An Interpolation Method That Minimizes an Energy Integral of Fractional Order

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Abstract. An interpolation method that minimizes an energy integral will be discussed. To be precise, given $N+1$ points (x_0, c_0), $(x_1, c_1), \ldots,$ (x_N, c_N) with $0 = x_0 < x_1 < \cdots < x_N = 1$ and $c_0 = c_N = 0$, we shall be interested in finding a sufficiently smooth function u on $[0, 1]$ that passes through these $N + 1$ points and minimizes the energy integral $E_\alpha(u) := \int_0^1 |u^{(\alpha)}(x)|^2 dx$, where $u^{(\alpha)}$ denotes the fractional derivative of u of order α. As suggested in [1], a Fourier series approach as well as functional analysis arguments can be used to show that such a function exists and is unique. An iterative procedure to obtain the function will be presented and some examples will be given here.

1 Introduction

Many interpolation methods have been developed for many decades. For recent results in interpolation, see for instance [3, 7, 8, 9, 15] and the references therein. In [1], Alghofari discussed the following interpolation problem: Given $N + 1$ points (x_0, c_0), $(x_1, c_1), \ldots,$ (x_N, c_N) with $0 = x_0 < x_1 < \cdots < x_N = 1$ and $c_0 = c_N = 0$, find a continuously differentiable function u on $[0, 1]$ that passes through these $N + 1$ points and minimizes the energy integral

$$E_2(u) := \int_0^1 |u''(x)|^2 dx.$$

To solve the problem, Alghofari used a Fourier series approach as well as functional analysis arguments. In particular, he showed that the problem has a unique solution and gave a hint to approximate the solution.

In this note, we shall generalize Alghofari's results by replacing the energy integral $E_2(u)$ with

$$E_\alpha(u) := \int_0^1 |u^{(\alpha)}(x)|^2 dx,$$

where $u^{(\alpha)}$ denotes the fractional derivative of u of order $\alpha \geq 0$. We show that for $\alpha > \frac{1}{2}$, the problem has a unique solution u which is continuous on $[0, 1]$.

D. Kapur (Ed.): ASCM 2007, LNAI 5081, pp. 151–162, 2008.

In addition, an iterative procedure to obtain the solution will be presented and some examples will be given.

Note that for $\alpha = 2$, $E_2(u)$ represents the curvature (or the strain energy of bending) of u and the solution to the problem is a cubic spline (see [4, 5, 12]). For $\alpha = 1$, $E_1(u)$ represents the tension (or the potential energy of axial load) of u and the solution is a piecewise linear function. From this point of view, the interpolation that we discuss here can be considered as a generalization of the polynomial spline interpolation. Our results may be related to those in [14].

2 The Problem and Its Solution

We begin with the classical Fourier series discussion. Let $u : [0,1] \to \mathbb{R}$ be a continuous function with $u(0) = u(1) = 0$. If, for instance, u is piecewise smooth, then u may be expressed as a Fourier sine series

$$u(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x, \quad x \in [0,1],$$

where

$$a_n = 2\int_0^1 u(x) \sin n\pi x \, dx, \quad n = 1, 2, 3, \ldots.$$

Parseval's identity states that $2\int_0^1 |u(x)|^2 dx = \sum_{n=1}^{\infty} a_n^2$. Further, if u is of class $C^{(k-1)}$ and $u^{(k-1)}$ is piecewise smooth (so that $u^{(k)}$ exists except at finitely many points and is piecewise continuous), then the Fourier sine coefficients a_n's satisfy the condition

$$\sum_{n=1}^{\infty} n^{2k} a_n^2 < \infty. \tag{1}$$

Conversely, if the coefficients a_n's satisfy the condition (1), then the functions $u, \ldots, u^{(k-1)}$ are absolutely continuous and $u^{(k)}$ is square integrable with

$$\|u^{(k)}\|_2^2 := 2\int_0^1 |u^{(k)}(x)|^2 dx = \pi^{2k} \sum_{n=1}^{\infty} n^{2k} a_n^2$$

(see, for instance, [6, 13]). All these tell us that we may identify $u^{(k)}$ with the square summable sequence $(n^k a_n)$. Here $n^k a_n$'s are the Fourier coefficients of $u^{(k)}$, from which we can recover $u^{(k)}$ almost everywhere through the formula

$$u^{(k)}(x) = \pi^k \sum_{n=1}^{\infty} n^k a_n \sin(n\pi x + k\tfrac{\pi}{2}).$$

Note that $\pi^k n^k \sin(n\pi x + k\frac{\pi}{2})$ is nothing but the k-th derivative of $\sin(n\pi x)$.

