Proceedings of IICMA 2009
Topic, pp. xx–xx.

# FRACTIONAL INTEGRAL OPERATORS ON LEBESGUE AND MORREY SPACES

Hendra Gunawan and Idha Sihwaningrum

**Abstract.** In this survey paper, we will present some results on fractional integral operators, also known as the Riesz potentials, especially their boundedness property on Lebesgue spaces and (generalized) Morrey spaces, including in the non-homogeneous setting. In addition, some results on related operators — such as the Hardy-Littlewood maximal operator — and some applications will be pointed out.

*Key words and Phrases*: Fractional integral operators, Lebesgue spaces, Morrey spaces, non-homogeneous spaces

## 1. FRACTIONAL INTEGRAL OPERATORS

In harmonic analysis, there are several main operators studied extensively. Among these operators are the Hardy-Littlewood maximal operator, singular integral operators, fractional integral operators, and fractional maximal operators. In this paper, we shall focus on fractional integral operators and their boundedness on Lebesgue and Morrey spaces (on $\mathbb{R}^n$ and on non-homogeneous spaces).

For $0 < \alpha < n$, the (classical) fractional integral operator $I_\alpha$, also known as the *Riesz potential* (of degree $\alpha$) is defined by

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy.$$

Note that the integral makes sense when, for instance, $f$ is bounded and compactly supported.

The operator $I_\alpha$ was first studied by Hardy and Littlewood in the 1920's [13, 15] and extended by Sobolev [33] in the 1930's. A well-known result for $I_\alpha$ is the Hardy-Littlewood-Sobolev inequality.

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*2000 Mathematics Subject Classification*: 42B20, 42B35, 47G10, 31B10, 26A33

**Theorem 1.1.** [Hardy-Littlewood; Sobolev] *For $1 < p < \frac{n}{\alpha}$, we have the inequality*

$$\|I_\alpha f\|_q \le C_p \|f\|_p, \tag{1}$$

*that is, $I_\alpha$ is bounded from Lebesgue spaces $L^p$ to $L^q$, provided that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.*

Note that there is also a weak version of this result, but we shall not discuss it here.

Through its Fourier transform, the operator $I_\alpha$ can be recognized as a multiple of the Laplacian to the power of $-\frac{\alpha}{2}$, that is,

$$I_\alpha f = \kappa(n,\alpha).(-\Delta)^{-\frac{\alpha}{2}} f.$$

As an immediate consequence of the Hardy-Littlewood-Sobolev inequality (1), one has the following estimate for the Newtonian potential $(-\Delta)^{-1}$:

$$\|(-\Delta)^{-1} f\|_{np/(n-2)} \le C_p \|f\|_p,$$

for $1 < p < \frac{n}{2}$, $n \ge 3$. Here $u := (-\Delta)^{-1} f$ is a solution of the Poisson equation $-\Delta u = f$. From (1) we can also prove Sobolev's embedding theorems (see [34]).

Around the 1970's, the Hardy-Littlewood-Sobolev inequality is extended from Lebesgue spaces to Morrey spaces. For $1 \le p < \infty$ and $0 \le \lambda \le n$, the (classical) Morrey space $L^{p,\lambda} = L^{p,\lambda}(\mathbb{R}^n)$ is defined to be the space of all functions $f \in L^p_{\rm loc}(\mathbb{R}^n)$ for which

$$\|f\|_{p,\lambda} := \sup_{B=B(a,r)} \Big( \frac{1}{r^\lambda} \int_B |f(y)|^p \, dy \Big)^{1/p} < \infty,$$

where $B(a,r)$ denotes the (open) ball centered at $a \in \mathbb{R}^n$ with radius $r > 0$ [18]. Here $\|\cdot\|_{p,\lambda}$ defines a semi-norm on $L^{p,\lambda}$. Note particularly that $L^{p,0} = L^p$ and $L^{p,n} = L^\infty$. For the structure of Morrey spaces and their generalisations, see the works of Campanato [3], Peetre [25], Zorko [38], and the references therein.

