A 2-D INTERPOLATION METHOD THAT MINIMIZES AN ENERGY INTEGRAL

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Abstract

This paper will present a method of interpolation which is used to construct a surface -expressed as a continuous function of two variables- passing some arbitrary points on a square domain. The function must minimize an energy integral of fractional order. To construct such a function, the double Fourier sine series as well as the functional analysis arguments are used. An iterative procedure to obtain the solution is also presented.

Keywords **:** *interpolation, energy integral, double Fourier sine series.*

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1. INTRODUCTION

Newton (1675), Lagrange (1795) and Hermite (1870) pioneered and developed the theory and methods of interpolation. In this decade, interpolation methods continue to grow rapidly. Some of them including the problem of energy minimizing interpolation can be seen in [1, 5, 10] and other references.

In [7], a method to construct a surface $u(x , y)$ on a square domain $D = \{ (x , y) \mid 0 \le x \le 1 , 0 \le y \le 1 \}$, passing the MN points (x_i , y_j , c_{ij}), where $0 < x_1 < \ldots < x_M < 1$, $0 < y_1 < \ldots < y_N < 1$ and $u(0, y) = u(1 , y) = u(x , 0) = u(x , 1) = 0$ has been proposed. The surface must minimize the energy integral with fractional order:

$$E_\beta(u) := \int_0^1 \int_0^1 \left| (-\Delta)^{\beta/2} u \right|^2 dx\, dy \ .$$

For $\beta = 2$, the integral is nothing but the total curvature of u on D. In reality, the available data of (x_i , y_j , c_{ij}) points in the domain, are not always homogeneous, but may scatter randomly. This paper will develop an interpolation method in which the number of known points is K and scatter randomly in region D.

2. FUNCTIONAL ANALYSIS FOR EXISTENCE OF SOLUTION

The problem of this research is how to construct a surface which is expressed as a two variable continuous function $z = u(x,y)$ such that it passes K given points (x_i , y_i , c_i) where $0 \le x \le 1$, $0 \le y \le 1$, with $u(0 , y) = u(1 , y) = u(x , 0) = u(x , 1) = 0$, and minimizes the energy integral of fractional order $E_\beta(u)$, where

$$E_\beta(u) := \int_0^1\int_0^1 \left|(-\Delta)^{\beta/2} u\right|^2 dx\,dy \quad . \qquad \text{... (1)}$$

To find such a function, we will use the *double Fourier sine series*

$$u(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[a_{mn} \sin m\pi x . \sin n\pi y\right].$$

Then, we have

$$E_\beta(u) := \int_0^1\int_0^1 \left|(-\Delta)^{\beta/2} u\right|^2 dxdy = \frac{1}{4}\pi^{2\beta} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}^2 \left(m^2 + n^2\right)^\beta .$$

Hence the coefficient of double Fourier sine series a_{mn} must satisfy

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}{}^2 \left(m^2 + n^2\right)^\beta < \infty \quad .$$

To solve the problem, first we define

$$W_\beta = \{ u(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[a_{mn} \sin m\pi x \ \sin n\pi y\right] \mid \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \left(m^2 + n^2\right)^\beta a_{mn}{}^2 < \infty \}.$$

It has been shown in [4] that $u(x,y)$ is continuous for $\beta > 1$, and W_β is *Hilbert* space with the *inner product*:

$$\langle u,v\rangle = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\ b_{mn} \left(m^2 + n^2\right)^\beta \qquad \text{for } u ,v \in W_\beta .$$

Furthermore we also define

$$V = \{ u(x,y) \in W_\beta \mid u(x_i , y_i) = 0 ,\ i = 1 , \ldots , K \}$$

and

$$U = \{ u(x,y) \in W_\beta \mid u(x_i , y_i) = c_i ,\ i = 1 , \ldots , K \}$$

and obtain the following lemma.

Lemma**:** *V is* a *closed subspace, while U is closed, convex, and not empty subset of* W_β .

Proof: The proof that V is closed, U closed and convex is similar to that in [7], so in this paper we will only show that U is not an empty set.