Inspired by the above facts, we may define the fractional derivative of u of order $\alpha \geq 0$, denoted by $u^{(\alpha)}$, almost everywhere by the following formula

$$u^{(\alpha)}(x) = \pi^\alpha \sum_{n=1}^{\infty} n^\alpha a_n \sin(n\pi x + \alpha\tfrac{\pi}{2}),$$

provided that $\sum_{n=1}^{\infty} n^{2\alpha} a_n^2 < \infty$. Notice that $\pi^\alpha n^\alpha \sin(n\pi x + \alpha\frac{\pi}{2})$ is the fractional derivative of $\sin n\pi x$ of order α (see [11]). Here we may check that the family $\{\sin(n\pi x + \alpha\frac{\pi}{2})\} : n \in \mathbb{N}\}$ forms an orthogonal system and that

$$2\int_0^1 |u^{(\alpha)}(x)|^2 dx = \pi^{2\alpha} \sum_{n=1}^{\infty} n^{2\alpha} a_n^2.$$

Accordingly, $u^{(\alpha)}$ is a square integrable function on $[0,1]$, which may be identified with the square summable sequence $(n^\alpha a_n)$.

Our problem is the following. Given $N+1$ points (x_0, c_0), $(x_1, c_1), \ldots, (x_N, c_N)$ with $0 = x_0 < x_1 < \cdots < x_N = 1$ and $c_0 = c_N = 0$, we wish to find an interpolant u which is continuous on $[0,1]$ and minimizes the energy integral

$$E_\alpha(u) := \int_0^1 |u^{(\alpha)}(x)|^2 dx. \tag{2}$$

To solve this problem, we consider the space $W = W_\alpha$ consisting of all functions u on $[0,1]$ of the form $u(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x$ with $\sum_{n=1}^{\infty} n^{2\alpha} a_n^2 < \infty$. On W, we define the inner product

$$\langle u, v\rangle := \sum_{n=1}^{\infty} n^{2\alpha} a_n b_n,$$

where a_n's and b_n's are the coefficients of u and v, respectively. Here minimizing the integral $\int_0^1 |u^{(\alpha)}(x)|^2 dx$ in W is equivalent to minimizing the sum $\sum_{n=1}^{\infty} n^{2\alpha} a_n^2 =: \|u\|^2$. With respect to the above inner product, W is complete, that is, $(W, \langle\cdot,\cdot\rangle)$ is a Hilbert space. Indeed, given a Cauchy sequence in W, one may show that it is convergent to an element in W.

Hereafter, we shall assume that $\alpha > \frac{1}{2}$, unless otherwise stated. As we shall see, this is not only a sufficient condition but also necessary to have a continuous solution. Let us first prove the following lemma.

Lemma 2.1. *Suppose that $\|u_m - u\| \to 0$ as $m \to \infty$. Then, (u_m) converges uniformly to u on $[0,1]$. More generally, $\big(u_m^{(\beta)}\big)$ converges uniformly to $u^{(\beta)}$ on $[0,1]$ for $0 \le \beta < \alpha - \frac{1}{2}$.*

Proof. For $m \in \mathbb{N}$, let $a_{m,n}$'s and a_n's be the coefficients of u_m and u. Let $0 \le \beta < \alpha - \frac{1}{2}$. Then, for each $x \in [0,1]$, we have

$$\begin{aligned} |u_m^{(\beta)}(x) - u^{(\beta)}(x)| &= \Big|\pi^\beta \sum_{n=1}^{\infty} n^\beta (a_{m,n} - a_n)\sin n\pi x\Big| \\ &\le \pi^\beta \Big[\sum_{n=1}^{\infty} n^{2\alpha}(a_{m,n} - a_n)^2\Big]^{1/2} \Big[\sum_{n=1}^{\infty} \frac{\sin^2 n\pi x}{n^{2(\alpha-\beta)}}\Big]^{1/2} \\ &\le C\|u_m - u\|, \end{aligned}$$

where C is independent of x. Hence $\big(u_m^{(\beta)}\big)$ converges uniformly to $u^{(\beta)}$. □

