Surfaces with Minimum Energy and Prescribed Nodes

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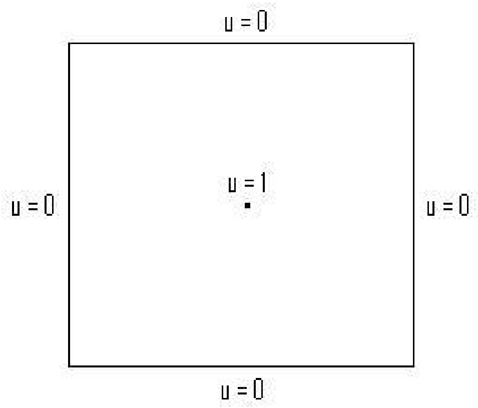
Isn't It an Interesting Surface?

The 2-D Problem

Find a cts fn $u : [0,1]^2 \to \mathbb{R}$ which minimizes an energy functional $E_\alpha(u)$ and satisfies the boundary and interior conditions:



The Energy Functional $E_\alpha(u)$

For $\alpha \geq 0$, the energy functional $E_\alpha(u)$ is given by

$$E_\alpha(u) := \int_0^1 \int_0^1 |(-\Delta)^{\frac{\alpha}{2}} u(x,y)|^2 dx\, dy,$$

where $-\Delta$ denotes the positive definite Laplacian in $\mathbb{R}^2$.

Note: $(-\Delta)^{\frac{\alpha}{2}}$ is known as the *fractional Laplacian* of order α. We shall define $(-\Delta)^{\frac{\alpha}{2}} u$ through its Fourier series (later).

The Special Case: $\alpha = 2$

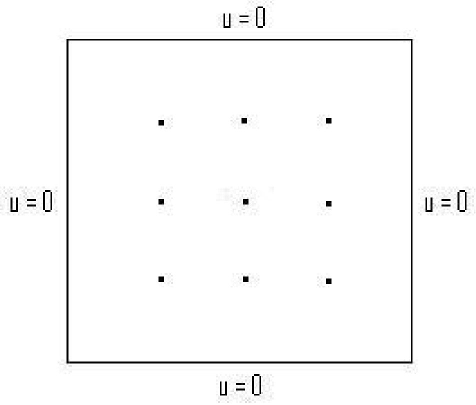
For $\alpha = 2$, the energy functional $E_2(u)$ given by

$$E_2(u) := \int_0^1 \int_0^1 |(-\Delta)u(x,y)|^2 dx\, dy,$$

represents the curvature (or the strain energy of bending) of u.

The Interior Conditions

In general, the interior conditions may be prescribed at $M \times N$ points (x_i, y_j), say $u(x_i, y_j) = c_{ij}$, $i = 1, \dots, M$, $j = 1, \dots, N$. E.g., for $M = N = 3$:



The Method

Since we are looking for a function $u(x, y)$ which vanishes at the boundary, we write $u(x, y)$ as a double Fourier sine series, that is,

$$u(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} \sin m\pi x \sin n\pi y,$$

where $a_{m,n}$'s are the Fourier coefficients.

For $\alpha = 2$, we have

$$(-\Delta)u(x,y) = \pi^2 \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m^2+n^2)a_{m,n}\sin m\pi x \sin n\pi y.$$

We see here that the Fourier cofficients of $(-\Delta)u$ are $\pi^2(m^2+n^2)$ $\times$ the Fourier coefficients of u.

The Definition of $(-\Delta)^{\frac{\alpha}{2}}u$

For $\alpha \geq 0$, we define $(-\Delta)^{\frac{\alpha}{2}}u$ by

$$(-\Delta)^{\frac{\alpha}{2}}u(x,y) = \pi^{\alpha}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(m^2+n^2)^{\frac{\alpha}{2}}a_{m,n}\sin m\pi x \sin n\pi y.$$

Parseval's Identity

Theorem:

$$\int_0^1 \int_0^1 |(-\Delta)^{\frac{\alpha}{2}} u(x,y)|^2 dx\, dy = \frac{\pi^{2\alpha}}{4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (m^2 + n^2)^{\alpha} a_{m,n}^2.$$

The Problem Reformulated

The problem may now be reformulated as follows: find a cts fn $u(x,y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} \sin m\pi x \sin n\pi y$ which minimizes

$$E_\alpha(u) := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (m^2 + n^2)^\alpha a_{m,n}^2$$

and satisfies the interior conditions

$$u(x_i, y_j) = c_{ij},\ i = 1, \ldots, M,\ j = 1, \ldots, N.$$

An Existence and Uniqueness Theorem

Theorem:

The problem has a unique continuous solution if and only if $\alpha > 1$.

