Fractional Integrals on Lebesgue and Morrey Spaces (of Non-homogeneous Type)

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Outline

1. FRACTIONAL INTEGRAL OPERATORS

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In harmonic analysis, there are several main operators studied extensively. Among these operators are:

- The Hardy-Littlewood Maximal Operator
- Singular Integral Operators
- Fractional Integral Operators
- Fractional Maximal Operators

In this talk, we shall focus on Fractional Integral Operators and their boundedness on Lebesgue and Morrey spaces (on $\mathbb{R}^n$ and on non-homogeneous spaces).

Fractional Integral Operators

For $0 < \alpha < n$, the (classical) fractional integral operator I_α (a.k.a. *Riesz potential*) is defined by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

Note that the integral makes sense when, for instance, f is bounded and compactly supported.

This operator was first studied by Hardy and Littlewood in the 1920's [12, 14] and extended by Sobolev [27] in the 1930's.

The Boundedness of I_α on Lebesgue Spaces

A well-known result for I_α is the Hardy-Littlewood-Sobolev inequality.

Theorem 2.1 (Hardy-Littlewood; Sobolev)

For $1 < p < \frac{n}{\alpha}$, we have the inequality

$$\|I_\alpha f\|_q \leq C_p \|f\|_p, \tag{1}$$

that is, I_α is bounded from L^p to L^q, provided that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

Note. There is also a weak version of this result, but we shall not discuss it here.

Through its Fourier transform, the operator I_α can be recognized as a multiple of the Laplacian to the power of $-\frac{\alpha}{2}$, that is,

$$I_\alpha f = \kappa(n,\alpha).(-\Delta)^{-\frac{\alpha}{2}} f.$$

As an immediate consequence of the HLS inequality (1), one has the following estimate for the Newtonian potential $(-\Delta)^{-1}$:

$$\|(-\Delta)^{-1} f\|_{np/(n-2)} \leq C_p \|f\|_p,$$

for $1 < p < \frac{n}{2},\ n \geq 3$.

Here $u := (-\Delta)^{-1} f$ is a solution of the Poisson equation $-\Delta u = f$.

From (1) we can also prove Sobolev's embedding theorems (see [28]).

The (Classical) Morrey Spaces

For $1 \le p < \infty$ and $0 \le \lambda \le n$, the (classical) Morrey space $L^{p,\lambda} = L^{p,\lambda}(\mathbb{R}^n)$ is defined to be the space of all functions $f \in L^p_{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{p,\lambda} := \sup_{B=B(a,r)} \left(\frac{1}{r^\lambda} \int_B |f(y)|^p \, dy \right)^{1/p} < \infty,$$

where $B(a,r)$ denotes the (open) ball centered at $a \in \mathbb{R}^n$ with radius $r > 0$ [17].

Here $\|\cdot\|_{p,\lambda}$ defines a semi-norm on $L^{p,\lambda}$. Note particularly that $L^{p,0} = L^p$ and $L^{p,n} = L^\infty$.

For the structure of Morrey spaces and their generalisations, see the works of S. Campanato, 1964 [3], J. Peetre, 1969 [24], C.T. Zorko, 1986 [31], and the references therein.

The Boundedness of I_α on Morrey Spaces

As stated in [24], S. Spanne proved that I_α is bounded from $L^{p,\lambda}$ to $L^{q,\lambda q/p}$ for $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $0 \le \lambda < n$.

A stronger result was obtained by D.R. Adams, 1975 [1] and reproved by F. Chiarenza and M. Frasca, 1987 [4].

Theorem 2.2 (Adams; Chiarenza-Frasca)

For $1 < p < \frac{n}{\alpha}$ and $0 \le \lambda < n - \alpha p$, we have the inequality

$$\|I_\alpha f\|_{q,\lambda} \le C_{p,\lambda} \|f\|_{p,\lambda}$$

provided that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$.

The Hardy-Littlewood Maximal Operator

The proof usually involves the properties of the Hardy-Littlewood maximal operator M, defined by the formula

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy,$$

where $|B(x,r)| = c\, r^n$ is the Lebesgue measure of $B(x,r)$.

The operator M is known to be bounded on L^p for $1 < p \leq \infty$ [13].

Chiarenza and Frasca [4] proved that M is also bounded on Morrey spaces.

Theorem 2.3 (Chiarenza-Frasca)

The inequality

$$\|Mf\|_{p,\lambda} \le C_{p,\lambda}\|f\|_{p,\lambda}$$

holds for $p > 1$ and $0 \le \lambda < n$.