Corollary 2.2. *If $u \in W$, then $u^{(\beta)}$ is continuous for $0 \le \beta < \alpha - \frac{1}{2}$. In particular, every function in W is continuous.*

Proof. For each β with $0 \le \beta < \alpha - \frac{1}{2}$, $u^{(\beta)}$ is a limit, and hence a uniform limit, of its partial sums. Now since the partial sums are continuous, $u^{(\beta)}$ too must be continuous. □

Now consider the subspace V of W consisting of all functions u that vanish at $x_i,\ i = 1, ..., N-1$; that is,

$$V := \{u \in W \,:\, u(x_i) = 0,\ i = 1, \ldots, N-1\}.$$

Meanwhile, let U be the subset of W given by

$$U := \{u \in W \,:\, u(x_i) = c_i,\ i = 1, \ldots, N-1\}.$$

Then, as for the case $\alpha = 2$ discussed in [1], we have:

Lemma 2.3. *V is closed, while U is nonempty, closed and convex.*

Proof. Let u be the limit of a convergent sequence (u_m) in V. Then, for each $i = 1, \ldots, N-1$, it follows from Lemma 2.1 that $u(x_i) = 0$ because $u_m(x_i) = 0$ for every $m \in \mathbb{N}$. Therefore V is closed. Similarly, U is closed. Next, it is nonempty because one can easily find a function $u_0(x) = \sum_{j=1}^{N-1} b_j \sin j\pi x$ satisfying the following system of equations

$$\sum_{j=1}^{N-1} b_j \sin j\pi x_i = c_i, \quad i = 1, \ldots, N-1.$$

Finally, if u_1 and u_2 in U, then $\alpha u_1 + \beta u_2 \in U$ provided that $\alpha + \beta = 1$. This tells us particularly that U is convex. □

The following theorem is a generalization of Alghofari's result [1].

Theorem 2.4. *The minimization problem (2) has a unique solution in W, and the solution is given by*

$$u = u_0 - \mathrm{proj}_V(u_0),$$

where u_0 is an arbitrary element of U and $\mathrm{proj}_V(u_0)$ denotes the orthogonal projection of u_0 on V.

Proof. Let u_0 be an element in U. Then, for any $v \in V$, $u_0 - v$ is also in U. Since U is a convex subset of W, there must exist a unique element $v_0 \in V$ such that $\|u_0 - v_0\|$ is of smallest norm [2]. Thus $u := u_0 - v_0$ is the unique solution in W for our minimization problem (2). By the theory of best approximation in Hilbert spaces, the element $v_0 \in V$ for which $\|u_0 - v_0\|$ is minimized is the orthogonal projection of u_0 on V, that is, $v_0 = \mathrm{proj}_V(u_0)$. □

As we have indicated before, to find an element in U is easy. What is rather difficult is to find an orthonormal basis for V. In the next section, we develop a procedure to find an initial element in U and an orthonormal basis for V, and to obtain the minimum solution iteratively through finite computations.

3 The Procedure to Obtain the Solution

Given $N+1$ points (x_0, c_0), $(x_1, c_1), \ldots$, (x_N, c_N) with $0 = x_0 < x_1 < \cdots < x_N = 1$, we can obtain (or approximate) the solution to (2) in W through the following steps.

Step 1. To obtain an initial element in U, we solve the system of equations

$$\sum_{j=1}^{N-1} b_j \sin j\pi x_i = c_i, \quad i = 1, \ldots, N-1,$$

for the coefficients b_j's. The $(N-1) \times (N-1)$ matrix $[\sin j\pi x_i]_{i,j}$ is always nonsingular (see [2]), and so the above system has a solution. Having found b_j's, we put $u_0(x) = \sum_{j=1}^{N-1} b_j \sin j\pi x$.