As stated in [25], Spanne proved that $I_\alpha$ is bounded from $L^{p,\lambda}$ to $L^{q,\lambda q/p}$ for $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $0 \le \lambda < n$. Later on, a stronger result was obtained by Adams [1], and reproved by Chiarenza and Frasca [4].

**Theorem 1.2.** [Adams; Chiarenza-Frasca] *For $1 < p < \frac{n}{\alpha}$ and $0 \le \lambda < n - \alpha p$, we have the inequality*

$$\|I_\alpha f\|_{q,\lambda} \le C_{p,\lambda} \|f\|_{p,\lambda}$$

*provided that $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$.*

The proof usually involves the properties of the Hardy-Littlewood maximal operator $M$, defined by the formula

$$Mf(x) := \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)| \, dy, \tag{2}$$

where $|B(x,r)| = c\,r^n$ is the Lebesgue measure of $B(x,r)$. As the operator $M$ is known to be bounded on $L^p$ for $1 < p \le \infty$ (see [14]), Chiarenza and Frasca [4] proved that $M$ is bounded on Morrey spaces.

## 2. THE BOUNDEDNESS ON GENERALISED MORREY SPACES

For $1 \le p < \infty$ and a suitable function $\phi : (0,\infty) \to (0,\infty)$, we define the generalised Morrey space $\mathcal{M}_{p,\phi} = \mathcal{M}_{p,\phi}(\mathbf{R}^n)$ to be the space of all functions $f \in L^p_{\mathrm{loc}}(\mathbf{R}^n)$ for which

$$\|f\|_{p,\phi} := \sup_{B=B(a,r)} \frac{1}{\phi(r)} \left( \frac{1}{|B|} \int_B |f(y)|^p dy \right)^{1/p} < \infty.$$

Note that if $\phi(t) = t^{(\lambda-n)/p}$, $0 \le \lambda \le n$, we have $\mathcal{M}_{p,\phi} = L^{p,\lambda}$ — the classical Morrey space. Unless otherwise stated, $\phi$ satisfies the following two conditions:

(2.1) $\quad \frac{1}{2} \le \frac{r}{s} \le 2 \Rightarrow \frac{1}{C_1} \le \frac{\phi(r)}{\phi(s)} \le C_1$ (*the doubling condition*);

(2.2) $$\int_r^\infty \frac{\phi^p(t)}{t}\, dt \le C_2\, \phi^p(r) \text{ for } 1 < p < \infty.$$

For any function $\phi$ that satisfies the doubling condition, we have

$$\int_{2^k r}^{2^{k+1} r} \frac{\phi(t)}{t}\, dt \sim \phi(2^k r)$$

for every integer $k$ and $r > 0$.

In 1994, Nakai [19] proved the boundedness of the Hardy- Littlewood maximal operator on generalised Morrey spaces.

**Theorem 2.1.** [Nakai] *The inequality*

$$\|Mf\|_{p,\phi} \le C_{p,\phi} \|f\|_{p,\phi}$$

*holds for* $1 < p < \infty$.

With the use of Theorem 2.1, Nakai [19] obtained the following result for $I_\alpha$.

**Theorem 2.2.** [Nakai] *Suppose that* $\psi : (0,\infty) \to (0,\infty)$ *satisfies the doubling condition and that* $\int_r^\infty t^{\alpha-1}\phi(t)\, dt \le C r^\alpha \phi(r) \le C\psi(r)$. *Then, for* $1 < p < q < \infty$, *we have*

$$\|I_\alpha f\|_{q,\psi} \le C \|f\|_{p,\phi},$$

*whenever* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

This result can be viewed as an extension of Spanne's result. Meanwhile, the following theorem can be considered as an extension of Adams-Chiarenza-Frasca's result.

**Theorem 2.3.** [Gunawan and Eridani [11]] *Suppose that, in addition to the condition* (2.1) *and* (2.2), $\phi$ *satisfies the inequality* $\phi(t) \le Ct^{\beta}$ *for* $-\frac{n}{p} \le \beta < -\alpha$, $1 < p < \frac{n}{\alpha}$. *Then, we have*

$$\|I_\alpha f\|_{q,\phi^{p/q}} \le C_{p,\beta}\|f\|_{p,\phi}$$

*where* $q = \frac{\beta p}{\alpha+\beta}$.

