Fractional integral operators on generalized Morrey spaces of non-homogeneous type[1]

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Abstract

We prove here the boundedness of the fractional integral operator I_α on generalized Morrey spaces in the non-homogeneous setting, which is analogous to Nakai's result [8] in the homogeneous case. Our proof makes use of the covering lemma of Sawano [12]. As a consequence of our result, we obtain an Olsen inequality for a multiplication operator involving I_α in the non-homogeneous setting.

Keywords: Fractional integral operators, Olsen inequality, non-homogeneous spaces, generalized Morrey spaces.

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1 Introduction

Suppose that $\mathbb{R}^d$ is equipped with a non-negative Radon measure μ satisfying the *growth condition*, that is, there exists a constant $C > 0$ and $0 < n \le d$ such that

$$\mu(Q) \le C\,\ell^n$$

for every d-dimensional cube Q with center $c_Q \in \mathbb{R}^d$ and side length $\ell > 0$. Here we consider only the cubes with sides parallel to the coordinate axes.

It has been widely known that the growth condition replaces the role of the *doubling condition* in homogeneous spaces. A non-negative measure μ satisfies the doubling condition if there exists a constant $C > 0$ such that for every cube Q with side length ℓ we have

$$\mu(2Q) \le C\,\mu(Q),$$

where $2Q$ denotes the cube concentric to Q with side length 2ℓ.

The analysis on non-homogeneous spaces has been developed since the works of Nazarov *et al.* [9] and Tolsa [17]. Their success — in replacing the doubling condition by the growth condition — has inspired other people to work on spaces of non-homogeneous type.

Our object of study here is the fractional integral operator I_α, which is defined — in the non-homogeneous setting — by

$$I_\alpha f(x) := \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}}\,d\mu(y),$$

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where $0 < \alpha < n \leq d$. Notice that when $n = d$ and μ is the usual Lebesgue measure, we recover the classical fractional integral operator introduced by Hardy and Littlewood [5, 6] and Sobolev [16]. See [1, 2, 8, 11] and many other literatures for various results on fractional integral operators in the classical version.

In [3], García-Cuerva and Martell studied the boundedness of I_α on Lebesgue spaces of non-homogeneous type. In particular, they obtained the following result. (Note that throughout the paper, we denote the norm of f in the space X by $\|f : X\|$.)

Theorem 1.1. [3] *If $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, then there exists a constant $C > 0$ such that*

$$\|I_\alpha f : L^q(\mu)\| \leq C \, \|f : L^p(\mu)\|,$$

that is, I_α is bounded from $L^p(\mu)$ to $L^q(\mu)$.

Our goal in this paper is to prove the boundedness of I_α on generalized Morrey spaces of non-homogeneous type, which is analogous to Nakai's result [8] in the homogeneous setting. As a consequence, we obtain an Olsen inequality for a multiplication operator involving I_α in the non-homogeneous version. For related results, see [4, 13, 14].

The inequality was first introduced — in the homogeneous setting — by Olsen [10]; and was generalized later by Kurata *et al.* [7]. They used the inequality to study the behaviour of the solution to a Schrödinger equation with a small perturbed potential W. Notice that $I_\alpha = (-\Delta)^{-\alpha/2}$, where $-\Delta$ is the Laplacian operator.

2 The boundedness of I_α and an Olsen inequality

For $k > 1$, let kQ denote a cube concentric to Q with side length k times side length of Q; $\mathcal{Q}(\mu)$ is the set of all cubes with positive μ-measure; and C stands for a constant which may differ from one line to another. For $1 \leq p < \infty$ and an *almost decreasing* function $\phi : (0, \infty) \to (0, \infty)$, we define the generalized non-homogeneous Morrey spaces $\mathcal{M}^{p,\phi}(k, \mu) = \mathcal{M}^{p,\phi}(\mathbb{R}^d, k, \mu)$ to be the set of all $f \in L^p_{\rm loc}(\mu)$ such that

$$\|f : \mathcal{M}^{p,\phi}(k, \mu)\| := \sup_Q \frac{1}{\phi(\mu(kQ))} \left(\frac{1}{\mu(kQ)} \int_Q |f(y)|^p \, d\mu(y) \right)^{1/p} < \infty.$$

Another version of such spaces can be found in [15]. Recall that $\phi : (0, \infty) \to (0, \infty)$ is almost decreasing [almost increasing] if there exists a constant C such that for $s < t$, we have $\phi(s) \geq C\, \phi(t)$ [$\phi(s) \leq C\, \phi(t)$].