Suppose that $u(x,y)$ passes K given points: (x_1 , y_1 , c_1), (x_2 , y_2 , c_2), (x_3 , y_3 , c_3), . . . , (x_K , y_K , c_K). If $x_i \neq x_j$ and $y_s \neq y_t$ for each i, j, s, and t, (see Figure-1), then by reordering x_i and y_i such that $x_i < x_{i+1}$ and $y_i < y_{i+1}$, we have the fact that $u(x,y)$ passes the K^2 points (x_i, y_j, c_{ij}) with $i = 1,. . ., K$ and $j = 1,. . ., K$, (see Figure-2). Based on the results in [8], the existence of a solution function $u(x,y)$ passing the K^2 points is guaranteed and this implies that this function also passes the given K points.

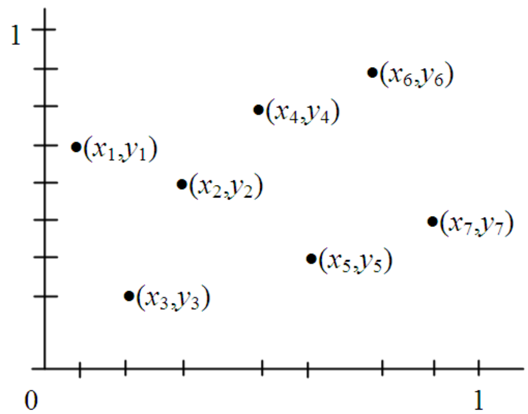


Figure-1

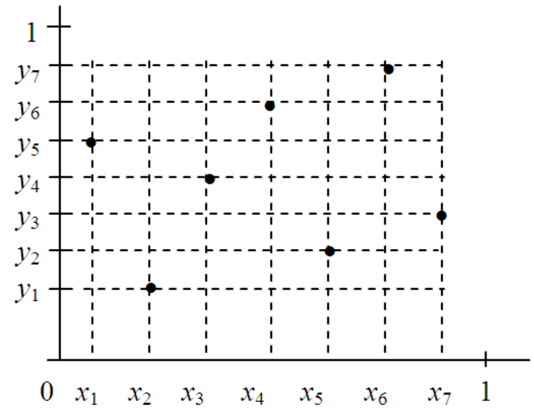


Figure-2

On the other hand, if $x_i = x_j$ for some i and j, and or $y_s = y_t$ for some s and t, (see Figure-3), then by reordering x_i and y_i we will have $M < K$ and $N < K$ such that $0 < x_1 < x_2 < . . .< x_M < 1$ dan $0 < y_1 < y_2 < \ . . . < y_N < 1$, (see Figure-4). By referring to [8] then the existence of $u(x,y)$ passing MN points is guaranteed, and this implies that the function also passes the K points.

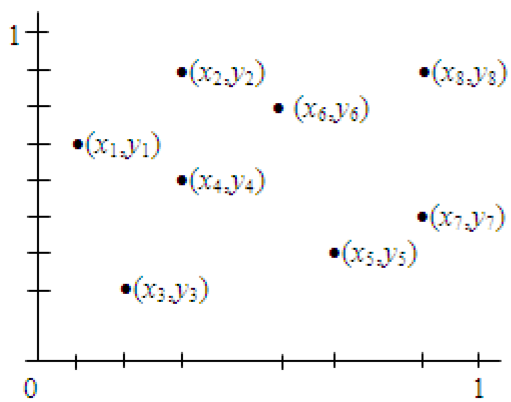


Figure-3

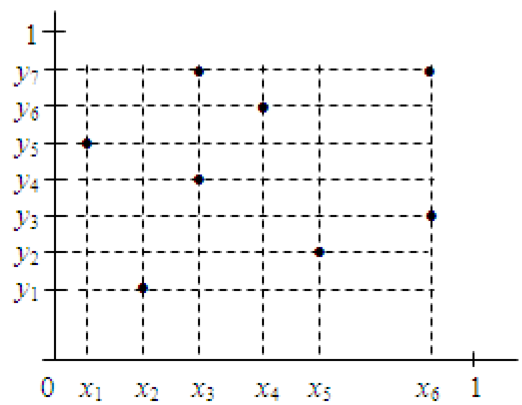


Figure-4

From both cases above, the existence of $u(x,y) \in U$ is guaranteed, and so that U is not empty . ■

Theorem: *The minimization problem* (1) *has a unique solution in* W_β *, and this solution is* $u = u_0 - proj_V(u_0)$ *with* u_0 *is an arbitrary element of* U*, and* $proj_V(u_0)$ *is an orthogonal projection of* u_0 *to* V*.*