Step 2. To obtain a basis for V, we consider the system of equations

$$\sum_{n=1}^{\infty} a_n \sin n\pi x_i = 0, \quad i = 1, \ldots, N-1,$$

each of which contains infinitely many unknowns a_n's. However, we can tackle this system by writing it as

$$\sum_{j=1}^{N-1} a_j \sin j\pi x_i = -\sum_{n=N}^{\infty} a_n \sin n\pi x_i, \quad i = 1, \ldots, N-1.$$

From this we can express $a_1, \ldots, a_{N-1}$ in terms of $a_n,\ n \geq N$.

Now if $(a_1, \ldots, a_{N-1}, a_N, a_{N+1}, a_{N+2} \ldots)$ stands for $\sum_{n=1}^{\infty} a_n \sin n\pi x$, then by expressing $a_1, \ldots, a_{N-1}$ in terms of a_n with $n \geq N$, every element in V can be expressed as

$$\begin{aligned} a_N(*, \ldots, *, 1, 0, 0, 0, \ldots) + a_{N+1}(*, \ldots, *, 0, 1, 0, 0, \ldots) + \\ + a_{N+2}(*, \ldots, *, 0, 0, 1, 0, \ldots) + a_{N+2}(*, \ldots, *, 0, 0, 0, 1, \ldots) + \cdots, \end{aligned}$$

where the first $N-1$ terms marked by asterisks come from $a_1, \ldots, a_{N-1}$. The following sequence form a basis for V:

$$\begin{aligned} &v_1 := (*, \ldots, *, 1, 0, 0, 0, \ldots),\ v_2 := (*, \ldots, *, 0, 1, 0, 0, \ldots), \\ &v_3 := (*, \ldots, *, 0, 0, 1, 0, \ldots),\ v_4 := (*, \ldots, *, 0, 0, 0, 1, \ldots), \ldots. \end{aligned}$$

Step 3. The minimum solution u is given by $u = u_0 - \text{proj}_V(u_0)$. To find (or approximate) it, we compute the orthogonal projection of u_0 on the subspace $V_m := \text{span}\{v_1, \ldots, v_m\}$ for $m = 1, 2, 3, \ldots$ iteratively. (But since v_n's may not be orthogonal, we might need to orthogonalize them first.) Now if $u_m := u_0 - \text{proj}_{V_m}(u_0)$, then the sequence (u_m) approximates the minimum solution u. Indeed, $\|u_m\|$ gets smaller and $\|u_m - u\| \to 0$ as $m \to \infty$.

In practice, we may stop the iteration process at u_M basically when we have $\|u_M - u_{M-1}\| < \epsilon$ for a given value of ϵ. Note that the larger the value of α the faster the convergence of (u_m).

To illustrate how our procedure works, we present a few examples. The first one is simple; the reader can follow the computations in details.

Example 3.1. (a) Suppose that we wish to find a continuous, piecewise smooth function u on $[0, 1]$ that minimizes the integral

$$E_1(u) := \int_0^1 |u'(x)|^2 \, dx, \tag{3}$$

subject to the condition that $u(0) = u(1) = 0$ and $u\big(\frac{1}{2}\big) = 1$.

For this, consider the subspace V of W consisting of all functions u that vanish at $\frac{1}{2}$; that is,

$$V := \{u \in W \, : \, u\big(\tfrac{1}{2}\big) = 0\},$$

and the subset U of W given by

$$U := \{u \in W \, : \, u\big(\tfrac{1}{2}\big) = 1\}.$$

Our initial approximation is $u_0(x) = \sin n\pi x$. Next, if $v(x) := \sum_{n=1}^{\infty} a_n \sin n\pi x$ is in V, then $v\big(\frac{1}{2}\big) = 0$ is equivalent to

$$a_1 - a_3 + a_5 - a_7 + - \cdots = 0,$$

for which we get

$$a_1 = a_3 - a_5 + a_7 - a_9 + - \cdots.$$

Hence, every element $(a_1, a_2, a_3, a_4, a_5, \ldots)$ in V can be expressed as

$$\begin{gathered} a_2(0,1,0,0,0,\ldots) + a_3(1,0,1,0,0,\ldots) + \\ + \, a_4(0,0,0,1,0,\ldots) + a_5(-1,0,0,0,1,\ldots) + \cdots. \end{gathered}$$

