Fractional integral operators and Olsen inequalities on non-homogeneous spaces

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Abstract. We prove the boundedness of the fractional integral operator I_α on generalized Morrey spaces of non-homogeneous type. In addition, we also present Olsen-type inequalities for a multiplication operator involving I_α. Our proof uses a result of García-Cuerva and Martell [3].

Keywords: Fractional integral operators, Olsen inequality, non-homogeneous spaces, generalized Morrey spaces

1. Introduction

Let $B(a, r)$ be a ball centered at $a \in \mathbb{R}^d$ with radius $r > 0$ and, for $k > 0$, $B(a, kr)$ denote a ball concentric to $B(a, r)$ with radius kr. A Borel measure μ on $\mathbb{R}^d$ is called *a doubling measure* if it satisfies the so-called *doubling condition*, that is, there exists a constant $C > 0$ such that

$$\mu(B(a, 2r)) \leq C\mu(B(a, r))$$

for every ball $B(a, r)$. The doubling condition is a key feature for a *homogeneous* (metric) measure space. Many classical theories in Fourier analysis have been generalized to the homogeneous setting without too much difficulties. In the last decade, however, some researchers found that many results are still true without the assumption of the doubling condition on μ (see, for instance, [2, 8, 10, 13]). This fact has encouraged other researchers to study various theories in the non-homogeneous setting.

By a *non-homogeneous* space we mean a (metric) measure space — here we will consider only $\mathbb{R}^d$ — equipped with a Borel measure μ satisfying the *growth* *condition* of order n with $0 < n \le d$ (instead of the doubling condition), that is, there exists a constant $C > 0$ such that

$$\mu(B(a,r)) \le C\, r^n$$

for every ball $B(a,r)$. Unless otherwise stated, throughout this paper we shall always work in the non-homogeneous setting.

Our main object of study in this paper is the fractional integral operator I_α, defined for $0 < \alpha < n \le d$ by the formula

$$I_\alpha f(x) := \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}}\, d\mu(y).$$

Notice that if $n = d$ and μ is the usual Lebesgue measure on $\mathbb{R}^d$, then I_α is the classical fractional integral operator introduced by Hardy and Littlewood [5, 6] and Sobolev [11].

The boundedness of I_α on Lebesgue spaces in the non-homogeneous setting has been studied by García-Cuerva, Gatto, and Martell in [2, 3]. One of their main results is the following theorem. (As in [10], we write $\|f : X\|$ to denote the norm of f in the space X.)

Theorem 1.1 [3] *If* $1 < p < \frac{n}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$*, then there exists a constant* $C > 0$ *such that*

$$\|I_\alpha f : L^q(\mu)\| \le C\, \|f : L^p(\mu)\|,$$

that is, I_α *is bounded from* $L^p(\mu)$ *to* $L^q(\mu)$*.*

Our goal here is to prove the boundedness of the fractional integral operator I_α on generalized Morrey spaces in the non-homogeneous setting. In addition, we will also derive Olsen-type inequalities for a multiplication operator involving I_α. Our results are analogous to Nakai's [7] and Gunawan and Eridani's [4] in the homogeneous case.

2. The boundedness of I_α on generalized Morrey spaces

For $1 \le p < \infty$ and a function $\phi : (0,\infty) \to (0,\infty)$, we define the generalized Morrey space $\mathcal{M}^{p,\phi}(\mu) = \mathcal{M}^{p,\phi}(\mathbb{R}^d, \mu)$ to be the space of all functions $f \in L^p_{\rm loc}(\mu)$ for which

$$\|f : \mathcal{M}^{p,\phi}(\mu)\| := \sup_{B=B(a,r)} \frac{1}{\phi(r)} \left(\frac{1}{r^n} \int_B |f(x)|^p d\mu(x) \right)^{1/p} < \infty.$$

(See [1] for definition of the analogous spaces in the homogeneous case.) Here we assume that the function ϕ always satisfies the following two conditions:

(2.1) There exists $C_1 > 1$ suct that $\frac{1}{C_1} \le \frac{\phi(t)}{\phi(r)} \le C_2$ whenever $1 \le \frac{t}{r} \le 2$.