Observe that when $\phi(t) = t^{(\lambda-n)/p}$, $0 \le \lambda < n - \alpha p$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$, Theorem 2.3 reduces to Theorem 1.2.

## 3. GENERALISED FRACTIONAL INTEGRAL OPERATORS

For a given function $\rho : (0,\infty) \to (0,\infty)$, we define the (generalised) fractional integral operator $T_\rho$ by

$$T_\rho f(x) := \int_{\mathbf{R}^n} \frac{\rho(|x-y|)}{|x-y|^n} f(y)\, dy$$

For $\rho(t) = t^\alpha$, $0 < \alpha < n$, we have $T_\rho = I_\alpha$ — the classical fractional integral operator. The operator $T_\rho$ was first studied by Nakai, 2000 [20]. Recent results on $T_\rho$ can be found in [6, 7, 10, 21, 22].

A slight modification of Theorem 3.3 may be formulated for $T_\rho$ as follows.

**Theorem 3.1.** [Gunawan and Eridani [11]] *Suppose that* $\rho(t) \le C_1 t^\alpha$ *for some* $0 < \alpha < n$, *and, in addition to the condition* (2.1) *and* (2.2), $\phi$ *satisfies the inequality* $\phi(t) \le C_2 t^\beta$ *for* $-\frac{n}{p} \le \beta < -\alpha$, $1 < p < \frac{n}{\alpha}$. *Then, for* $q = \frac{\beta p}{\alpha+\beta}$, *we have*

$$\|T_\rho f\|_{q,\phi^{p/q}} \le C_{p,\beta}\|f\|_{p,\phi},$$

*that is,* $T_\rho$ *is bounded from* $\mathcal{M}_{p,\phi}$ *to* $\mathcal{M}_{q,\phi^{p/q}}$.

Another generalisation of Theorem 1.2 is the following result.

**Theorem 3.2.** [Gunawan [10]] *Suppose that, in addition to the condition* (2.1) *and* (2.2), $\phi$ *is surjective. If* $\rho$ *satisfies the doubling condition and*

$$\int_0^r \frac{\rho(t)}{t}\, dt \le C\phi(r)^{(p-q)/q} \quad \text{and} \quad \int_r^\infty \frac{\rho(t)\phi(t)}{t}\, dt \le C\phi(r)^{p/q},$$

*for* $1 < p < q < \infty$, *then we have*

$$\|T_\rho f\|_{q,\phi^{p/q}} \le C_{p,\phi}\|f\|_{p,\phi}.$$

## 4. OLSEN INEQUALITIES

In studying a Schrödinger equation

$$(-\Delta + V(x) + W(x))u(x) = f(x)$$

with perturbed potentials $W$ on $\mathbb{R}^n$ (particularly for $n = 3$), Olsen [24] proved the following result.

**Theorem 4.1.** [Olsen] *For $1 < p < \frac{n}{\alpha}$ and $0 \le \lambda < n - \alpha p$, we have*

$$\|W \cdot I_\alpha f\|_{p,\lambda} \le C_{p,\lambda} \|W\|_{(n-\lambda)/\alpha,\lambda} \|f\|_{p,\lambda},$$

*that is, $W \cdot I_\alpha$ is bounded on $L^{p,\lambda}$, provided that $W \in L^{(n-\lambda)/\alpha,\lambda}$.*

As a consequence of Theorem 4.1, we see that for $1 < p < \frac{n}{2}$, $n \ge 3$, the estimate

$$\|W \cdot (-\Delta)^{-1} f\|_{p,\lambda} \le C_{p,\lambda}\, \|W\|_{(n-\lambda)/2,\lambda} \|f\|_{p,\lambda},$$

holds provided that $W \in L^{(n-\lambda)/2,\lambda}$, $0 \le \lambda < n - 2p$. In particular, when $\lambda = 0$, one has

$$\|W \cdot (-\Delta)^{-1} f\|_p \le C_p\, \|W\|_{n/2} \|f\|_p$$

provided that $W \in L^{n/2}$.