The Proof of Theorem 2.2

For each $x \in \mathbb{R}^n$, write $I_\alpha f(x) = I_1(x) + I_2(x)$ where

$$I_1(x) := \int_{|x-y|<R} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy; \quad I_2(x) := \int_{|x-y|\geq R} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy,$$

with R being an arbitrary positive number.

For I_1, we have the following estimate:

$$
\begin{aligned}
|I_1(x)| &\le \sum_{k=-\infty}^{-1} \int_{2^k R \le |x-y| < 2^{k+1} R} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy \\
&\le \sum_{k=-\infty}^{-1} (2^k R)^{\alpha-n} \int_{B(x,2^{k+1}R)} |f(y)|\, dy \\
&\le C\, Mf(x) \sum_{k=-\infty}^{-1} (2^k R)^{\alpha} \\
&\le C\, R^{\alpha} Mf(x).
\end{aligned}
$$

To estimate I_2, we proceed as follows:

$$\begin{aligned}
|I_2(x)| &\le \sum_{k=0}^{\infty} \int_{2^k R \le |x-y| < 2^{k+1} R} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, dy \\
&\le \sum_{k=0}^{\infty} (2^k R)^{\alpha - n} \int_{B(x,2^{k+1}R)} |f(y)|\, dy \\
&\le C \sum_{k=0}^{\infty} (2^k R)^{\alpha - \frac{n}{p}} \Big(\int_{B(x,2^{k+1}R)} |f(y)|^p dy \Big)^{1/p} \\
&\le C \sum_{k=0}^{\infty} (2^k R)^{\alpha + \frac{\lambda - n}{p}} \|f\|_{p,\lambda} \\
&\le C\, R^{\alpha + \frac{\lambda - n}{p}} \|f\|_{p,\lambda}.
\end{aligned}$$

The last inequality holds because $\sum_{k=0}^{\infty} 2^{k(\alpha + \frac{\lambda - n}{p})}$ converges, for we have $\alpha + \frac{\lambda - n}{p} < 0$.

Combining the two estimates, we get

$$|I_\alpha f(x)| \le C\,R^\alpha[Mf(x) + R^{\frac{\lambda-n}{p}}\|f\|_{p,\lambda}].$$

Assuming that $f \neq 0$ and Mf is finite everywhere, we choose $R = \left(\frac{Mf(x)}{\|f\|_{p,\lambda}}\right)^{p/(\lambda-n)}$. Then, we have

$$\begin{aligned} |I_\alpha f(x)| &\le C\,[Mf(x)]^{1-\alpha p/(n-\lambda)}\|f\|_{p,\lambda}^{\alpha p/(n-\lambda)} \\ &= C\,[Mf(x)]^{p/q}\|f\|_{p,\lambda}^{1-p/q}. \end{aligned}$$

The desired inequality then follows from this and the boundedness of M on L^p. □

Generalised Morrey Spaces

For $1 \leq p < \infty$ and a suitable function $\phi : (0,\infty) \to (0,\infty)$, we define the generalised Morrey space $\mathcal{M}_{p,\phi} = \mathcal{M}_{p,\phi}(\mathbf{R}^n)$ to be the space of all functions $f \in L^p_{\rm loc}(\mathbf{R}^n)$ for which

$$\|f\|_{p,\phi} := \sup_{B=B(a,r)} \frac{1}{\phi(r)} \left(\frac{1}{|B|} \int_B |f(y)|^p dy \right)^{1/p} < \infty.$$

Note: If $\phi(t) = t^{(\lambda-n)/p}$, $0 \leq \lambda \leq n$, we have $\mathcal{M}_{p,\phi} = L^{p,\lambda}$ — the classical Morrey space.

Unless otherwise stated, ϕ satisfies the following two conditions:

(2.1) $\frac{1}{2} \le \frac{r}{s} \le 2 \Rightarrow \frac{1}{C_1} \le \frac{\phi(r)}{\phi(s)} \le C_1$ (*the doubling condition*);

(2.2) $\displaystyle\int_r^\infty \frac{\phi^p(t)}{t}\,dt \le C_2\,\phi^p(r)$ for $1 < p < \infty$.

For any function ψ that satisfies the doubling condition, we have

$$\int_{2^k r}^{2^{k+1}r} \frac{\psi(t)}{t}\,dt \sim \psi(2^k r)$$

for every integer k and $r > 0$.