(2.2) There exists $C_2 > 0$ such that $\int_r^\infty t^{\alpha-1}\phi(t)\,dt \le C_2\, r^\alpha \phi(r)$ for every $r > 0$.

Note that condition (2.1) is referred to as the *doubling condition* (with the *doubling constant* C_1). As a consequence of this condition, there exists a constant $C > 1$ such that

$$\frac{1}{C}\phi(r) \le \int_r^{2r} \phi(t)\,dt \le C\,\phi(r)$$

for every $r > 0$. Here and throughout the rest of this paper, C denotes a positive constant, which may differ from line to line.

Theorem 2.1 *Suppose that* $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, *and* $\psi : (0,\infty) \to (0,\infty)$ *satisfies* $r^\alpha \phi(r) \le C\psi(r)$ *(with* C *is independent of* r*). Then, we have*

$$\|I_\alpha f : \mathcal{M}^{q,\psi}(\mu)\| \le C\,\|f : \mathcal{M}^{p,\phi}(\mu)\|,$$

whenever $f \in \mathcal{M}^{p,\phi}(\mu)$.

Proof. For $a \in \mathbb{R}^d$ and $r > 0$, let $B = B(a,r)$ and $\tilde{B} = B(a,2r)$. We decompose $f \in \mathcal{M}^{p,\phi}(\mu)$ as $f = f_1 + f_2 := f\chi_{\tilde{B}} + f\chi_{\tilde{B}^c}$. Then observe that $f_1 \in L^p(\mu)$, with the norm

$$\|f_1 : L^p(\mu)\| = \left(\int_{\tilde{B}} |f(x)|^p d\mu(x)\right)^{1/p} \le C\,r^{n/p}\phi(r)\,\|f : \mathcal{M}^{p,\phi}(\mu)\|.$$

The $L^p(\mu)$-$L^q(\mu)$ boundedness of I_α then gives us

$$\begin{aligned}
\left(\frac{1}{r^n}\int_B |I_\alpha f_1(x)|^q d\mu(x)\right)^{1/q} &\le \frac{1}{r^{n/q}}\,\|I_\alpha f_1 : L^q(\mu)\| \\
&\le \frac{C}{r^{n/q}}\,\|f_1 : L^p(\mu)\| \\
&\le C\,r^{n/p-n/q}\phi(r)\,\|f : \mathcal{M}^{p,\phi}(\mu)\| \\
&= C\,r^\alpha\phi(r)\,\|f : \mathcal{M}^{p,\phi}(\mu)\| \\
&\le C\,\psi(r)\,\|f : \mathcal{M}^{p,\phi}(\mu)\|.
\end{aligned}$$

Next, for each $x \in B$, we estimate $I_\alpha f_2(x)$ as follows:

$$
\begin{aligned}
|I_\alpha f_2(x)| &\le \int_{\tilde{B}^c} \frac{|f(y)|}{|x-y|^{n-\alpha}} d\mu(y) \\
&\le \int_{|x-y|\ge r} \frac{|f(y)|}{|x-y|^{n-\alpha}} d\mu(y) \\
&= \sum_{k=0}^{\infty} \int_{2^k r \le |x-y| \le 2^{k+1} r} \frac{|f(y)|}{|x-y|^{n-\alpha}} d\mu(y) \\
&\le \sum_{k=0}^{\infty} \frac{1}{(2^k r)^{n-\alpha}} \int_{B(x,2^{k+1}r)} |f(y)|\, d\mu(y) \\
&\le C \sum_{k=0}^{\infty} (2^k r)^\alpha \left[\frac{\int_{B(x,2^{k+1}r)} |f(y)|^p d\mu(y)}{(2^{k+1}r)^n} \right]^{\frac{1}{p}} \left[\frac{\mu(B(x,2^{k+1}r))}{(2^{k+1}r)^n} \right]^{1-\frac{1}{p}} \\
&\le C \, \|f : \mathcal{M}^{p,\phi}(\mu)\| \sum_{k=0}^{\infty} (2^k r)^\alpha \phi(2^k r).
\end{aligned}
$$