In 2002, Kurata *et al.* [17] extended Olsen's result by proving that, for some $p > 1$ and a function $\phi$ satisfying several conditions (including the doubling condition), the operator $W \cdot I_\alpha$ is bounded on generalised Morrey spaces $\mathcal{M}_{p,\phi}$, provided that $W \in \mathcal{M}_{s_1,\phi} \cap \mathcal{M}_{s_2,\phi}$ for some indices $s_1$ and $s_2$. Their estimate, however, is rather complicated. We shall here present simpler estimates for $W \cdot I_\alpha$ on generalised Morrey spaces.

**Theorem 4.2.** [Gunawan and Eridani [11]] *Suppose that, in addition to the condition* (2.1) *and* (2.2)*, $\phi$ satisfies the inequality $\phi(t) \le Ct^\beta$ for $-\frac{n}{p} \le \beta < -\alpha$, $1 < p < \frac{n}{\alpha}$. Then, we have*

$$\|W \cdot I_\alpha f\|_{p,\phi} \le C_{p,\beta} \|W\|_{s,\phi^{p/s}} \|f\|_{p,\phi}$$

*provided that $W \in \mathcal{M}_{s,\phi^{p/s}}$ where $s = -\frac{\beta p}{\alpha}$.*

We use Hölder's inequality and Theorem 3.2 to prove Theorem 4.2. Another estimate for $W \cdot I_\alpha$ is provided by the following theorem — which its proof is found directly without using Theorem 3.2.

**Theorem 4.2.** [Gunawan and Eridani [11]] *Suppose that $\phi$ satisfies the doubling condition and the inequality*

$$\int_r^\infty t^{\alpha-1} \phi(t)\, dt \le C\, r^\alpha \phi(r).$$

*Then, for $1 < p < \frac{n}{\alpha}$, we have*

$$\|W \cdot I_\alpha f\|_{p,\phi} \le C_{p,\phi} \|W\|_{n/\alpha} \|f\|_{p,\phi},$$

*provided that $W \in L^{n/\alpha}$.*

## 5. THE BOUNDEDNESS ON NONHOMOGENEOUS SPACES

From now on, $\mathbb{R}^n$ will be replaced by $\mathbb{R}^d$ since the letter $n$ will be used for another purpose. For convenience, we shall also use cubes $Q(x, r)$ instead of balls $B(x, r)$. Let $\mathbb{R}^d$ be endowed with a (positive) Radon measure $\mu$ which satisfies the growth condition of order $n$ ($0 < n \le d$), that is there exists $C > 0$ such that

$$\mu(Q) \le C\, r^n$$

holds for any cube $Q \subseteq \mathbb{R}^d$ with side length $2r$. Here, the sides of $Q$ are parallel to the coordinate axes. Some researchers call such $(\mathbb{R}^d, \mu)$ the space of non-homogeneous type as it does not necessarily to satisfy the doubling condition:

$$\mu(Q(x, 2r)) \le C\, \mu(Q(x, r)),$$

which is the essential property of homogeneous spaces.

For almost three decades, the homogeneous spaces play an important role in the development of many theories in Harmonic Analysis. (See for example the work of [5, 35].) This is due to the fact that most of the central results in the Euclidean setting can be generalised without too much difficulties to the homogeneous setting – Verdera, 2002 [37]. In recent years, however, researchers found that many results still hold in non-homogeneous spaces, as confirmed by Nazarov et.al., [23], Tolsa, [36], and Garca-Cuerva and Gatto [8].

In the non-homogeneous context, Garcia-Cuerva and Martell [9] defined the fractional integral operator $I^n_\alpha$ ($0 < \alpha < n \le d$) by

$$I^n_\alpha f(x) := \int_{\mathbb{R}^d} \frac{f(y)}{|x - y|^{n-\alpha}}\, d\mu(y).$$

**Theorem 5.1.** [Garcia-Cuerva and Martell [9]] The operator $I^n_\alpha$ is bounded from $L^p(\mu)$ to $L^q(\mu)$ for $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$.