In 1994, E. Nakai [18] proved the boundedness of the Hardy-Littlewood maximal operator on generalised Morrey spaces.

Theorem 3.1 (Nakai)

The inequality

$$\|Mf\|_{p,\phi} \le C_{p,\phi}\|f\|_{p,\phi}$$

holds for $1 < p < \infty$.

The Boundedness of I_α on Generalised Morrey Spaces. I

Theorem 3.2 (Nakai)

Suppose that $\psi : (0, \infty) \to (0, \infty)$ satisfies the doubling condition and that $\int_r^\infty t^{\alpha-1}\phi(t)\,dt \le Cr^\alpha \phi(r) \le C\psi(r)$. Then, for $1 < p < q < \infty$, we have

$$\|I_\alpha f\|_{q,\psi} \le C\|f\|_{p,\phi},$$

whenever $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

Note. This result can be viewed as an extension of Spanne's result.

The Boundedness of I_α on Generalised Morrey Spaces. II

The following theorem can be considered as an extension of Adams-Chiarenza-Frasca's result.

Theorem 3.3 (G, Eridani (2009))

Suppose that, in addition to the condition (2.1) and (2.2), ϕ satisfies the inequality $\phi(t) \le Ct^\beta$ for $-\frac{n}{p} \le \beta < -\alpha,\ 1 < p < \frac{n}{\alpha}$. Then, we have

$$\|I_\alpha f\|_{q,\phi^{p/q}} \le C_{p,\beta}\|f\|_{p,\phi}$$

where $q = \frac{\beta p}{\alpha+\beta}$.

Remark. Observe that when $\phi(t) = t^{(\lambda-n)/p}$, $0 \le \lambda < n - \alpha p$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$, Theorem 3.3 reduces to Theorem 2.2.

The Generalised Fractional Integral Operator T_ρ

For a given function $\rho : (0, \infty) \to (0, \infty)$, we define the (generalised) fractional integral operator T_ρ by

$$T_\rho f(x) := \int_{\mathbf{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y)\, dy$$

For $\rho(t) = t^\alpha,\ 0 < \alpha < n$, we have $T_\rho = I_\alpha$ — the classical fractional integral operator.

The operator T_ρ was first studied by Nakai, 2000 [19]. Recent results on T_ρ can be found in the works of Eridani, G, and/or Nakai [5, 6, 9, 20, 21].

The Boundedness of T_ρ on Generalised Morrey spaces

A slight modification of Theorem 3.3 may be formulated for T_ρ as follows.

Theorem 4.1 (G, Eridani)

Suppose that $\rho(t) \le C_1 t^\alpha$ for some $0 < \alpha < n$, and, in addition to the condition (2.1) and (2.2), ϕ satisfies the inequality $\phi(t) \le C_2 t^\beta$ for $-\frac{n}{p} \le \beta < -\alpha,\ 1 < p < \frac{n}{\alpha}$. Then, for $q = \frac{\beta p}{\alpha+\beta}$, we have

$$\|T_\rho f\|_{q,\phi^{p/q}} \le C_{p,\beta} \|f\|_{p,\phi},$$

that is, T_ρ is bounded from $\mathcal{M}_{p,\phi}$ to $\mathcal{M}_{q,\phi^{p/q}}$.

Another generalisation of Theorem 2.2 is the following result of [9].

Theorem 4.2 (G (2003))

Suppose that, in addition to the condition (2.1) and (2.2), ϕ is surjective. If ρ satisfies the doubling condition and

$$\int_0^r \frac{\rho(t)}{t}\,dt \le C\phi(r)^{(p-q)/q} \quad \text{and} \quad \int_r^\infty \frac{\rho(t)\phi(t)}{t}dt \le C\phi(r)^{p/q},$$

for $1 < p < q < \infty$, then we have

$$\|T_\rho f\|_{q,\phi^{p/q}} \le C_{p,\phi}\|f\|_{p,\phi}.$$

Olsen Inequality

In studying a Schrödinger equation

$$(-\Delta + V(x) + W(x))u(x) = f(x)$$

with perturbed potentials W on $\mathbb{R}^n$ (particularly for $n = 3$), P.A. Olsen, 1995 [23] proved the following result.

