Fractional integral operators on generalized Morrey spaces of non-homogeneous type

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Abstract
We prove here the boundedness of the fractional integral operator $I_\alpha$ on generalized Morrey spaces in the non-homogeneous setting, which is analogous to Nakai’s result [8] in the homogeneous case. Our proof makes use of the covering lemma of Sawano [12]. As a consequence of our result, we obtain an Olsen inequality for a multiplication operator involving $I_\alpha$ in the non-homogeneous setting.

Keywords: Fractional integral operators, Olsen inequality, non-homogeneous spaces, generalized Morrey spaces.

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1 Introduction
Suppose that $\mathbb{R}^d$ is equipped with a non-negative Radon measure $\mu$ satisfying the growth condition, that is, there exists a constant $C > 0$ and $0 < n \leq d$ such that

$$\mu(Q) \leq C \ell^n$$

for every $d$-dimensional cube $Q$ with center $c_Q \in \mathbb{R}^d$ and side length $\ell > 0$. Here we consider only the cubes with sides parallel to the coordinate axes.

It has been widely known that the growth condition replaces the role of the doubling condition in homogeneous spaces. A non-negative measure $\mu$ satisfies the doubling condition if there exists a constant $C > 0$ such that for every cube $Q$ with side length $\ell$ we have

$$\mu(2Q) \leq C \mu(Q),$$

where $2Q$ denotes the cube concentric to $Q$ with side length $2\ell$.

The analysis on non-homogeneous spaces has been developed since the works of Nazarov et al. [9] and Tolsa [17]. Their success — in replacing the doubling condition by the growth condition — has inspired other people to work on spaces of non-homogeneous type.

Our object of study here is the fractional integral operator $I_\alpha$, which is defined — in the non-homogeneous setting — by

$$I_\alpha f(x) := \int_{\mathbb{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}} d\mu(y),$$

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where $0 < \alpha < n \leq d$. Notice that when $n = d$ and $\mu$ is the usual Lebesgue measure, we recover the classical fractional integral operator introduced by Hardy and Littlewood [5, 6] and Sobolev [16]. See [1, 2, 8, 11] and many other literatures for various results on fractional integral operators in the classical version.

In [3], García-Cuerva and Martell studied the boundedness of $I_\alpha$ on Lebesgue spaces of non-homogeneous type. In particular, they obtained the following result. (Note that throughout the paper, we denote the norm of $f$ in the space $X$ by $\|f : X\|$.)

**Theorem 1.1.** [3] If $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, then there exists a constant $C > 0$ such that

$$\|I_\alpha f : L^q(\mu)\| \leq C \|f : L^p(\mu)\|,$$

that is, $I_\alpha$ is bounded from $L^p(\mu)$ to $L^q(\mu)$.

Our goal in this paper is to prove the boundedness of $I_\alpha$ on generalized Morrey spaces of non-homogeneous type, which is analogous to Nakai’s result [8] in the homogeneous setting. As a consequence, we obtain an Olsen inequality for a multiplication operator involving $I_\alpha$ in the non-homogeneous version. For related results, see [4, 13, 14].

The inequality was first introduced — in the homogeneous setting — by Olsen [10]; and was generalized later by Kurata et al. [7]. They used the inequality to study the behaviour of the solution to a Schrödinger equation with a small perturbed potential $W$. Notice that $I_\alpha = (-\Delta)^{-\alpha/2}$, where $-\Delta$ is the Laplacian operator.

## 2 The boundedness of $I_\alpha$ and an Olsen inequality

For $k > 1$, let $kQ$ denote a cube concentric to $Q$ with side length $k$ times side length of $Q$; $Q(\mu)$ is the set of all cubes with positive $\mu$-measure; and $C$ stands for a constant which may differ from one line to another. For $1 \leq p < \infty$ and an *almost decreasing* function $\phi : (0, \infty) \to (0, \infty)$, we define the generalized non-homogeneous Morrey spaces $\mathcal{M}^{p, \phi}(k, \mu) = \mathcal{M}^{p, \phi}(\mathbb{R}^d, k, \mu)$ to be the set of all $f \in L^p_{\text{loc}}(\mu)$ such that

$$\|f : \mathcal{M}^{p, \phi}(k, \mu)\| := \sup_Q \frac{1}{\phi(\mu(kQ))} \left( \frac{1}{\mu(kQ)} \int_Q |f(y)|^p \, d\mu(y) \right)^{1/p} < \infty.$$

Another version of such spaces can be found in [15]. Recall that $\phi : (0, \infty) \to (0, \infty)$ is almost decreasing [almost increasing] if there exists a constant $C$ such that for $s < t$, we have $\phi(s) \geq C \phi(t)$ [$\phi(s) \leq C \phi(t)$].