The Hardy-Littlewood maximal operator (2) is known to be unbounded on $L^p(\mu)$ (see [23]). The proof of the non-homogeneous version of Hardy-Littlewood-Sobolev inequality (in Theorem 5.1) employs the boundedness of the maximal operator $M^n$ on $L^p(\mu)$, where

$$M^n f(x) := \sup_{r>0} \frac{1}{r^n} \int_{B(x,r)} |f(y)|\, d\mu(y). \tag{3}$$

In order to extend the result further, Sawano [26] introduced the maximal operator $M_k$:

$$M_k f(x) := \sup_{Q \ni x} \frac{1}{\mu(k\,Q)} \int_Q f(y) d\mu(y)$$

for $k > 1$. Here, $k\,Q$ stands for the cube with the same centre as $Q$ and side length $2kr$. In line with the choice of the parameter $k$ in the definition $M_k$, Sawano and Tanaka [29] defined the Morrey spaces of non-homogeneous type using two parameters. Suppose that $1 \le s \le p < \infty$ and $\mathcal{Q}$ denotes the set of all cubes with positive $\mu$-measure. The Morrey space $L^{p,s}(k,\mu) = L^{p,s}(\mathbb{R}^d, k, \mu)$ is defined to be the space of all functions $f \in L^p_{\rm loc}(\mu)$ such that

$$\|f\|_{p,s,k,\mu} := \sup_{Q \in \mathcal{Q}(\mu)} (\mu(kQ))^{\frac{1}{p}-\frac{1}{s}} \Big[\int_Q |f(y)|^s \, d\mu(y)\Big]^{1/s} < \infty.$$

From the definition of this semi-norm, it is clear that $L^{p,p}(k,\mu) = L^p(\mu)$ and for $1 \le s_1 \le s_2 \le p < \infty$, we have $L^{p,p}(k,\mu) \subseteq L^{p,s_1}(k,\mu) \subseteq L^{p,s_2}(k,\mu)$. Moreover, for $k_1, k_2 > 1$, we have $L^{p,s}(k_1,\mu) \approx L^{p,s}(k_2,\mu)$, that is, $L^{p,s}(k_1,\mu)$ and $L^{p,s}(k_2,\mu)$ coincide as a set and their norm are mutually equivalent. The last property lead us to the following theorem on the fractional integral operator $I^n_\alpha$.

**Theorem 5.2.** [Sawano and Tanaka] *Let $1 < s \le p < \infty$, $1 < t \le q$, $\frac{t}{q} = \frac{s}{p}$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Then we have*

$$\|I^n_\alpha f\|_{q,t,k,\mu} \le C \|f\|_{p,s,k,\mu}.$$

The boundedness of $I^n_\alpha$ from $L^p(\mu)$ to $L^q(\mu)$ then follows from the above theorem when $s = p$ and $t = q$. We refer the reader to [28] for more results on the boundedness of $I^n_\alpha$ on non-homogeneous Morrey spaces.

We shall now state our results on the boundedness of $I^n_\alpha$ on the generalised non-homogeneous Morrey spaces — which are defined in two versions. The first version is defined suitable with the definition of $M^n$, while the second one is defined suitable with that of $M_k$. For $1 \le p < \infty$, define the (first version) generalized non-homogeneous Morrey space $M^{p,\phi}(\mu)$ to be the set of all $f \in L^p_{\rm loc}(\mu)$ such that

$$\|f\|_{p,\phi,\mu} := \sup_{r>0} \frac{1}{\phi(r)} \Big[\frac{1}{r^n} \int_{Q(x,r)} |f(y)|^p d\mu(y)\Big]^{1/p} < \infty.$$

Here we assume that $\phi$ satisfies the doubling condition.