From this we get the following basis for V:

$$\begin{aligned} &v_1 := (0,1,0,0,0,\ldots), \;\; v_2 := (1,0,1,0,0,\ldots), \\ &v_3 := (0,0,0,1,0,\ldots), \;\; v_4 := (-1,0,0,0,1,\ldots), \;\; \ldots. \end{aligned}$$

If one carries out Step 3 as prescribed, one will get $u_1 = (1, 0, 0, 0, 0, \ldots)$, $u_2 = u_3 = \frac{9}{10}(1, 0, -\frac{1}{3^2}, 0, 0, \ldots)$, and so on. The limiting solution is

$$u = \frac{8}{\pi^2}(1, 0, -\frac{1}{3^2}, 0, \frac{1}{5^2}, \ldots).$$

Alternatively, one can compute the orthogonal complement of u_0 with respect to V directly as follows. If $u = (b_1, b_2, b_3, \ldots)$ is orthogonal to V, then $u \perp v_m$ for each $m \in \mathbb{N}$, and so $b_2, b_4, b_6, \ldots$ must be equal to 0 and

$$b_1 = -3^2 b_3 = 5^2 b_5 = -7^2 b_7 = \cdots.$$

Hence $u = b_1(1, 0, -\frac{1}{3^2}, 0, \frac{1}{5^2}, \ldots)$, that is,

$$u(x) = b_1\Big(\sin \pi x - \frac{1}{3^2}\sin 3\pi x + \frac{1}{5^2}\sin 5\pi x - \frac{1}{7^2}\sin 7\pi x + - \cdots\Big).$$

But $u\big(\frac{1}{2}\big) = 1$ gives

$$b_1 = \Big(\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}\Big)^{-1} = \frac{8}{\pi^2},$$

and therefore

$$u(x) = \frac{8}{\pi^2}\Big(\sin \pi x - \frac{1}{3^2}\sin 3\pi x + \frac{1}{5^2}\sin 5\pi x - \frac{1}{7^2}\sin 7\pi x + - \cdots\Big).$$

Notice that this is nothing but the Fourier sine series of the piecewise linear function f given by

$$f(x) = \begin{cases} 2x, & 0 \le x \le \frac{1}{2} \\ 2(1-x), & \frac{1}{2} < x \le 1. \end{cases}$$

The difference between the sequence (u_m) and the Fourier partial sums is that each u_m passes through the point $\big(\frac{1}{2}, 1\big)$ while the Fourier partial sums do not.

(b) In general, given $N+1$ points (x_0, c_0), $(x_1, c_1), \ldots, (x_N, c_N)$ with $0 = x_0 < x_1 < \cdots < x_N = 1$, the solution to the minimization problem (2) for $\alpha = 1$ is the Fourier sine series of the piecewise linear function f for which $f(x_i) = c_i$ and f is linear on each subinterval $[x_{i-1}, x_i]$.

For example, let $x_i = \frac{i}{4}$, $i = 0, \ldots, 4$, and $c_0 = 0$, $c_1 = \frac{7}{10}$, $c_2 = 1$, $c_3 = \frac{3}{10}$, $c_4 = 0$. With a computer program, we apply our procedure and get a sequence (u_m) that approximates the solution in W. We stop the iterations at u_M basically when $\|u_M - u_{M-1}\| < \epsilon$. For $\epsilon = 0.01$, the iterations stop at u_{182}. Figure 1 shows the graphs of u_0 (blue), u_5 (green), u_{30} (brown), and $u_M = u_{182}$ (red), which clearly indicate that the limiting series must be that of the piecewise linear function passing through the points (x_i, c_i), $i = 0, \ldots, 4$.

For $\alpha = 1$, one may observe that the piecewise linear function f that passes through the given points always solves the minimization problem (2). This follows from the following fact.