One may observe that for any two values of $k$, the norms are mutually equivalent; and thus $\mathcal{M}^{p, \phi}(k_1, \mu) \approx \mathcal{M}^{p, \phi}(k_2, \mu)$ for any $k_1, k_2 > 1$. Since the value of $k$ does not matter, we write $\mathcal{M}^{p, \phi}(\mu)$ for $\mathcal{M}^{p, \phi}(k, \mu)$, where $k$
may vary from one part to another. Furthermore, if \(1 < p < q < \infty\), then
\[
\| f : \mathcal{M}^1,\phi(\mu) \| \leq \| f : \mathcal{M}^p,\phi(\mu) \| \leq \| f : \mathcal{M}^q,\phi(\mu) \|.
\]
Accordingly, we have
\[
\mathcal{M}^q,\phi(\mu) \subseteq \mathcal{M}^p,\phi(\mu) \subseteq \mathcal{M}^1,\phi(\mu).
\]
We assume, from now on, that \(r^{1/p}\phi(r)\) is almost increasing, and that there exists a positive constant \(C\) such that \(\int_r^\infty t^{\alpha/n-1}\phi(t)dt \leq Cr^{\alpha/n}\phi(r)\) for every \(r > 0\). Then we have the following theorem.

**Theorem 2.1.** Let \(1 < p < \frac{n}{\alpha}\) and \(\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}\). Assume that \(\psi : (0, \infty) \rightarrow (0, \infty)\) is almost decreasing, and there exists a constant \(C > 0\) (which is independent of \(r\)) such that \(r^{\alpha/n}\phi(r) \leq C\psi(r)\). Then, we have
\[
\| I_\alpha f : \mathcal{M}^q,\psi(\mu) \| \leq C \| f : \mathcal{M}^p,\phi(\mu) \|,
\]
that is, \(I_\alpha\) is bounded from \(\mathcal{M}^p,\phi(\mu)\) to \(\mathcal{M}^q,\psi(\mu)\).

**Proof.** Let \(Q \in \mathcal{Q}(\mu)\) be fixed. For \(f \in \mathcal{M}^p,\phi(\mu)\), we set
\[
f = f_1 + f_2 = f_\chi_{2KQ} + f_\chi_{\mathbb{R}^d \setminus KQ},
\]
where \(K\) is sufficiently large. It follows that \(f_1 \in L^p(\mu)\) with the norm
\[
\| f_1 : L^p(\mu) \| \leq [\mu(2KQ)]^{1/p}\phi(\mu(2KQ)) \| f : \mathcal{M}^p,\phi(\mu) \|.
\]
As a consequence, we have
\[
\left( \frac{1}{\mu(2KQ)} \int_Q |I_\alpha f_1(x)|^q d\mu(y) \right)^{1/q} \leq \frac{1}{[\mu(2KQ)]^{1/q}} \| I_\alpha f_1 : L^q(\mu) \|
\leq \frac{C}{[\mu(2KQ)]^{1/q}} \| f_1 : L^p(\mu) \|
\leq C\psi(\mu(2KQ)) \| f : \mathcal{M}^p,\phi(\mu) \|.
\]

To estimate \(I_\alpha f_2\), fix \(x \in Q\). Observe that we can find \(\lambda > 1\) such that for each \(y \in \mathbb{R}^d \setminus KQ\), we have \(\lambda R \supseteq Q\) whenever \(R \supseteq \{x, y, cQ\}\). (Note that the value of \(\lambda\) is relatively small when \(K\) is large.) With this \(\lambda\), we have
\[
\frac{1}{|x - y|^{n-\alpha}} \leq C_\lambda \sup_R \mu(\lambda R)^{\alpha/n-1}
\]
for some constant \(C_\lambda\) independent of \(x\) and \(y\). Here the supremum is taken over all \(R \in \mathcal{Q}(\mu)\) such that \(x, y, cQ \in R\). Furthermore, let us set
\[
A_j := \left\{ y \in \mathbb{R}^d \setminus KQ : 2^j < \inf_R \mu(\lambda R) \leq 2^{j+1} \right\}.
\]
Then, by using the covering lemma of Sawano [12, Lemma 12], there exist \(Q_j^{(1)}, \ldots, Q_j^{(N)}\) such that \(A_j \subset \bigcup_{m=1}^{N} \sqrt[2]{Q_j^{(m)}}\) and \(2^j < \mu(\lambda Q_j^{(m)}) \leq 2^{j+1}\); and so we get the pointwise estimate for \(I_{\alpha} f_2\):

\[
|I_{\alpha} f_2(x)| = \int_{\mathbb{R}^d \setminus KQ} \frac{|f(y)|}{|x-y|^{n-\alpha}} \, d\mu(y) \\
\leq C \sum_{j \in \mathbb{Z}} 2^{(\alpha/n-1)j} \int_{A_j} |f(y)| \, d\mu(y) \\
\leq C \sum_{j \in \mathbb{Z}} \sum_{m=1}^{N} 2^{(\alpha/n-1)j} \int_{\sqrt[2]{Q_j^{(m)}}} |f(y)| \, d\mu(y) \\
\leq C \sum_{j \in \mathbb{Z}} \sum_{m=1}^{N} 2^{(\alpha/n)j} \phi(2^{j}) \|f : M^{1,\phi}(\mu)\|.
\]