Note about the Proof

The main ideas of the proof are to (1) work in the right space, (2) find an initial function that satisfies the condition, and then (3) compute the orthogonal complement of this function on the subspace of functions that vanish at the given nodes.

While the existence and the uniqueness of the solution follows from Hilbert space arguments, the continuity is guaranteed by the value of the exponent $\alpha > 1$.

Ingredients of the Proof

$$W := \{u(x,y) = \sum_{m,n} a_{m,n} \sin m\pi x \sin n\pi y : \sum_{m,n} (m^2+n^2)^\alpha a_{m,n}^2 < \infty\}$$

is a Hilbert space w.r.t. the inner product

$$\langle u, v \rangle := \sum_{m,n} (m^2 + n^2)^\alpha a_{m,n} b_{m,n}.$$

$U := \{u \in W \; : \; u(x_i, y_j) = c_{ij}, \; i = 1, \ldots, M, \; j = 1, \ldots, N\}$ is a nonempty, closed and convex subset of W.

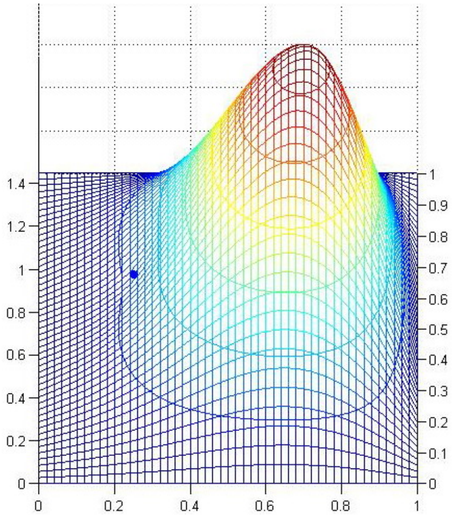
$V := \{u \in W \; : \; u(x_i, y_j) = 0, \; i = 1, \ldots, M, \; j = 1, \ldots, N\}$ is a closed subspace of W.

The solution is given by $u = u_0 - \mathrm{proj}_V(u_0)$, where u_0 is an arbitrary member of U.

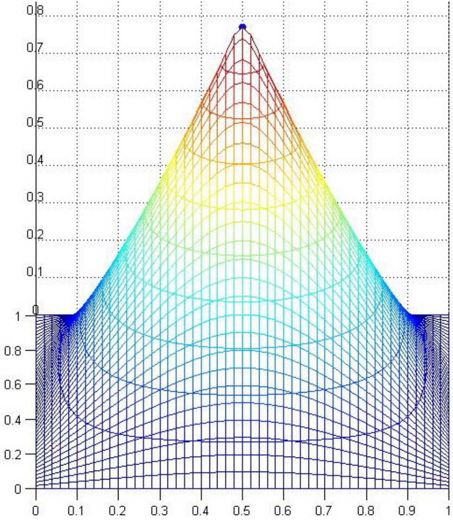
For $1 < \alpha < 1.5$ and one interior point is prescibed, the solution looks like one shown at the beginning.

For $\alpha \geq 1.5$ or more interior points are prescribed, the solution may look like the following pictures.

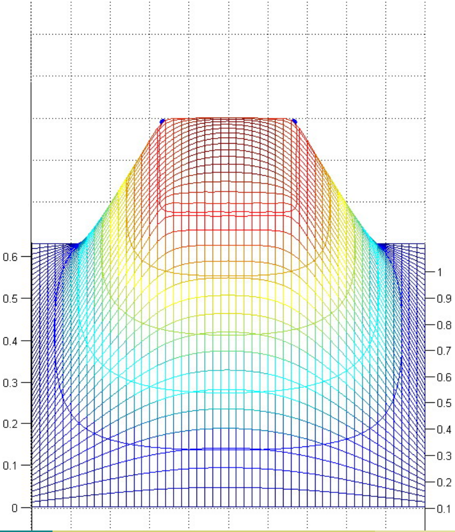
The surface passing through (0.5,0.25,0.25) and (0.5,0.75,1.25) with minimum $E_2(u)$



The surface passing through (0.5,0.5,1) with minimum $E_{1.5}(u)$



A surface passing through four prescribed points with minimum $E_{1.5}(u)$



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