3. PROCEDURE FOR CONSTRUCTING THE FUNCTION: AN EXAMPLE

We shall describe the procedure to construct the function through an example.
Suppose that we want to find the function $u(x,y)$ passing $K = 3$ given points :
A (x_1, y_1, c_1) , B (x_2, y_2, c_2) dan C (x_3, y_3, c_3) . Based on lemma above, there is initial function $u_0(x,y)$ passing $K^2 = 9$ points (x_i, y_j, c_{ij}), namely:

$$u_0(x,y) = \sum_{m=1}^{3}\sum_{n=1}^{3}\left[a_{mn} \sin m\pi x . \sin n\pi y\right]$$
$$= S_{11}(x,y) + a_{12}\, S_{12}(x,y) + a_{22}\, S_{22}(x,y) + a_{21}\, S_{21}(x,y) + a_{13}\, S_{13}(x,y) + a_{23}\, S_{23}(x,y) + a_{33}\, S_{33}(x,y) + a_{32}\, S_{32}(x,y) + a_{31}\, S_{31}(x,y).$$

where $S_{mn}(x,y) = \sin m\pi x \ \sin n\pi y$. This means that:

$$\begin{vmatrix}
S_{11}(x_1,y_1) & S_{12}(x_1,y_1) & S_{22}(x_1,y_1) & S_{21}(x_1,y_1) & S_{13}(x_1,y_1) & S_{23}(x_1,y_1) & S_{33}(x_1,y_1) & S_{32}(x_1,y_1) & S_{31}(x_1,y_1) \\
S_{11}(x_2,y_1) & S_{12}(x_2,y_1) & S_{22}(x_2,y_1) & S_{21}(x_2,y_1) & \dots & & & & S_{31}(x_2,y_1) \\
S_{11}(x_3,y_1) & S_{12}(x_3,y_1) & S_{22}(x_3,y_1) & S_{21}(x_3,y_1) & \dots & & & & S_{31}(x_3,y_1) \\
S_{11}(x_1,y_2) & S_{12}(x_1,y_2) & S_{12}(x_1,y_2) & S_{12}(x_1,y_2) & \dots & & & & S_{31}(x_1,y_2) \\
S_{11}(x_2,y_2) & S_{12}(x_2,y_2) & S_{22}(x_2,y_2) & S_{21}(x_2,y_2) & \dots & & & & S_{31}(x_2,y_2) \\
S_{11}(x_3,y_2) & S_{12}(x_3,y_2) & S_{22}(x_3,y_2) & S_{21}(x_3,y_2) & \dots & & & & S_{31}(x_3,y_2) \\
S_{11}(x_1,y_3) & S_{12}(x_1,y_3) & S_{12}(x_1,y_3) & S_{12}(x_1,y_3) & \dots & & & & S_{31}(x_3,y_2) \\
S_{11}(x_2,y_3) & S_{12}(x_2,y_3) & S_{12}(x_2,y_3) & S_{12}(x_2,y_3) & \dots & & & & S_{31}(x_2,y_3) \\
S_{11}(x_3,y_3) & S_{12}(x_3,y_3) & S_{22}(x_3,y_3) & S_{11}(x_3,y_3) & \dots & & & & S_{31}(x_3,y_3)
\end{vmatrix} \neq 0$$

The following will show the existence of a more simple initial function $u_0(x,y)$ that only contains three terms in accordance with the number of known points.

Note that the row vector-1, row vector-5, and row vector-9:

$$S_{11}(x_1,y_1) \quad S_{12}(x_1,y_1) \quad S_{22}(x_1,y_1) \quad S_{21}(x_1,y_1) \quad S_{13}(x_1,y_1) \quad S_{23}(x_1,y_1) \quad S_{33}(x_1,y_1) \quad S_{32}(x_1,y_1) \quad S_{31}(x_1,y_1),$$
$$S_{11}(x_2,y_2) \quad S_{12}(x_2,y_2) \quad S_{22}(x_2,y_2) \quad S_{21}(x_2,y_2) \quad S_{13}(x_2,y_2) \quad S_{23}(x_2,y_2) \quad S_{33}(x_2,y_2) \quad S_{32}(x_2,y_2) \quad S_{31}(x_2,y_2),$$
$$S_{11}(x_3,y_3) \quad S_{12}(x_3,y_3) \quad S_{22}(x_3,y_3) \quad S_{21}(x_3,y_3) \quad S_{13}(x_3,y_3) \quad S_{23}(x_3,y_3) \quad S_{33}(x_3,y_3) \quad S_{32}(x_3,y_3) \quad S_{31}(x_3,y_3)$$

are linearly independent.