One may observe that for any two values of k, the norms are mutually equivalent; and thus $\mathcal{M}^{p,\phi}(k_1, \mu) \approx \mathcal{M}^{p,\phi}(k_2, \mu)$ for any $k_1, k_2 > 1$. Since the value of k does not matter, we write $\mathcal{M}^{p,\phi}(\mu)$ for $\mathcal{M}^{p,\phi}(k, \mu)$, where k

may vary from one part to another. Furthermore, if $1 < p < q < \infty$, then $\|f : \mathcal{M}^{1,\phi}(\mu)\| \leq \|f : \mathcal{M}^{p,\phi}(\mu)\| \leq \|f : \mathcal{M}^{q,\phi}(\mu)\|$. Accordingly, we have $\mathcal{M}^{q,\phi}(\mu) \subseteq \mathcal{M}^{p,\phi}(\mu) \subseteq \mathcal{M}^{1,\phi}(\mu)$.

We assume, from now on, that $r^{1/p}\phi(r)$ is almost increasing, and that there exists a positif constant C such that $\int_r^\infty t^{\alpha/n-1}\phi(t)dt \leq Cr^{\alpha/n}\phi(r)$ for every $r > 0$. Then we have the following theorem.

Theorem 2.1. *Let $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Assume that $\psi : (0,\infty) \to (0,\infty)$ is almost decreasing, and there exists a constant $C > 0$ (which is independent of r) such that $r^{\alpha/n}\phi(r) \leq C\psi(r)$. Then, we have*

$$\|I_\alpha f : \mathcal{M}^{q,\psi}(\mu)\| \leq C\,\|f : \mathcal{M}^{p,\phi}(\mu)\|,$$

that is, I_α is bounded from $\mathcal{M}^{p,\phi}(\mu)$ to $\mathcal{M}^{q,\psi}(\mu)$.

Proof. Let $Q \in \mathcal{Q}(\mu)$ be fixed. For $f \in \mathcal{M}^{p,\phi}(\mu)$, we set $f = f_1 + f_2 = f\chi_{K\,Q} + f\chi_{\mathbb{R}^d\setminus K\,Q}$, where K is sufficiently large. It follows that $f_1 \in L^p(\mu)$ with the norm

$$\|f_1 : L^p(\mu)\| \leq [\mu(2KQ)]^{1/p}\phi(\mu(2KQ))\,\|f : \mathcal{M}^{p,\phi}(\mu)\|.$$

As a consequence, we have

$$\begin{aligned}\left(\frac{1}{\mu(2KQ)}\int_Q |I_\alpha f_1(x)|^q d\mu(y)\right)^{1/q}&\\ &\leq \frac{1}{[\mu(2KQ)]^{1/q}}\,\|I_\alpha f_1 : L^q(\mu)\|\\ &\leq \frac{C}{[\mu(2KQ)]^{1/q}}\,\|f_1 : L^p(\mu)\|\\ &\leq C[\mu(2KQ)]^{1/p-1/q}\phi(\mu(2KQ))\,\|f : \mathcal{M}^{p,\phi}(\mu)\|\\ &\leq C\psi(\mu(2KQ))\,\|f : \mathcal{M}^{p,\phi}(\mu)\|.\end{aligned}$$

To estimate $I_\alpha f_2$, fix $x \in Q$. Observe that we can find $\lambda > 1$ such that for each $y \in \mathbb{R}^d \setminus K\,Q$, we have $\lambda R \supseteq Q$ whenever $R \supseteq \{x, y, c_Q\}$. (Note that the value of λ is relatively small when K is large.) With this λ, we have

$$\frac{1}{|x-y|^{n-\alpha}} \leq C_\lambda \sup_R\,[\mu(\lambda R)]^{\alpha/n-1}$$

for some constant C_λ independent of x and y. Here the supremum is taken over all $R \in \mathcal{Q}(\mu)$ such that $x, y, c_Q \in R$. Furthermore, let us set

$$A_j := \left\{y \in \mathbb{R}^d \setminus K\,Q : 2^j < \inf_R \mu(\lambda R) \leq 2^{j+1}\right\}.$$

Then, by using the covering lemma of Sawano [12, Lemma 12], there exist $Q_j^{(1)}, \dots, Q_j^{(N)}$ such that $A_j \subset \bigcup_{m=1}^{N} \sqrt{\lambda} Q_j^{(m)}$ and $2^j < \mu(\lambda Q_j^{(m)}) \le 2^{j+1}$; and so we get the pointwise estimate for $I_\alpha f_2$:

$$\begin{aligned}
|I_\alpha f_2(x)| &= \int_{\mathbb{R}^d \setminus KQ} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, d\mu(y) \\
&\le C \sum_{\substack{j\in\mathbb{Z} \\ A_j \ne \emptyset}} 2^{(\alpha/n-1)j} \int_{A_j} |f(y)|\, d\mu(y) \\
&\le C \sum_{\substack{j\in\mathbb{Z} \\ A_j \ne \emptyset}} \sum_{m=1}^{N} 2^{(\alpha/n-1)j} \int_{\sqrt{\lambda} Q_j^{(m)}} |f(y)|\, d\mu(y) \\
&\le C \sum_{\substack{j\in\mathbb{Z} \\ A_j \ne \emptyset}} \sum_{m=1}^{N} 2^{j\alpha/n} \phi(2^j)\, \|f : \mathcal{M}^{1,\phi}(\mu)\|.
\end{aligned}$$