Since $\phi(t)$ and t^α satisfy the doubling condition, we have

$$
\left(2^k r\right)^\alpha \phi\left(2^k r\right) \le C \int_{2^k r}^{2^{k+1} r} t^{\alpha-1} \phi(t) dt,
$$

for $k = 0, 1, 2, \ldots$. Hence we obtain the pointwise estimate

$$
\begin{aligned}
|I_\alpha f_2(x)| &\le C \, \|f : \mathcal{M}^{p,\phi}(\mu)\| \int_r^\infty t^{\alpha-1}\phi(t) dt \\
&\le C \, r^\alpha \phi(r) \, \|f : \mathcal{M}^{p,\phi}(\mu)\| \\
&\le C \, \psi(r) \, \|f : \mathcal{M}^{p,\phi}(\mu)\|.
\end{aligned}
$$

Taking the q-th power and integrating over B, we get

$$
\begin{aligned}
\left(\frac{1}{r^n} \int_B |I_\alpha f_2(x)|^q d\mu(x) \right)^{1/q} &\le C \, r^{-n/q} \psi(r) \, \|f : \mathcal{M}^{p,\phi}(\mu)\| \left(\int_B d\mu(x) \right)^{1/q} \\
&= C \, r^{-n/q} \psi(r) \, \|f : \mathcal{M}^{p,\phi}(\mu)\| \, [\mu(B(a,r))]^{1/q} \\
&\le C \, \psi(r) \, \|f : \mathcal{M}^{p,\phi}(\mu)\|.
\end{aligned}
$$

The desired inequality follows from this and the previous estimate. □

3. Olsen-type Inequalities

As in the case of homogeneous spaces (see [4]), we present here Olsen-type inequalities for a multiplication operator involving the fractional integral operator I_α in the non-homogeneous setting. Such an inequality is useful when one studies a perturbed Schrödinger equation (see [9]). The key is that $I_\alpha = (-\Delta)^{-\alpha/2}$, where $-\Delta$ denotes the Laplacian in $\mathbb{R}^d$ (see [12]).

Before we present our main result in this section, let us first state an Olsen-type inequality that follows directly from Theorem 1.1 and Hölder's inequality.

Theorem 3.1 *For $1 < p < \frac{n}{\alpha}$, the inequality*

$$\|WI_\alpha f : L^p(\mu)\| \le C\,\|W : L^{n/\alpha}(\mu)\|\,\|f : L^p(\mu)\|,$$

holds provided that $W \in L^{n/\alpha}(\mu)$.

Proof. Let q satisfy $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. By Hölder inequality, we have

$$\int_{\mathbb{R}^d} |WI_\alpha f(x)|^p\,d\mu(x) \le \left(\int_{\mathbb{R}^d} |W(x)|^{\frac{pq}{q-p}}\,d\mu(x)\right)^{\frac{q-p}{q}} \left(\int_{\mathbb{R}^d} |I_\alpha f(x)|^q\,d\mu(x)\right)^{\frac{p}{q}}.$$

Taking the p-th roots, we get

$$\left(\int_{\mathbb{R}^d} |WI_\alpha f(x)|^p\,d\mu(y)\right)^{\frac{1}{p}} \le \left(\int_{\mathbb{R}^d} |W(x)|^{\frac{n}{\alpha}}\,d\mu(x)\right)^{\frac{\alpha}{n}} \left(\int_{\mathbb{R}^d} |I_\alpha f(x)|^q\,d\mu(x)\right)^{\frac{1}{q}}.$$

By the boundedness of I_α from $L^p(\mu)$ to $L^q(\mu)$ (Theorem 1.1), we obtain

$$\|WI_\alpha f : L^p(\mu)\| \le C\,\|W : L^{n/\alpha}(\mu)\|\,\|f : L^p(\mu)\|,$$

as desired. □

Our theorem below generalizes the previous Olsen-type inequality. The proof uses Theorem 2.1 and Hölder's inequality.

Theorem 3.2 *For $1 < p < \frac{n}{\alpha}$, the inequality*

$$\|WI_\alpha f : \mathcal{M}^{p,\phi}(\mu)\| \le C\,\|W : L^{n/\alpha}(\mu)\|\,\|f : \mathcal{M}^{p,\phi}(\mu)\|,$$

holds provided that $W \in L^{n/\alpha}(\mu)$.

