac{1}{2}, \frac{5}{2}, 0, 4, \frac{9}{2}, 4, \frac{5}{2})$	$(\frac{33}{2}, 22, 33, 44, 33, 22, 11)$
4	$(1, \frac{1}{2}, 1, 0, \frac{3}{2}, 2, \frac{3}{2})$	$(16, 24, 32, 48, 36, 24, 12)$
5	$(2, 2, 3, 3, 0, 1, 1)$	$(15, \frac{45}{2}, 30, 45, \frac{75}{2}, 25, \frac{25}{2})$
6	$(4, 5, 7, 9, 5, 0, \frac{1}{2})$	$(13, \frac{39}{2}, 26, 39, \frac{65}{2}, 26, 13)$
7	$(8, 11, 15, 21, 15, 8, 0)$	$(9, \frac{27}{2}, 18, 27, \frac{45}{2}, 18, \frac{27}{2})$

In type E_8, we have $\delta = (46, 68, 91, 135, 110, 84, 57, 29)$. By removing the j_0-th vertex from the Dynkin diagram we obtain the following list.

j_0	δ_0	δ_1
1	$(0, \frac{21}{2}, \frac{21}{2}, 20, 18, 15, 11, 6)$	$(46, \frac{115}{2}, \frac{161}{2}, 115, 92, 69, 46, 23)$
2	$(\frac{7}{2}, 0, 6, \frac{15}{2}, 8, \frac{15}{2}, 6, \frac{7}{2})$	$(\frac{85}{2}, 68, 85, \frac{255}{2}, 102, \frac{153}{2}, 51, \frac{51}{2})$
3	$(\frac{1}{2}, 3, 0, 5, 6, 6, 5, 3)$	$(\frac{91}{2}, 65, 91, 130, 104, 78, 52, 26)$

4	$(1, \frac{1}{2}, 1, 0, 2, 3, 3, 2)$	$(45, \frac{135}{2}, 90, 135, 108, 81, 54, 27)$
5	$(2, 2, 3, 3, 0, \frac{3}{2}, 2, \frac{3}{2})$	$(44, 66, 88, 132, 110, \frac{165}{2}, 55, \frac{55}{2})$
6	$(4, 5, 7, 9, 5, 0, 1, 1)$	$(42, 63, 84, 126, 105, 84, 56, 28)$
7	$(8, 11, 15, 21, 15, 8, 0, \frac{1}{2})$	$(38, 57, 76, 114, 95, 76, 57, \frac{57}{2})$
8	$(17, \frac{49}{2}, 33, 48, \frac{75}{2}, 26, \frac{27}{2}, 0)$	$(29, \frac{87}{2}, 58, 87, \frac{145}{2}, 58, \frac{87}{2}, 29)$

2.6 F_4. Here we have $\delta = (8, 15, 21, 11)$. Removing the j_0-th vertex from the Dynkin diagram will give the following result.

j_0	δ_0	δ_1
1	$(0, 3, 5, 3)$	$(8, 12, 16, 8)$
2	$(\frac{1}{2}, 0, 1, 1)$	$(\frac{15}{2}, 15, 20, 10)$
3	$(1, 1, 0, \frac{1}{2})$	$(7, 14, 21, \frac{21}{2})$
4	$(\frac{5}{2}, 4, \frac{9}{2}, 0)$	$(\frac{11}{2}, 11, \frac{33}{2}, 11)$

2.7 G_2. The special element is $\delta = (5, 3)$. Removing either vertex from the Dynkin diagram will yield a diagram for A_1. When the 1st vertex is removed, we get $\delta_0 = (0, \frac{1}{2})$ and $\delta_1 = (5, \frac{5}{2})$. When the 2nd vertex is removed, we get $\delta_0 = (\frac{1}{2}, 0)$ and $\delta_1 = (\frac{9}{2}, 3)$.

3. Computing λ_{j_0}

Having found δ_1, we only need to divide it by $\langle \delta_1, \alpha_{j_0} \rangle$ to get λ_{j_0}. Knowing the Cartan integers $\langle \alpha_i, \alpha_j \rangle$ makes it easy to calculate $\langle \delta_1, \alpha_{j_0} \rangle$.

Below are the values of $\langle \delta_1, \alpha_{j_0} \rangle$ for root systems of type $\mathsf{A} - \mathsf{G}$.

For type $\mathsf{A} - \mathsf{D}$, we have the following result.

$$\mathsf{A}_l\ (l \geq 1)\ : \quad \langle \delta_1, \alpha_{j_0} \rangle = \tfrac{1}{2}(l+1), \quad j_0 = 1, \ldots, l.$$

$$\mathsf{B}_l\ (l \geq 2)\ : \quad \langle \delta_1, \alpha_{j_0} \rangle = \begin{cases} \frac{1}{2}(2l - j_0), & j_0 = 1, \ldots, l-1; \\ l, & j_0 = l. \end{cases}$$

$$\mathsf{C}_l\ (l \geq 3)\ : \quad \langle \delta_1, \alpha_{j_0} \rangle = \tfrac{1}{2}(2l - j_0 + 1), \quad j_0 = 1, \ldots, l.$$

$$\mathsf{D}_l\ (l \geq 4)\ : \quad \langle \delta_1, \alpha_{j_0} \rangle = \begin{cases} \frac{1}{2}(2l - j_0 - 1), & j_0 = 1, \ldots, l-2; \\ l-1, & j_0 = l-1, l. \end{cases}$$

For type E – G, we have the following result. (The values of $\langle \delta_1, \alpha_{j_0} \rangle$ are listed as j_0 increases from 1 to l.)

E_6 : $6, \frac{11}{2}, \frac{9}{2}, \frac{7}{2}, \frac{9}{2}, 6.$

E_7 : $\frac{17}{2}, 7, \frac{11}{2}, 4, 5, \frac{13}{2}, 9.$

E_8 : $\frac{23}{2}, \frac{17}{2}, \frac{13}{2}, \frac{9}{2}, \frac{11}{2}, 7, \frac{19}{2}, \frac{29}{2}.$

F_4 : $4, \frac{5}{2}, \frac{7}{2}, \frac{11}{2}.$

G_2 : $\frac{5}{2}, \frac{3}{2}.$

The above results clearly verify the fundamental dominant weights of root systems of type A – G listed in [4, p. 69].

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