This mean that there are at least two columns ij and kl such that for these ij and kl we have

$$\begin{vmatrix}
S_{ij}(x_1,y_1) & S_{kl}(x_1,y_1) & S_{st}(x_1,y_1) \\
S_{ij}(x_2,y_2) & S_{kl}(x_2,y_2) & S_{st}(x_2,y_2) \\
S_{ij}(x_3,y_3) & S_{kl}(x_3,y_3) & S_{st}(x_3,y_3)
\end{vmatrix} \neq 0 \quad \text{for } st \neq ij \text{ and } st \neq kl.$$

So there are at least $9 - 2 = 7$ pairs with determinant $\neq 0$.

In this case, the initial function $u_0(x, y)$ passing $K = 3$ points is a function with three terms:

$$u_0(x,y) = a_{ij} \sin i\pi x . \sin j\pi y + a_{kl} \sin k\pi x . \sin l\pi y + a_{st} \sin m\pi x . \sin n\pi y \qquad \dots(*)$$

Furthermore, to determine the basis of

$$V = \{\, u(x,y) \in W_\beta \mid u(x_1, y_1) = 0 ,\ u(x_2, y_2) = 0 ,\ u(x_3, y_3) = 0 \,\},$$

take an arbitrary function: $u(x\,,y) \in V$:

$$u(x\,,y) = a_{11}\sin\pi x.\sin\pi y + a_{12}\sin\pi x.\sin 2\pi y + a_{22}\sin 2\pi x.\sin 2\pi y + a_{21}\sin 2\pi x.\sin\pi y + a_{13}\sin\pi x.\sin 3\pi y + a_{23}\sin 2\pi x.\sin 3\pi y + a_{33}\sin 3\pi x.\sin 3\pi y + a_{32}\sin 3\pi x.\sin 2\pi y + a_{31}\sin 3\pi x.\sin\pi y + a_{14}\sin\pi x.\sin 4\pi y + a_{24}\sin 2\pi x.\sin 4\pi y + a_{34}\sin 3\pi x.\sin 4\pi y + \ldots$$

Since $u(x_i\,,y_i) = 0$ for $i = 1, 2, 3$, then we have three equations:

$$0 = a_{11}\sin\pi x_i.\sin\pi y_i + a_{12}\sin\pi x_i.\sin 2\pi y_i + a_{22}\sin 2\pi x_i.\sin 2\pi y_i + a_{21}\sin 2\pi x_i.\sin\pi y_i + a_{13}\sin\pi x_i.\sin 3\pi y_i + a_{23}\sin 2\pi x_i.\sin 3\pi y_i + a_{33}\sin 3\pi x_i.\sin 3\pi y_i + a_{32}\sin 3\pi x_i.\sin 2\pi y_i + a_{31}\sin 3\pi x_i.\sin\pi y_i + a_{14}\sin\pi x_i.\sin 4\pi y_i + a_{24}\sin 2\pi x_i.\sin 4\pi y_i + \ldots$$

If the notation of $\sin m\pi x_k.\sin n\pi y_k$ is simplified by s_{mn}^k then we have

$$0 = a_{11}.s_{11}^1 + a_{12}.s_{12}^1 + a_{22}.s_{22}^1 + a_{21}.s_{21}^1 + a_{13}.s_{13}^1 + a_{23}.s_{23}^1 + a_{33}.s_{33}^1 + a_{32}.s_{32}^1 + a_{31}.s_{31}^1 + \ldots$$
$$0 = a_{11}.s_{11}^2 + a_{12}.s_{12}^2 + a_{22}.s_{22}^2 + a_{21}.s_{21}^2 + a_{13}.s_{13}^2 + a_{23}.s_{23}^2 + a_{33}.s_{33}^2 + a_{32}.s_{32}^2 + a_{31}.s_{31}^2 + \ldots$$
$$0 = a_{11}.s_{11}^3 + a_{12}.s_{12}^3 + a_{22}.s_{22}^3 + a_{21}.s_{21}^3 + a_{13}.s_{13}^3 + a_{23}.s_{23}^3 + a_{33}.s_{33}^3 + a_{32}.s_{32}^3 + a_{31}.s_{31}^3 + \ldots$$

From (*), we assume that a_{ij}, a_{kl} , and a_{st} are the dependent variables, while a_{mn} are the independent variables. Other steps such as finding the basis, orthogonalization process, finding solution function, and the calculation of energy are similar to the steps described in [7].