Theorem 5.1 (Olsen)

For $1 < p < \frac{n}{\alpha}$ and $0 \le \lambda < n - \alpha p$, we have

$$\|W \cdot I_\alpha f\|_{p,\lambda} \le C_{p,\lambda} \|W\|_{(n-\lambda)/\alpha,\lambda} \|f\|_{p,\lambda},$$

that is, $W \cdot I_\alpha$ is bounded on $L^{p,\lambda}$, provided that $W \in L^{(n-\lambda)/\alpha,\lambda}$.

As a consequence of Theorem 4.1, we see that for $1 < p < \frac{n}{2},\ n \geq 3$, the estimate

$$\|W \cdot (-\Delta)^{-1} f\|_{p,\lambda} \leq C_{p,\lambda} \, \|W\|_{(n-\lambda)/2,\lambda} \|f\|_{p,\lambda},$$

holds provided that $W \in L^{(n-\lambda)/2,\lambda},\ 0 \leq \lambda < n - 2p$.

In particular, when $\lambda = 0$, one has

$$\|W \cdot (-\Delta)^{-1} f\|_{p} \leq C_{p} \, \|W\|_{n/2} \|f\|_{p}$$

provided that $W \in L^{n/2}$.

In 2002, K. Kurata *et al.* [16] extended Olsen's result by proving that, for some $p > 1$ and a function ϕ satisfying several conditions (including the doubling condition), the operator $W \cdot I_\alpha$ is bounded on generalised Morrey spaces $\mathcal{M}_{p,\phi}$, provided that $W \in \mathcal{M}_{s_1,\phi} \cap \mathcal{M}_{s_2,\phi}$ for some indices s_1 and s_2.

Their estimate, however, is rather complicated. We shall here present simpler estimates for $W \cdot I_\alpha$ on generalised Morrey spaces.

An Estimate for $W \cdot I_\alpha$

Theorem 5.2 (G, Eridani)

Suppose that, in addition to the condition (2.1) and (2.2), ϕ satisfies the inequality $\phi(t) \le Ct^\beta$ for $-\frac{n}{p} \le \beta < -\alpha,\ 1 < p < \frac{n}{\alpha}$. Then, we have

$$\|W \cdot I_\alpha f\|_{p,\phi} \le C_{p,\beta} \|W\|_{s,\phi^{p/s}} \|f\|_{p,\phi}$$

provided that $W \in \mathcal{M}_{s,\phi^{p/s}}$ where $s = -\frac{\beta p}{\alpha}$.

Proof. Use Hölder's inequality and Theorem 2.5. □

Another Estimate for $W \cdot I_\alpha$

Theorem 5.3 (G, Eridani)

Suppose that ϕ satisfies the doubling condition and the inequality

$$\int_r^\infty t^{\alpha-1}\phi(t)\,dt \le C\,r^\alpha \phi(r).$$

Then, for $1 < p < \frac{n}{\alpha}$, we have

$$\|W \cdot I_\alpha f\|_{p,\phi} \le C_{p,\phi}\|W\|_{n/\alpha}\|f\|_{p,\phi},$$

provided that $W \in L^{n/\alpha}$.

Proof. For $a \in \mathbf{R}^n$ and $r > 0$, let $B = B(a,r)$, $\tilde{B} = B(a,2r)$, and write $f = f_1 + f_2 := f\chi_{\tilde{B}} + f\chi_{\tilde{B}^c}$. We observe that $f_1 \in L^p$ with

$$\|f_1\|_p = \Big(\int_{\mathbf{R}^n} |f_1(y)|^p dy\Big)^{1/p} = \Big(\int_{\tilde{B}} |f(y)|^p dy\Big)^{1/p} \le C\, r^{n/p}\phi(r)\|f\|_{p,\phi}.$$

Hence, by applying Theorem 4.1 for $\lambda = 0$, we get

$$\begin{aligned}\Big(\int_B |W \cdot I_\alpha f_1(x)|^p dx\Big)^{1/p} &\le \|W \cdot I_\alpha f_1\|_p \le C\,\|W\|_{n/\alpha}\|f_1\|_p \\ &\le C\, r^{n/p}\phi(r)\|W\|_{n/\alpha}\|f\|_{p,\phi},\end{aligned}$$

whence

$$\frac{1}{\phi(r)}\Big(\frac{1}{|B|}\int_B |W \cdot I_\alpha f_1(x)|^p dx\Big)^{1/p} \le C\,\|W\|_{n/\alpha}\|f\|_{p,\phi}.$$