Recall that $\mu(Q) \le \mu(\lambda R)$ for every $R \supseteq \{x, y, c_Q\}$. So, if $A_j \ne \emptyset$, then $\mu(Q) \le \inf_R \mu(\lambda R) \le 2^{j+1}$. Since $2^{j\alpha/n}\phi(2^j) \le C \int_{2^{j-1}}^{2^j} t^{\alpha/n-1}\phi(t)dt$, we obtain

$$\begin{aligned}
\sum_{\substack{j\in\mathbb{Z} \\ A_j \ne \emptyset}} 2^{j\alpha/n}\phi(2^j) &\le C \sum_{\substack{j\in\mathbb{Z} \\ A_j \ne \emptyset}} \int_{2^{j-1}}^{2^j} t^{\alpha/n-1}\phi(t)\, dt \\
&\le C \int_{2^{-2}\mu(Q)}^{\infty} t^{\alpha/n-1}\phi(t)\, dt \\
&\le C \left(2^{-2}\mu(Q)\right)^{\alpha/n} \phi\left(2^{-2}\mu(Q)\right).
\end{aligned}$$

It then follows that

$$\begin{aligned}
&\frac{1}{\psi(\mu(2KQ))} \left(\frac{1}{\mu(2KQ)} \int_Q |I_\alpha f_2(x)|^q d\mu(y) \right)^{1/q} \\
&\le C\, \|f : \mathcal{M}^{1,\phi}(\mu)\| \frac{\left(2^{-2}\mu(Q)\right)^{\alpha/n} \phi\left(2^{-2}\mu(Q)\right) \left(2^{-2}\mu(Q)\right)^{1/q}}{\mu(2KQ)^{\alpha/n}\phi(\mu(2KQ))\mu(2KQ)^{1/q}} \\
&\le C\, \|f : \mathcal{M}^{1,\phi}(\mu)\| \frac{\left(2^{-2}\mu(Q)\right)^{1/p} \phi\left(2^{-2}\mu(Q)\right)}{\mu(2KQ)^{1/p}\phi(\mu(2KQ))} \\
&\le C\, \|f : \mathcal{M}^{p,\phi}(\mu)\|
\end{aligned}$$

because $r^{1/p}\phi(r)$ is almost increasing. Now, by the virtue of Minkowski's inequality, we get

$$\frac{1}{\psi(\mu(2KQ))}\left(\frac{1}{\mu(2KQ)}\int_Q |I_\alpha f(x)|^q d\mu(y)\right)^{1/q}$$
$$\leq \frac{1}{\psi(\mu(2KQ))}\left(\frac{1}{\mu(2KQ)}\int_Q |I_\alpha f_(x)|^q d\mu(y)\right)^{1/q} +$$
$$\frac{1}{\psi(\mu(2KQ))}\left(\frac{1}{\mu(2KQ)}\int_Q |I_\alpha f_2(x)|^q d\mu(y)\right)^{1/q}$$
$$\leq C\,\|f : \mathcal{M}^{p,\phi}(\mu)\|,$$

which yields the desired result. □

As a consequence of Theorem 2.1, we obtain an Olsen inequality involving a multiplication operator W.

Theorem 2.2. *Let* $1 < p < \frac{n}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *If* $r^{\alpha/n}\phi(r)$ *is almost decreasing, then*

$$\|WI_\alpha f : \mathcal{M}^{p,\phi}(\mu)\| \leq C\|W : L^{n/\alpha}(\mu)\|\|f : \mathcal{M}^{p,\phi}(\mu)\|,$$

provided that $W \in L^{n/\alpha}(\mu)$.