Proof. Let $\psi(r) = r^\alpha\phi(r)$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, so that Theorem 2.1 applies. Then, by Hölder's inequality, we have

$$\left(\frac{1}{r^n}\int_B |WI_\alpha f(x)|^p\,d\mu(x)\right)^{\frac{1}{p}} \le \left(\frac{1}{r^n}\int_B |W(x)|^{\frac{n}{\alpha}} d\mu(x)\right)^{\frac{\alpha}{n}} \times \left(\frac{1}{r^n}\int_B |I_\alpha f(x)|^q d\mu(x)\right)^{\frac{1}{q}},$$

for every ball $B = B(a,r)$. Multiplying both sides by $\frac{r^\alpha}{\psi(r)}\ \left(= \frac{1}{\phi(r)}\right)$, we get

$$\begin{aligned}
&\frac{1}{\phi(r)}\left(\frac{1}{r^n}\int_B |WI_\alpha f(x)|^p\,d\mu(x)\right)^{\frac{1}{p}} \\
&\le \left(\int_B |W(x)|^{\frac{n}{\alpha}} d\mu(x)\right)^{\frac{\alpha}{n}} \frac{1}{\psi(r)}\left(\frac{1}{r^n}\int_B |I_\alpha f(x)|^q d\mu(x)\right)^{\frac{1}{q}} \\
&\le \|W : L^{n/\alpha}(\mu)\|\,\|I_\alpha f : \mathcal{M}^{q,\psi}(\mu)\| \\
&\le C\,\|W : L^{n/\alpha}(\mu)\|\,\|f : \mathcal{M}^{p,\phi}(\mu)\|,
\end{aligned}$$

and so the desired inequality follows. □

`Remark`. The proof may also use Theorem 1.1 directly just like the proof of Theorem 2.1.

References

[1] ERIDANI, H. GUNAWAN, AND E. NAKAI, "On the generalized fractional integral operators", *Sci. Math. Jpn.* **60** (2004), 539–550.

[2] J. GARCÍA-CUERVA AND A.E. GATTO, "Boundedness properties of fractional integral operators associated to non-doubling measures", *Studia Math.* **162** (2004), no. 3, 245–261.

[3] J. GARCÍA-CUERVA AND J.M. MARTELL, "Two weight norm inequalities for maximal operators and fractional integrals on non-homogeneous spaces", *Indiana Univ. Math. J.* **50** (2001), no. 3, 1241–1280.

[4] H. GUNAWAN AND ERIDANI, "Fractional integrals and generalized Olsen inequalities", to appear in *Kyungpook Math. J.*

[5] G.H. HARDY AND J.E. LITTLEWOOD, "Some properties of fractional integrals. I", *Math. Zeit.* **27** (1927), 565–606.

[6] G.H. HARDY AND J.E. LITTLEWOOD, "Some properties of fractional integrals. II", *Math. Zeit.* **34** (1932), 403–439.

[7] E. NAKAI, “Hardy-Littlewood maximal operator, singular integral operators, and the Riesz potentials on generalized Morrey spaces”, *Math. Nachr.* **166** (1994), 95–103.

[8] F. NAZAROV, S. TREIL, AND A. VOLBERG, ”Weak type estimates and Cotlar inequalities for Calderon-Zygmund operators on non-homogeneous spaces”, *Internat. Math. Res. Notices* **9** (1998), 463–487.

[9] P.A. OLSEN, “Fractional integration, Morrey spaces and a Schrödinger equation”, *Comm. Partial Differential Equations* **20** (1995), 2005–2055.

[10] Y. SAWANO, T. SOBUKAWA, AND H. TANAKA, “Limiting case of the boundedness of fractional integral operators on non-homogeneous space”, *J. Inequal. Appl.* Art. ID 92470 (2006), 16p.

[11] S.L. SOBOLEV, “On a theorem in functional analysis” (Russian), *Mat. Sob.* **46** (1938), 471–497 [English translation in *Amer. Math. Soc. Transl. ser. 2* **34** (1963), 39–68].

[12] E.M. STEIN, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, 1970.

[13] X. TOLSA, “BMO, H^1, and Calderón-Zygmund operators for non doubling measures”, *Math. Ann.* **319** (2001), 89–149.