**Theorem 5.3.** [Sihwaningrum, *et al.* [30]] *Suppose that $\psi : (0,\infty) \to (0,\infty)$ satisfies the doubling condition and that $\int_r^\infty t^{\alpha-1}\phi(t)dt \le C\, r^\alpha \phi(r) \le C\, \psi(r)$. Then*

$$\|I^n_\alpha f\|_{q,\psi,\mu} \le C\, \|f\|_{p,\phi,\mu},$$

*for* $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$

Now, for $k > 1$, define the (second version) generalized non-homogeneous Morrey space $M^{p,\phi}(k,\mu)$ to be the set of all $f \in L^p_{\rm loc}(\mu)$ such that

$$\|f\|_{p,\phi,k,\mu} := \sup_{Q \ni \mathcal{Q}(\mu)} \frac{1}{\phi(\mu(kQ))} \Big[\frac{1}{\mu(kQ)} \int_Q |f(y)|^p d\mu(y)\Big]^{1/p} < \infty.$$

Here we assume that $\phi$ is almost decreasing, that is, there exists a constant $C$ such that $\phi(s) \geq C\phi(t)$ for $s \leq t$.

**Theorem 5.4.** [Gunawan, *et al.* [12]] *Let* $a = \frac{\alpha}{n}$ *and* $\phi$ *be surjective and* $\phi(t) \leq C\, t^b$ *where* $-\frac{1}{p} \leq b \leq -a < 0$. *Then,* $I^n_\alpha$ *is bounded from* $M^{p,\phi}(k,\mu)$ *to* $M^{q,\phi^{p/q}}(k,\mu)$ *where* $p > 1$ *and* $q = \frac{bp}{a+b}$.

The above result is analogous to Gunawan's [10] in the classical case; and thus, it can be view as an extension of Adams-Chiarenza-Frasca's.

In companion to Theorem 5.4, we also have a result which can be considered as an extension of Spanne's. Assume now that $r^{1/p}\phi(r)$ is almost increasing, and that there exists a positive constant $C$ such that $\int_r^\infty t^{\alpha/n-1}\phi(t)dt \leq Cr^{\alpha/n}\phi(r)$ for every $r > 0$.

**Theorem 5.5.** [Sihwaningrum, *et al.* [31, 32]] *Let* $1 < p < \frac{n}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Assume that* $\psi : (0,\infty) \to (0,\infty)$ *is almost decreasing, and there exists a constant* $C > 0$ *(which is independent of* $r$*) such that* $r^{\alpha/n}\phi(r) \leq C\psi(r)$. *Then, we have*

$$\|I_\alpha f\|_{q,\phi,k,\mu} \leq C\,\|f\|_{p,\phi,k,\mu},$$

*that is,* $I_\alpha$ *is bounded from* $\mathcal{M}^{p,\phi}(k,\mu)$ *to* $\mathcal{M}^{q,\psi}(k,\mu)$.

As a consequence, we obtain the following Olsen inequality.

**Theorem 5.6.** [Sihwaningrum, *et al.* [32]] *Let* $1 < p < \frac{n}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *If* $r^{\alpha/n}\phi(r)$ *is almost decreasing, then*

$$\|WI_\alpha f\|_{p,\phi,k\mu} \leq C\|W\|_{n/\alpha,\mu}\|f\|_{p,\phi,k,\mu},$$

*provided that* $W \in L^{n/\alpha}(\mu)$.

More results on $I^n_\alpha$ could be obtained in [27]. Meanwhile, a similar result for an analog of $T_\rho$ and Olsen-type inequalities in non-homogeneous setting can be found in [12].

**Acknowledgement.** We would like to thank IICMA 2009 Organizers for having invited me to present research advances on fractional integrals. The research is supported by 2008-2009 Fundamental Research Grant from DGHE, Republic of Indonesia.

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Hendra Gunawan: Bandung Institute of Technology, Bandung, Indonesia.
E-mail: hgunawan@math.itb.ac.id

Idha Sihwaningrum: Bandung Institute of Technology, Bandung, Indonesia.
E-mail: hanidha@students.itb.ac.id