Recall that \(\mu(Q) \leq \mu(\lambda R)\) for every \(R \supseteq \{x, y, c_Q\}\). So, if \(A_j \neq \emptyset\), then \(\mu(Q) \leq \inf_R \mu(\lambda R) \leq 2^{j+1}\). Since \(2^{(\alpha/n)j} \phi(2^{j}) \leq C \int_{2^{j-1}}^{2^j} t^{\alpha/n-1} \phi(t) \, dt\), we obtain

\[
\sum_{j \in \mathbb{Z}} 2^{(\alpha/n)j} \phi(2^{j}) \leq C \sum_{j \in \mathbb{Z}} \int_{2^{j-1}}^{2^j} t^{\alpha/n-1} \phi(t) \, dt \\
\leq C \int_{2^{j-1} \mu(Q)}^{\infty} t^{\alpha/n-1} \phi(t) \, dt \\
\leq C \left(2^{-2} \mu(Q)\right)^{\alpha/n} \phi \left(2^{-2} \mu(Q)\right).
\]

It then follows that

\[
\frac{1}{\psi(\mu(2KQ))} \left(\frac{1}{\mu(2KQ)} \int_{Q} |I_{\alpha} f_2(x)|^q \, d\mu(y)\right)^{1/q} \\
\leq C \|f : M^{1,\phi}(\mu)\| \frac{(2^{-2} \mu(Q))^{\alpha/n} \phi \left(2^{-2} \mu(Q)\right) (2^{-2} \mu(Q))^{1/q}}{\mu(2KQ)^{\alpha/n} \phi(\mu(2KQ)) \mu(2KQ)^{1/q}} \\
\leq C \|f : M^{1,\phi}(\mu)\| \frac{(2^{-2} \mu(Q))^{1/p} \phi \left(2^{-2} \mu(Q)\right)}{\mu(2KQ)^{1/p} \phi(\mu(2KQ))} \\
\leq C \|f : M^{p,\phi}(\mu)\|
\]

because \(t^{1/p} \phi(r)\) is almost increasing. Now, by the virtue of Minkowski’s
inequality, we get
\[
\frac{1}{\psi(\mu(2KQ))} \left( \frac{1}{\mu(2KQ)} \int_Q |I_{\alpha} f(x)|^q d\mu(y) \right)^{1/q} \leq \frac{1}{\psi(\mu(2KQ))} \left( \frac{1}{\mu(2KQ)} \int_Q |I_{\alpha} f_1(x)|^q d\mu(y) \right)^{1/q} + \frac{1}{\psi(\mu(2KQ))} \left( \frac{1}{\mu(2KQ)} \int_Q |I_{\alpha} f_2(x)|^q d\mu(y) \right)^{1/q} \leq C \| f : \mathcal{M}^{p,\phi}(\mu) \|,
\]
which yields the desired result.

As a consequence of Theorem 2.1, we obtain an Olsen inequality involving a multiplication operator $W$.

**Theorem 2.2.** Let $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. If $r^{\alpha/n} \phi(r)$ is almost decreasing, then
\[
\| WI_{\alpha} f : \mathcal{M}^{p,\phi}(\mu) \| \leq C \| W : L^{n/\alpha}(\mu) \| \| f : \mathcal{M}^{p,\phi}(\mu) \|,
\]
provided that $W \in L^{n/\alpha}(\mu)$.

**Proof.** For every cube $Q$, we use Hölder’s inequality to get
\[
\left( \frac{1}{\mu(kQ)} \int_Q |W I_{\alpha} f(x)|^p d\mu(x) \right)^{1/p} \leq \left( \frac{1}{\mu(kQ)} \int_Q |W(x)|^{n/\alpha} d\mu(x) \right)^{\alpha/n} \left( \frac{1}{\mu(kQ)} \int_Q |I_{\alpha} f(x)|^q d\mu(x) \right)^{1/q}.
\]
If $\psi(r) = r^{\alpha/n} \phi(r)$, then we could apply Theorem 2.1 to obtain
\[
\frac{1}{\phi(\mu(kQ))} \left( \frac{1}{\mu(kQ)} \int_Q |W I_{\alpha} f(x)|^p d\mu(x) \right)^{1/p} \leq \left( \int_Q |W(x)|^{n/\alpha} d\mu(x) \right)^{\alpha/n} \times \left( \frac{1}{\mu(kQ)} \int_Q |I_{\alpha} f(x)|^q d\mu(x) \right)^{1/q} \leq C \| W : L^{n/\alpha}(\mu) \| \| I_{\alpha} f : \mathcal{M}^{p,\phi}(\mu) \| \leq C \| W : L^{n/\alpha}(\mu) \| \| f : \mathcal{M}^{p,\phi}(\mu) \|.
\]
Now, by taking the supremum over all $Q$, the desired inequality follows. \qed
Remark. In the case that $\phi(t) = t^{-1/p}$, we have $M^{p,\phi}(k, \mu) = L^p(\mu)$. As a result, we obtain the Olsen inequality on Lebesgue spaces, which tells us that $WI_\alpha$ is a bounded operator on $L^p(\mu)$.

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