A concrete example to find the function is presented as follows.

Suppose that we wish to construct a function $u(x, y)$ passing three points A = (1/3 , 1/3 , 1), B = (1/2 , 2/3 , 2), and C = (2/3 , 1/2 , 1), with $u(0\,,y) = u(1\,,y) = u(x\,,0) = u(x\,,1) = 0$ and minimize (1) for β = 1,25.

The steps undertaken are as follows:

***Step-1*: Determining initial function** $u_0(x\,,y)$

By using the above concept of the existence of the initial function with three terms, we can determine the fixed coefficient of initial function $u_0(x\,,y)$.
Let these coefficients are a_{11} , a_{12} , and a_{13} . So we take

$$u_0\,(x,y) = a_{11}\sin\pi x.\sin\pi y + a_{12}\sin\pi x.\sin 2\pi y + a_{13}\sin\pi x.\sin 3\pi y$$

Since $u_0\,(x,y)$ passing A, B, and C, then we have

$$1 = \tfrac{3}{4}\,a_{11} + \tfrac{3}{4}\,a_{12} + 0\,a_{13}$$
$$2 = \tfrac{1}{2}\sqrt{3}\,a_{11} - \tfrac{1}{2}\sqrt{3}\,a_{12} + 0\,a_{13}$$
$$1 = \tfrac{1}{2}\sqrt{3}\,a_{11} - \tfrac{1}{2}\sqrt{3}\,a_{13}\,,$$

Whence : $a_{11} = 1.821367$, $a_{12} = -0.488034$, and $a_{13} = 0.66667$.

Hence the initial function is:

$$u_0(x\,,y) = 1.821367\sin\pi x.\sin\pi y - 0.488034\sin\pi x.\sin 2\pi y + 0.66667\sin\pi x.\sin 3\pi y$$

$$= \begin{bmatrix} 1.821367 & -0.488034 & 0.66667 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

***Step-2*: Determining Basis of** $V = \{u(x,y) \in W_{1.25} \mid u(x_1, y_1) = u(x_2, y_2) = u(x_3, y_3) = 0\}$.

Take arbitrary $u(x\,,y) \in V$ where

$$u(x\,,y) = a_{11}\sin\pi x.\sin\pi y + a_{12}\sin\pi x.\sin 2\pi y + a_{22}\sin 2\pi x.\sin 2\pi y + a_{21}\sin 2\pi x.\sin\pi y + a_{13}\sin\pi x.\sin 3\pi y + a_{23}\sin 2\pi x.\sin 3\pi y + a_{33}\sin 3\pi x.\sin 3\pi y + a_{32}\sin 3\pi x.\sin 2\pi y + a_{31}\sin 3\pi x.\sin\pi y + a_{14}\sin\pi x.\sin 4\pi y + a_{24}\sin 2\pi x.\sin 4\pi y + \ldots$$

Since $u(1/3\,,1/3) = 0$, $u(1/2\,,2/3) = 0$, and $u(2/3\,,1/2) = 0$ then

$$0 = \tfrac{3}{4}a_{11} + \tfrac{3}{4}a_{12} + \tfrac{3}{4}a_{22} + \tfrac{3}{4}a_{21} - \tfrac{3}{4}a_{14} - \tfrac{3}{4}a_{24} + \tfrac{3}{4}a_{44} - \tfrac{3}{4}a_{42} - \tfrac{3}{4}a_{41} + \ldots$$

$$0 = \tfrac{1}{2}\sqrt{3}\,a_{11} - \tfrac{1}{2}\sqrt{3}\,a_{12} + \tfrac{1}{2}\sqrt{3}\,a_{32} - \tfrac{1}{2}\sqrt{3}\,a_{31} + \tfrac{1}{2}\sqrt{3}\,a_{14} - \tfrac{1}{2}\sqrt{3}\,a_{34} - \tfrac{1}{2}\sqrt{3}\,a_{15} + \ldots$$