Fact 3.2. *On every interval $[a,b]$ where $u(a)$ and $u(b)$ are fixed, the integral $\int_a^b |u'(x)|^2 dx$ is minimized (among continuously differentiable functions u) if and only if u is linear.*

Proof. Let $m := \frac{u(b)-u(a)}{b-a}$. Then, by the Fundamental Theorem of Calculus, we have

$$\begin{aligned} \int_a^b |u'(x) - m|^2 dx &= \int_a^b |u'(x)|^2 dx - 2m \int_a^b u'(x)\, dx + m^2(b-a) \\ &= \int_a^b |u'(x)|^2 dx - 2m[u(b) - u(a)] + m^2(b-a) \\ &= \int_a^b |u'(x)|^2 dx - m^2(b-a). \end{aligned}$$

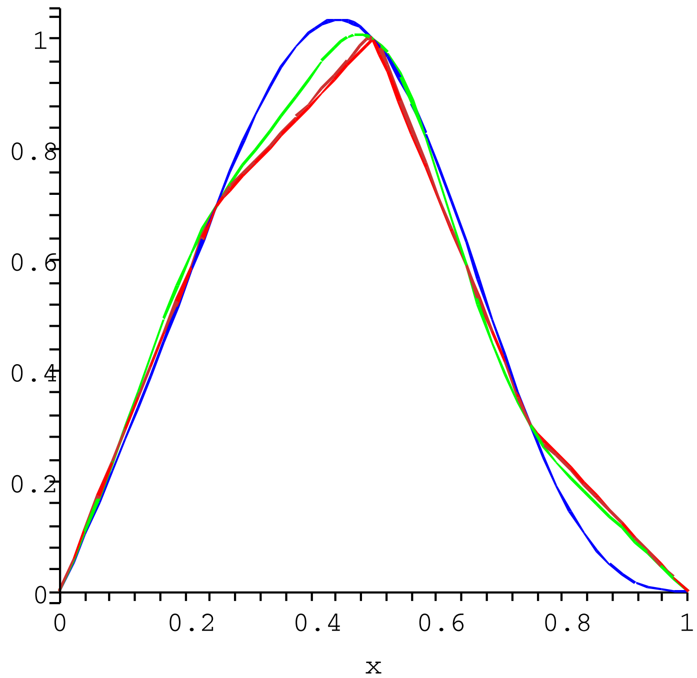

Fig. 1. $\alpha = 1;\ u(0) = 0,\ u\left(\frac{1}{4}\right) = \frac{7}{10},\ u\left(\frac{1}{2}\right) = 1,\ u\left(\frac{3}{4}\right) = \frac{3}{10},\ u(1) = 0;\ \epsilon = 0.01$

Hence $\int_a^b |u'(x)|^2 dx \geq \int_a^b m^2\, dx$, and $\int_a^b |u'(x)|^2 dx$ is minimized if and only if $u'(x) = m$ for every x in $[a, b]$; that is, if and only if u is linear. □

Example 3.3. Suppose that we wish to find a continuous function u that is twice differentiable almost everywhere on $[0, 1]$ and minimizes the integral

$$E_2(u) := \int_0^1 |u''(x)|^2\, dx, \tag{4}$$

subject to the condition that $u(0) = u(1) = 0$ and $u\left(\frac{1}{2}\right) = 1$.

Then, as in Example 3.1 (a), we will get

$$u(x) = \frac{96}{\pi^4}\Big(\sin \pi x - \frac{1}{3^4} \sin 3\pi x + \frac{1}{5^4} \sin 5\pi x - \frac{1}{7^4} \sin 7\pi x + - \cdots\Big),$$

which is the Fourier sine series of the cubic spline

$$f(x) = \begin{cases} 3x - 4x^3, & 0 \leq x \leq \frac{1}{2} \\ 3(1-x) - 4(1-x)^3, & \frac{1}{2} < x \leq 1. \end{cases}$$

In the next examples, we apply our procedure for fractional $\alpha's$ and we see what happens particularly when $\alpha > 1$ and $\frac{1}{2} < \alpha < 1$.