Proof. For every cube Q, we use Hölder's inequality to get

$$\left(\frac{1}{(\mu(kQ))}\int_Q |WI_\alpha f(x)|^p \, d\mu(x)\right)^{1/p}$$
$$\leq \left(\frac{1}{(\mu(kQ))}\int_Q |W(x)|^{n/\alpha} \, d\mu(x)\right)^{\alpha/n}\left(\frac{1}{(\mu(kQ))}\int_Q |I_\alpha f(x)|^q \, d\mu(x)\right)^{1/q}.$$

If $\psi(r) = r^{\alpha/n}\phi(r)$, then we could apply Theorem 2.1 to obtain

$$\frac{1}{\phi(\mu(kQ))}\left(\frac{1}{\mu(kQ)}\int_Q |WI_\alpha f(x)|^p \, d\mu(x)\right)^{1/p}$$
$$\leq \left(\int_Q |W(x)|^{n/\alpha} \, d\mu(x)\right)^{\alpha/n} \times$$
$$\frac{1}{\psi(\mu(kQ))}\left(\frac{1}{\mu(kQ)}\int_Q |I_\alpha f(x)|^q \, d\mu(x)\right)^{1/q}$$
$$\leq C\|W : L^{n/\alpha}(\mu)\|\|I_\alpha f : \mathcal{M}^{p,\phi}(\mu)\|$$
$$\leq C\|W : L^{n/\alpha}(\mu)\|\|f : \mathcal{M}^{p,\phi}(\mu)\|.$$

Now, by taking the supremum over all Q, the desired inequality follows. □

`Remark.` In the case that $\phi(t) = t^{-1/p}$, we have $\mathcal{M}^{p,\phi}(k,\mu) = L^p(\mu)$. As a result, we obtain the Olsen inequality on Lebesgue spaces, which tells us that WI_α is a bounded operator on $L^p(\mu)$.

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References

[1] D. Adams, 'A note on Riesz potentials', *Duke Math. J.* **42** (1975), 765–778.

[2] F. Chiarenza and M. Frasca, 'Morrey spaces and Hardy-Littlewood maximal function', *Rend. Mat.* **7** (1987), 273–279.

[3] J. García-Cuerva and J.M. Martell, 'Two weight norm inequalities for maximal operators and fractional integrals on non-homogeneous spaces', *Indiana Univ. Math. J.* **50** (2001), no. 3, 1241–1280.

[4] H. Gunawan and Eridani, 'Fractional integrals and generalized Olsen inequalities', to appear in *Kyungpook Math. J.*

[5] G.H. Hardy and J.E. Littlewood, 'Some properties of fractional integrals. I', *Math. Zeit.* **27** (1927), 565–606.

[6] G.H. Hardy and J.E. Littlewood, 'Some properties of fractional integrals. II', *Math. Zeit.* **34** (1932), 403–439.

[7] K. Kurata, S. Nishigaki and S. Sugano, 'Boundedness of integral operators on generalized Morrey spaces and its application to Schrödinger operators', *Proc. Amer. Math. Soc.* **128** (2002), 1125–1134.

[8] E. Nakai, 'Hardy-Littlewood maximal operator, singular integral operators, and the Riesz potentials on generalized Morrey spaces', *Math. Nachr.* **166** (1994), 95–103.

[9] F. Nazarov, S. Treil and A. Volberg, 'Weak type estimates and Cotlar inequalities for Calderón-Zygmund operators on nonhomogeneous spaces', *Internat. Math. Res. Notices* (1998), no. 9, 463–487.

[10] P.A. Olsen, 'Fractional integration, Morrey spaces and a Schrödinger equation', *Comm. Partial Differential Equations* **20** (1995), no. 11–12, 2005–2055.

[11] J. Peetre, 'On the theory of $\mathcal{L}_{p,\lambda}$ spaces', *J. Func. Anal.* **4** (1969), 71–87.

[12] Y. Sawano, 'l^q-valued extention of the fractional maximal operators for non-doubling measures via potential operators', *Int. J. Pure Appl. Math.* **26** (2006), no. 4, 505–523.

[13] Y. Sawano, T. Sobukawa and H. Tanaka, 'Limiting case of the boundedness of fractional integral operators on nonhomogeneous space', *J. Inequal. Appl.* **2006**, Art. ID 92470, 16pp.

[14] Y. Sawano, 'Generalized Morrey spaces for non-doubling measures', to appear in *Nonlinear Differential Equation Appl.*

[15] I. Sihwaningrum, H. P. Suryawan and H. Gunawan, 'Fractional integral operators and Olsen inequalities on non-homogeneous spaces', to appear in *Aust. J. Math. Anal. Appl.*

[16] S.L. Sobolev, 'On a theorem in functional analysis' (Russian), *Mat. Sob.* **46** (1938), 471–497 [English translation in *Amer. Math. Soc. Transl. ser. 2* **34** (1963), 39–68].

[17] X. Tolsa, 'BMO, H^1, and Calderón-Zygmund operators for non doubling measures', *Math. Ann.* **319** (2001), 89–149.

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