$$0 = \tfrac{1}{2}\sqrt{3}\,a_{11} - \tfrac{1}{2}\sqrt{3}\,a_{21} - \tfrac{1}{2}\sqrt{3}\,a_{13} + \tfrac{1}{2}\sqrt{3}\,a_{23} - \tfrac{1}{2}\sqrt{3}\,a_{43} + \tfrac{1}{2}\sqrt{3}\,a_{41} + \tfrac{1}{2}\sqrt{3}\,a_{15} + \ldots$$

and since coefficients of initial function $u_0(x\,,y)$ are a_{11} , a_{12} , and a_{13} , then we assume a_{11} , a_{12} , and a_{13} as the dependent variable such that we have :

$$a_{11} + a_{12} = -a_{22} - a_{21} + a_{14} + a_{24} - a_{44} + a_{42} + a_{41} - \ldots$$

$$a_{11} - a_{12} = -a_{32} + a_{31} - a_{14} + a_{34} + a_{15} + \ldots$$

$$a_{11} - a_{13} = +a_{21} - a_{23} + a_{43} - a_{41} - a_{15} + \ldots$$

With elimination, we obtain:

$$a_{11} = -\tfrac{1}{2}a_{22} - \tfrac{1}{2}a_{21} - \tfrac{1}{2}a_{32} + \tfrac{1}{2}a_{31} + \tfrac{1}{2}a_{24} + \tfrac{1}{2}a_{34} - \tfrac{1}{2}a_{44} + \tfrac{1}{2}a_{42} + \tfrac{1}{2}a_{41} + \tfrac{1}{2}a_{15} + \ldots$$

$$a_{12} = -\tfrac{1}{2}a_{22} - \tfrac{1}{2}a_{21} + \tfrac{1}{2}a_{32} - \tfrac{1}{2}a_{31} + a_{14} + \tfrac{1}{2}a_{24} - \tfrac{1}{2}a_{34} - \tfrac{1}{2}a_{44} + \tfrac{1}{2}a_{42} + \tfrac{1}{2}a_{41} - \tfrac{1}{2}a_{15} + \ldots$$

$$a_{13} = -\tfrac{1}{2}a_{22} - \tfrac{3}{2}a_{21} + a_{23} - \tfrac{1}{2}a_{32} + \tfrac{1}{2}a_{31} + \tfrac{1}{2}a_{24} + \tfrac{1}{2}a_{34} - \tfrac{1}{2}a_{44} - a_{43} + \tfrac{1}{2}a_{42} + \tfrac{3}{2}a_{41} + \tfrac{3}{2}a_{15} + \ldots$$

In matrix equation, we can write:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & \cdots \\ a_{21} & a_{22} & a_{23} & a_{24} & \cdots \\ a_{31} & a_{32} & a_{33} & a_{34} & \cdots \\ a_{41} & a_{42} & a_{43} & a_{44} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{bmatrix} = a_{22}\begin{bmatrix} -0.5 & -0.5 & -0.5 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{bmatrix} + a_{21}\begin{bmatrix} -0.5 & -0.5 & -1.5 & 0 & \cdots \\ 1 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{bmatrix}$$

$$+ a_{23}\begin{bmatrix} 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{bmatrix} + a_{33}\begin{bmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{bmatrix} + a_{32}\begin{bmatrix} -0.5 & 0.5 & -0.5 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{bmatrix}$$

$$+\ a_{31}\begin{bmatrix} 0.5 & -0.5 & 0.5 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \ddots \end{bmatrix} + a_{14}\begin{bmatrix} 0 & 1 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \ddots \end{bmatrix} + a_{24}\begin{bmatrix} 0.5 & 0.5 & 0.5 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \ddots \end{bmatrix}$$

$$+\ a_{34}\begin{bmatrix} 0.5 & -0.5 & 0.5 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \ddots \end{bmatrix} + a_{44}\begin{bmatrix} -0.5 & -0.5 & -0.5 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \ddots \end{bmatrix} + a_{43}\begin{bmatrix} 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & \dots \\ \dots & \dots & \dots & \dots & \ddots \end{bmatrix} + \dots$$

$$= \ a_{22}\, v_1 + a_{21}\, v_2 + a_{23}\, v_3 + a_{33}\, v_4 + \dots$$

such that the above matrix $\{ v_1 , v_2 , v_3 , v_4 , \dots \}$ is the basis of V.