Example 3.4. (a) Suppose that $\alpha = 1.5$ and we wish to find a sufficiently smooth function u on $[0,1]$ that minimizes the integral $E_\alpha(u)$, subject to the condition that $u(0) = u(1) = 0$ and $u\left(\frac{1}{2}\right) = 1$.

Compared to Example 3.1(a), the function u here must be smoother at $\frac{1}{2}$. (From Lemma 2.2, we know that u has the fractional derivative u^β of order $\beta < 1$ which is continuous on $[0,1]$.) With a computer program, we apply our procedure with $\epsilon = 0.01$ and the iterations stop at u_{52}. Note that the convergence of (u_m) here is faster than that in Example 3.1(a). Figure 2 shows the graph of the approximate solution.

(b) Suppose now that $\alpha = 0.6$ and we wish to find a continuous function u on $[0,1]$ that minimizes the integral $E_\alpha(u)$, subject to the condition that $u(0) = u(1) = 0$ and $u\left(\frac{1}{2}\right) = 1$.

As one would expect, the function u now will be less smooth at $\frac{1}{2}$. Again, with a computer program, we apply our procedure with $\epsilon = 0.05$ and the iterations stop at u_{76} (we use a relatively large value of ϵ because the rate of convergence of (u_m) is expected to be low for small α). The graph of the approximate solution is shown in Figure 3.

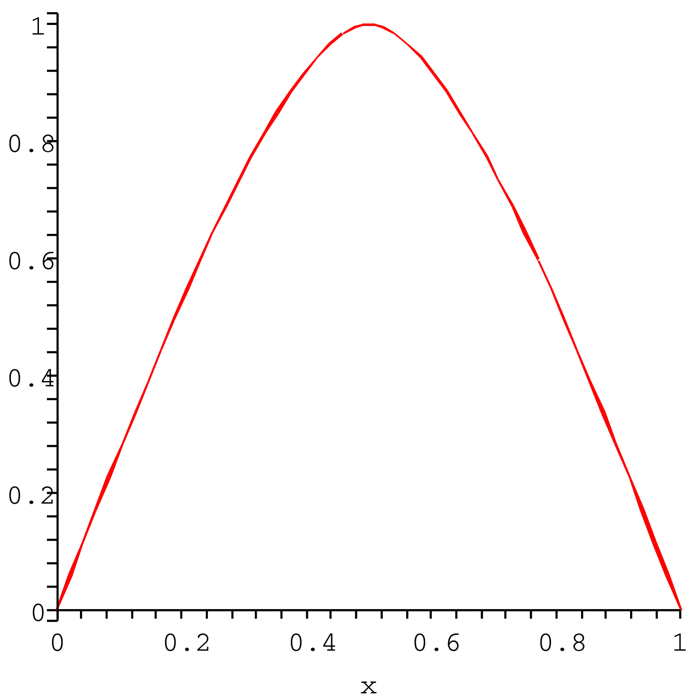

Fig. 2. $\alpha = 1.5;\ u(0) = 0,\ u\left(\frac{1}{2}\right) = 1,\ u(1) = 0;\ \epsilon = 0.01$

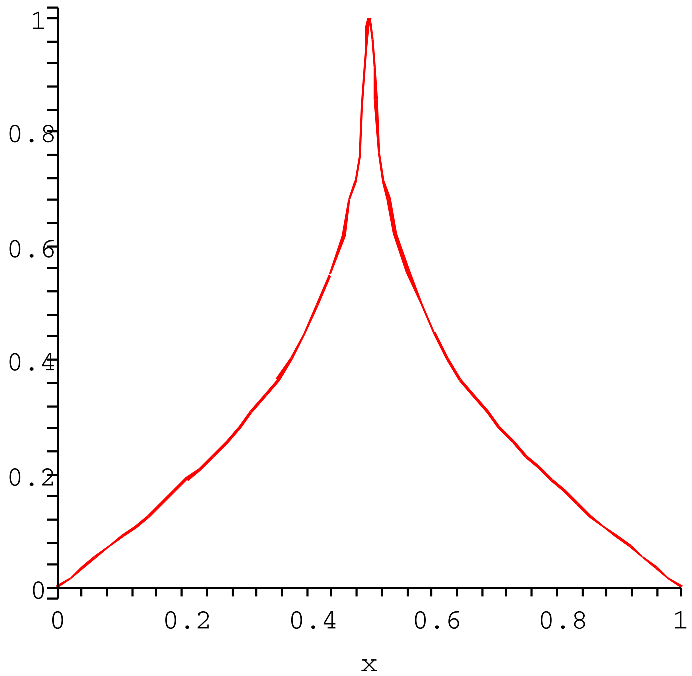

Fig. 3. $\alpha = 0.6;\ u(0) = 0,\ u(\frac{1}{2}) = 1,\ u(1) = 0;\ \epsilon = 0.05$