Next, for $x \in B$, we have

$$|I_\alpha f_2(x)| \le \int_{\tilde{B}^c} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy \le \int_{|x-y|\ge r} \frac{|f(y)|}{|x-y|^{n-\alpha}} dy$$

Then, we shall obtain

$$|I_\alpha f_2(x)| \le C\, \|f\|_{p,\phi} \int_r^\infty t^{\alpha-1}\phi(t)\, dt \le C\, r^\alpha \phi(r) \|f\|_{p,\phi}.$$

Hence

$$\Big(\frac{1}{|B|}\int_B |W.I_\alpha f_2(x)|^p dx\Big)^{\frac{1}{p}} \le C\, r^\alpha \phi(r)\|f\|_{p,\phi}\Big(\frac{1}{|B|}\int_B |W(x)|^p\, dx\Big)^{1/p}$$
$$\le C\, r^\alpha \phi(r)\|f\|_{p,\phi}\Big(\frac{1}{|B|}\int_B |W(x)|^{n/\alpha} dx\Big)^{\frac{\alpha}{n}}$$
$$\le C\,\phi(r)\|W\|_{n/\alpha}\|f\|_{p,\phi},$$

and so

$$\frac{1}{\phi(r)}\Big(\frac{1}{|B|}\int_B |W\cdot I_\alpha f_2(x)|^p dx\Big)^{1/p} \le C\,\|W\|_{n/\alpha}\|f\|_{p,\phi}.$$

The desired estimate follows from the two estimates via Minkowski inequality. □

An Estimate for $W \cdot T_\rho$

Theorem 5.4 (G, Eridani)

Suppose that $\rho(t) \le C_1 t^\alpha$ for some $0 < \alpha < n$, and, in addition to the condition (2.1) and (2.2), $\phi(t) \le C_2 t^\beta$ for $-\frac{n}{p} \le \beta < -\alpha$, $1 < p < \frac{n}{\alpha}$. Then, we have

$$\|W \cdot T_\rho f\|_{p,\phi} \le C_{p,\beta} \|W\|_{s,\phi^{p/s}} \|f\|_{p,\phi}$$

provided that $W \in \mathcal{M}_{s,\phi^{p/s}}$ where $s = -\frac{\beta p}{\alpha}$.

Proof. Use Hölder's inequality and Theorem 3.1. □

Another Estimate for $W \cdot T_\rho$

Theorem 5.5 (G, Eridani)

Suppose that, in addition to the condition (2.1) and (2.2), ϕ is surjective. If ρ satisfies the doubling condition and

$$\int_0^r \frac{\rho(t)}{t}\,dt \le C\phi(r)^{(p-q)/q} \quad \text{and} \quad \int_r^\infty \frac{\rho(t)\phi(t)}{t}\,dt \le C\phi(r)^{p/q},$$

for $1 < p < q < \infty$, then we have

$$\|W \cdot T_\rho f\|_{p,\phi} \le C_{p,\phi}\|W\|_{s,\phi^{p/s}}\|f\|_{p,\phi},$$

provided that $W \in \mathcal{M}_{s,\phi^{p/s}}$ where $\frac{1}{s} = \frac{1}{p} - \frac{1}{q}$.

Proof. Let $B = B(a,r)$ be an arbitrary ball in $\mathbf{R}^n$. By Hölder's inequality, we have

$$\frac{1}{|B|}\int_B |W\cdot T_\rho f(x)|^p dx \le \Big(\frac{1}{|B|}\int_B |W(x)|^s dx\Big)^{\frac{p}{s}} \Big(\frac{1}{|B|}\int_B |T_\rho f(x)|^q dx\Big)^{\frac{p}{q}}$$

with $\frac{p}{s}+\frac{p}{q}=1$. Now take the p-th roots and then divide both sides by $\phi(r)$ to get

$$\begin{aligned}\frac{1}{\phi(r)}\Big(\frac{1}{|B|}\int_B |W\cdot T_\rho f(x)|^p dx\Big)^{\frac{1}{p}} &\le \frac{1}{\phi(r)^{p/s}}\Big(\frac{1}{|B|}\int_B |W(x)|^s dx\Big)^{1/s} \\ &\quad\times \frac{1}{\phi(r)^{p/q}}\Big(\frac{1}{|B|}\int_B |T_\rho f(x)|^q dx\Big)^{\frac{1}{q}} \\ &\le C\,\|W\|_{s,\phi^{p/s}}\|T_\rho f\|_{q,\phi^{p/q}}.\end{aligned}$$

The desired inequality is obtained by taking the supremum over all balls B and using Theorem 3.2. □

From now on, $\mathbb{R}^n$ will be replaced by $\mathbb{R}^d$.
The letter n will be used for another purpose.