Step-3*: Orthogonalizing Basis of *V

Since the above basis is not orthogonal, then we need to orthogonalize this basis to become an orthogonal basis $\{ w_1 , w_2 , w_3 , w_4 , \dots \}$. Here

$$w_1 = v_1 = \begin{bmatrix} -0.5 & -0.5 & -0.5 & 0 & \dots \\ 0 & 1 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \ddots \end{bmatrix}$$

$$w_2 = v_2 - \frac{\langle v_2, w_1\rangle}{\langle w_1, w_1\rangle}.w_1 = v_2 - \frac{(2^{1.25}.\tfrac14 + 5^{1.25}.\tfrac14 + 10^{1.25}.\tfrac34)}{(2^{1.25}.\tfrac14 + 5^{1.25}.\tfrac14 + 10^{1.25}.\tfrac34 + 8^{1.25}.1)}\ w_1$$

$$= \begin{bmatrix} -0.112036 & -0.112036 & -1.11204 & 0 & \dots \\ 1 & -0.775929 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix}$$

and for $k \in \mathbf{N}$

$$w_k = v_k - \frac{\langle v_k, w_1\rangle}{\langle w_1, w_1\rangle}.w_1 - \frac{\langle v_k, w_2\rangle}{\langle w_2, w_2\rangle}.w_2 - \dots - \frac{\langle v_k, w_{k-1}\rangle}{\langle w_{k-1}, w_{k-1}\rangle}.w_{k-1}$$

with $\langle u,v\rangle = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(m^2+n^2\right)^{1.25} a_{mn}.b_{mn}$ is *inner product* on $W_{1.25}$.

***Step-4*: Determining an Approximate Function : $\hat{u}_n$**

From step-1 we have initial function $u_0(x,y) \in U$ is

$$u_0(x, y) = \begin{bmatrix} 1.821367 & -0.488034 & 0.66667 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{bmatrix}$$

The first approximate function from the first iteration is an orthogonal projection of u_0 to V_1 = span { w_1}:

$$\hat{\boldsymbol{u}}_{\mathbf{1}} = u_0 - \frac{\langle u_0, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 = \begin{bmatrix} 1.667 & -0.642 & 0.513 & 0 & \cdots \\ 0 & 0.308 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \ddots \end{bmatrix}$$

The second approximate function from the second iteration is an orthogonal projection of u_0 to V_2 = span { w_1 , w_2 }:

$$\hat{\boldsymbol{u}}_{\mathbf{2}} = u_0 - \frac{\langle u_0, w_1 \rangle}{\langle w_1, w_1 \rangle} w_1 - \frac{\langle u_0, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 = \hat{\boldsymbol{u}}_{\mathbf{1}} - \frac{\langle u_0, w_2 \rangle}{\langle w_2, w_2 \rangle} w_2 ,$$

and so on.

***Step-5*: Determining The Energy Integral Value:** $E_{1.25}(u)$

From (1) with $\beta = 1.25$ we have

$$E_{1.25}(u) := \int_0^1 \int_0^1 \left| (-\Delta)^{0.625} u \right|^2 dx\, dy = \frac{1}{4}.\pi^{2..5} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn}{}^2 \left(m^2 + n^2\right)^{1..25} ,$$

hence for solution function $\hat{\boldsymbol{u}}_{\mathbf{1}}$ **:**

$$E_{1.25}(\hat{\boldsymbol{u}}_{\mathbf{1}}) = \frac{1}{4}.\pi^{2.5} . [\ (1+1)^{1.25}.(1.667)^2 + (1+4)^{1.25}.(-0.642)^2 + (1+9)^{1.25}.(0.513)^2 + (4+4)^{1.25}.(0.308)^2\ ]$$

$$= 3.91109\ \pi^{2.5}$$

Without writing the constant $\pi^{2.5}$, with a computer program, we obtain the results as follows:

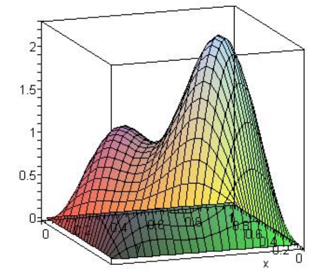
$E_{1.25}(\hat{\boldsymbol{u}}_2)$ =2.74487 $E_{1.25}(\hat{\boldsymbol{u}}_3)$ = 2.70349 $E_{1.25}(\hat{\boldsymbol{u}}_4)$ = 2.70349 . . .
$E_{1.25}(\hat{\boldsymbol{u}}_{1218})$ =1.42688 $E_{1.25}(\hat{\boldsymbol{u}}_{1219})$ = 1.42682 $E_{1.25}(\hat{\boldsymbol{u}}_{1220})$ = 1.42680
$E_{1.25}(\hat{\boldsymbol{u}}_{1221})$ = 1.42672.

Finally if we take an error factor $\varepsilon = 0.0001$, then we can stop at iteration-1221, with the last approximate function is $\hat{\boldsymbol{u}}_{1221}$ and the minimum energy is 1.4267.

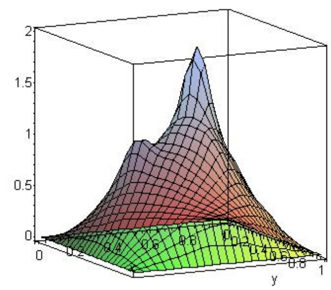
The second example for $\beta = 1.1$ produces an approximate function $\hat{\boldsymbol{u}}_{1220}$, and its minimum energy value is 1.01954 with an error factor $\varepsilon = 0.0001$.
The last example we take $\beta = 2.0$, and the approximate function is $\hat{\boldsymbol{u}}_{1222}$, whose minimum energy value is 4.8471 with an error factor $\varepsilon = 0.0001$.

The graphs of the approximate functions are:

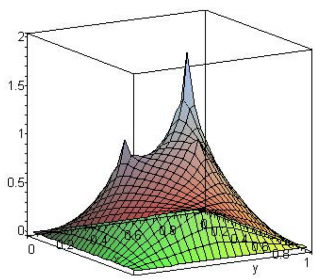
For β = 1.1 :



$\hat{u}_0$

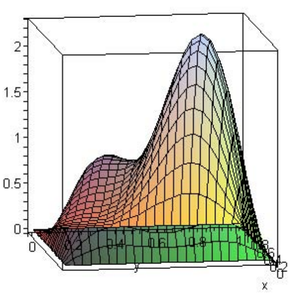


$\hat{u}_{147}$

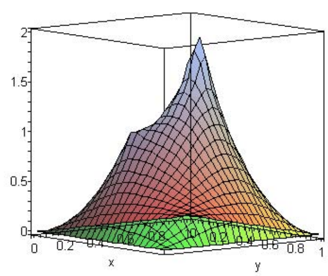


$\hat{u}_{1220}$

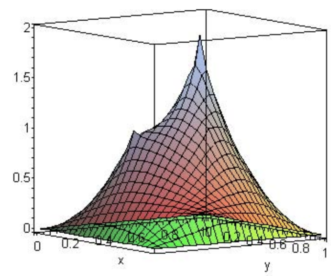
For β = 1.25 :



$\hat{u}_1$

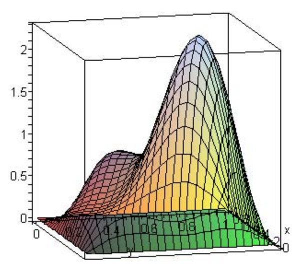


$\hat{u}_{250}$

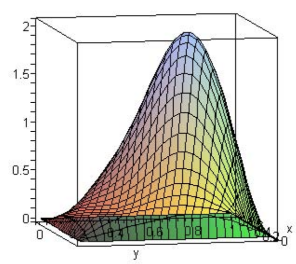


$\hat{u}_{1221}$

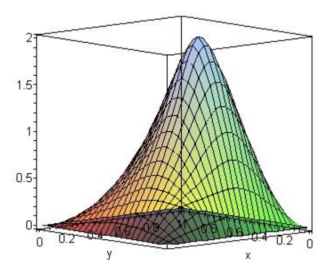
For β = 2.0 :



$\hat{u}_1$



$\hat{u}_8$



$\hat{u}_{1222}$

Note that from the three examples above, we see that different values of β give different function or surfaces that minimizes the energy. The larger the value of β, the smoother the surface.

Acknowledgement

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