Remark. Our procedure also works for an energy functional which is a linear combination of several E_α's with at least one of the $\alpha' s$ is greater than $\frac{1}{2}$. Moreover, we have been successful in extending our method to solve an analogous problem in 2-dimensional setting.

4 What Happens When $0 \le \alpha \le \frac{1}{2}$

Suppose that $0 \le \alpha \le \frac{1}{2}$ and we are trying to find a continuous function u on $[0, 1]$ that minimizes the integral $E_\alpha(u)$, subject to the condition that $u(0) = u(1) = 0$ and $u(\frac{1}{2}) = 1$.

To solve this problem, we consider the space W consisting of all functions u on $[0, 1]$ of the form $u(x) = \sum_{n=1}^{\infty} a_n \sin n\pi x$ with $\sum_{n=1}^{\infty} n^{2\alpha} a_n^2 < \infty$, equipped with the inner product

$$\langle u, v\rangle := \sum_{n=1}^{\infty} n^{2\alpha} a_n b_n,$$

where a_n's and b_n's are the coefficients of u and v, respectively.

As in Example 3.1, we consider the subspace V of W consisting of all functions u that vanish at $\frac{1}{2}$; that is,

$$V := \{u \in W \,:\, u(\tfrac{1}{2}) = 0\},$$

and the subset U of W given by

$$U := \{u \in W \,:\, u\big(\tfrac{1}{2}\big) = 1\}.$$

If $(a_1, a_2, a_3, \ldots) \in V$, then $a_1 - a_3 + a_5 - a_7 + - \cdots = 0$. Accordingly, the following vectors

$$v_1 := (0,1,0,0,0,\ldots),\ v_2 := (1,0,1,0,0,\ldots),$$
$$v_3 := (0,0,0,1,0,\ldots),\ v_4 := (-1,0,0,0,1,\ldots),\ \ldots$$

form a basis for V.

As we can see, the above vectors also span W. Indeed, if $u = (b_1, b_2, b_3, \ldots)$ is orthogonal to V, then $u \perp v_m$ for each $m \in \mathbb{N}$, and so $b_2, b_4, b_6, \ldots$ are all 0 and

$$b_1 = -3^{2\alpha} b_3 = 5^{2\alpha} b_5 = -7^{2\alpha} b_7 = \cdots.$$

Hence $u = b_1(1, 0, -\frac{1}{3^{2\alpha}}, 0, \frac{1}{5^{2\alpha}}, \ldots)$. But then

$$\|u\|^2 = b_1^2 \Big(1 + \frac{1}{3^{2\alpha}} + \frac{1}{5^{2\alpha}} + \cdots\Big) < \infty \text{ if and only if } b_1 = 0.$$

This means that $V^{\perp} = \{0\}$ or $V = W$. (In the infinite dimensional case, an equation like $a_1 - a_3 + a_5 - a_7 + - \cdots = 0$ does not have to define a hyperplane.)

Consequently, starting with our initial approximation $u_0 = (1,0,0,0,\ldots)$, we will end up with $u = (0,0,0,0,\ldots)$ or $u(x) = 0$ almost everywhere. Our procedure guarantees that the function u will satisfy $u\big(\frac{1}{2}\big) = 1$; but obviously u cannot be continuous at $\frac{1}{2}$.

Acknowledgement. H. Gunawan and F. Pranolo are supported by ITB Research Program No. 174/ K01.07/PL/2007. We are grateful to Dr. A.R. Alghofari for useful discussion about the subject. We also thank the reviewers for their comments on the earlier version of this paper.

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