For convenience, we shall also use cubes $Q(x, r)$ instead of balls $B(x, r)$.

The success of the development of many theories in Fourier Analysis in homogeneous spaces for almost three decades is due to the fact that most of the central results in the Euclidean setting can be generalised without too much difficulties to the homogeneous setting – Verdera, 2002 [30].

In recent years, however, researchers found that many results still hold without the assumption of doubling condition on the measure, as confirmed by Nazarov et.al., 1998 [22], Tolsa, 1999 [29], and Garca-Cuerva and Gatto, 2004 [7].

Let μ be a (positive) Radon measure on $\mathbb{R}^d$. Then $(\mathbb{R}^d, \mu)$ is a *non-homogeneous space* if μ satisfies the growth condition of order n $(0 < n \leq d)$, that is,

$$\mu(Q) \leq C\,[\ell(Q)]^n,$$

for any cube $Q \subseteq \mathbb{R}^d$ with the sides parallel to the coordinate axis. Here $\ell(Q)$ stands for the side length of Q.

The Boundedness of I_α^n on Non-homogeneous Lebesgue Spaces

In the non-homogeneous context, we define the fractional integral operator I_α^n $(0 < \alpha < n \leq d)$ by

$$I_\alpha^n f(x) := \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}} \, d\mu(y),$$

(see Garcia-Cuerva & Martell, 2001 [8]).

Theorem 6.1 (Garcia-Cuerva & Martell)

The operator I_α^n is bounded from $L^p(\mu)$ to $L^q(\mu)$ for $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

Generalised Non-homogeneous Morrey Spaces. I

For $1 \leq p < \infty$, define the non-homogeneous Morrey space $M^{p,\phi}(\mu)$ to be the set of all $f \in L^p_{\rm loc}(\mu)$ such that

$$\|f\|_{p,\phi,\mu} := \sup_{r>0} \frac{1}{\phi(r)} \Big[\frac{1}{r^n} \int_{Q(x,r)} |f(y)|^p d\mu(y)\Big]^{1/p} < \infty.$$

The Boundedness of I_α^n on Non-homogeneous Morrey Spaces. I

Theorem 6.2 (Sihwaningrum, Suryawan, G (2009))

Suppose that $\psi : (0,\infty) \to (0,\infty)$ *satisfies the doubling condition and that* $\int_r^\infty t^{\alpha-1}\phi(t)dt \le C\, r^\alpha \phi(r) \le C\, \psi(r)$*. Then*

$$\|I_\alpha^n f\|_{q,\psi,\mu} \le C\, \|f\|_{p,\phi,\mu},$$

for $1 < p < \frac{n}{\alpha},\ \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$

Generalised Non-homogeneous Morrey Spaces. II

For $k > 1$, define the non-homogeneous Morrey space $M^{p,\phi}(k,\mu)$ to be the set of all $f \in L^p_{\text{loc}}(\mu)$ such that

$$\|f\|_{p,\phi,k,\mu} := \sup_{Q=Q(x,r)} \frac{1}{\phi(\mu(kQ))} \left[\frac{1}{\mu(kQ)} \int_Q |f(y)|^p d\mu(y) \right]^{1/p} < \infty.$$

The reason why we have $k > 1$ is due to the result of Nazarov *et al.* [22] about the boundedness of the Hardy-Littlewood maximal operator $M_k f(x) := \sup_{Q \ni x} \frac{1}{\mu(kQ)} \int_Q f(y)\, d\mu(y)$.

The Boundedness of I_α^n on Non-homogeneous Morrey Spaces. II

Theorem 6.3 (G, Sawano, Sihwaningrum (2009))

Let $a = \frac{\alpha}{n}$ *and* ϕ *be surjective and* $\phi(t) \le C\, t^b$ *where* $-\frac{1}{p} \le b \le -a < 0$. *Then,* I_α^n *is bounded from* $M^{p,\phi}(k,\mu)$ *to* $M^{q,\phi^{p/q}}$ *where* $p > 1$ *and* $q = \frac{bp}{a+b}$.

A similar result for an analog of T_ρ and Olsen-type inequalities in non-homogeneous setting can be found in [11, 25].

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