EKSISTENSI INTERPOLAN SINUSOIDA BERDIMENSI DUA

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A B S T R A K

Dalam makalah ini akan dibahas tentang bukti dari syarat cukup bagi eksistensi interpolan sinusoida berdimensi dua yaitu berupa fungsi berbentuk *deret sinus Fourier ganda* yang diketahui melalui beberapa buah titik (x_i , y_j , c_{ij}), dengan $0 < x_i < 1$, dan $0 < y_i < 1$. Pembuktian akan dilakukan melalui dekomposisi matriks yang memuat dua buah variabel menjadi perkalian dari pada dua buah matriks yang masing-masing memuat satu variabel, serta menunjukkan bahwa kedua matriks tersebut adalah nonsingulir.

Kata Kunci: Interpolan, sinusoida, dekomposisi.

1. PENDAHULUAN

Dalam rangka menjamin keberadaan (eksistensi) suatu fungsi dalam masalah intepolasi, maka untuk menunjukan adanya interpolan berbentuk sinusoida berdimensi dua, yaitu deret sinus *Fourier ganda*: $u(x,y) = \sum_{m=1}^{M}\sum_{n=1}^{N}\left[a_{mn} \sin m\pi x . \sin n\pi y\right]$ yang melalui K buah titik: (x_i , y_j , c_{ij}), dengan $0 < x_i < 1$, dan $0 < y_i < 1$, penulis akan membaginya dalam dua tahap. Tahap-1 terlebih dahulu akan membahas tentang eksistensi fungsi $u(x,y)$ yang melalui sejumlah $K = (M.N)$ buah titik homogen: (x_i , y_j , c_{ij}) dengan $0 < x_1 < . . . < x_M < 1$ dan $0 < y_1 < . . . < y_N < 1$. Selanjutnya dalam tahap-2 akan dibahas tentang eksistensi fungsi $u(x,y)$ yang melalui K buah buah titik sembarang: (x_i , y_i , c_i) dengan $0 < x_i < 1$, dan $0 < y_i < 1$ untuk $i = 1, 2, . . . , K$.

Dalam masalah satu dimensi, yaitu bagaimana menjamin eksistensi fungsi berbentuk deret sinus Fourier $u(x,y) = \sum_{m=1}^{N}\left[a_m \sin m\pi x\right]$ yang melalui N buah titik: (x_i , c_i) dengan $0 < x_i < 1$, telah dibuktikan dalam [3] dan digunakan dalam [5] bahwa berlaku:

$$\begin{vmatrix} \sin \pi x_1 & \sin 2\pi x_1 & \sin 3\pi x_1 & \dots & \sin N\pi x_1 \\ \sin \pi x_2 & \sin 2\pi x_2 & \sin 3\pi x_2 & \dots & \sin N\pi x_2 \\ \sin \pi x_3 & \sin 2\pi x_3 & \sin 3\pi x_3 & \dots & \sin N\pi x_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sin \pi x_N & \sin 2\pi x_N & \sin 3\pi x_N & \dots & \sin N\pi x_N \end{vmatrix} \neq 0$$

2. Eksistensi fungsi *u*(*x*,*y*) yang Melalui (*M.N*) Buah Titik

Untuk mengetahui eksistensi fungsi $u(x,y) := \sum_{m=1}^{M}\sum_{n=1}^{N}\left[a_{mn} \sin m\pi x . \sin n\pi y\right]$ yang melalui $(M.N)$ buah titik: (x_i , y_j , c_{ij}), di mana $0 < x_1 < . . . < x_M < 1$ dan $0 < y_1 < . . . < y_N < 1$, terlebih dahulu kita substitusikan nilai (x_i , y_j , c_{ij}) tersebut pada fungsi $u(x,y)$ di atas sehingga membentuk *sistem persamaan linear* dengan $(M.N)$ buah persamaan dan $(M.N)$ buah variabel: [1, 4]

$$u(x_i,y_j) = \sum_{m=1}^{M}\sum_{n=1}^{N}\left[a_{mn}\sin m\pi x_i.\sin n\pi y_j\right] \quad = \quad c_{ij}$$

Dalam bentuk persamaan matriks, dapat ditulis:

$$\begin{bmatrix}
\sin\pi x_1.\sin\pi y_1 & \cdots & \sin M\pi x_1.\sin\pi y_1 & \sin\pi x_1.\sin 2\pi y_1 & \cdots & \sin M\pi x_1.\sin 2\pi y_1 & \cdots & \cdots & \sin\pi x_1.\sin N\pi y_1 & \cdots & \sin M\pi x_1.\sin N\pi y_1 \\
\vdots & & & & & & & & & & \vdots \\
\sin\pi x_M.\sin\pi y_1 & \cdots & \sin M\pi x_M.\sin\pi y_1 & \sin\pi x_M \sin 2\pi y_1 & \cdots & \sin M\pi x_M.\sin 2\pi y_1 & \cdots & \cdots & \sin\pi x_M.\sin N\pi y_1 & \cdots & \sin M\pi x_M.\sin N\pi y_1 \\
\sin\pi x_1.\sin\pi y_2 & \cdots & \sin M\pi x_1.\sin\pi y_2 & \sin\pi x_1 \sin 2\pi y_2 & \cdots & \sin M\pi x_1.\sin 2\pi y_2 & \cdots & \cdots & \sin\pi x_M.\sin N\pi y_2 & \cdots & \sin M\pi x_M.\sin N\pi y_M \\
\vdots & & & & & & & & & & \vdots \\
\sin\pi x_M.\sin\pi y_2 & \cdots & \sin M\pi x_M.\sin\pi y_2 & \sin\pi x_M \sin 2\pi y_2 & \cdots & \sin M\pi x_M.\sin 2\pi y_2 & \cdots & \cdots & & \cdots & \\
\vdots & & & & & & & & & & \vdots \\
\vdots & & & & & & & & & & \vdots \\
\sin\pi x_1.\sin\pi y_N & \cdots & \sin M\pi x_1.\sin\pi y_N & \sin\pi x_1 \sin 2\pi y_N & \cdots & \sin M\pi x_1.\sin 2\pi y_2 & \cdots & \cdots & & \cdots & \\
\vdots & & & & & & & & & \ddots & \vdots \\
\sin\pi x_M.\sin\pi y_N & \cdots & \sin M\pi x_M.\sin\pi y_N & \sin\pi x_M \sin 2\pi y_N & \cdots & \sin M\pi x_M.\sin 2\pi y_2 & \cdots & \cdots & & \cdots & \sin M\pi x_M.\sin N\pi y_N
\end{bmatrix}
\bullet
\begin{bmatrix} a_{11} \\ \cdots \\ a_{M1} \\ a_{12} \\ \cdots \\ a_{M2} \\ \cdots \\ \cdots \\ a_{1N} \\ \cdots \\ a_{MN} \end{bmatrix}
=
\begin{bmatrix} c_{11} \\ \cdots \\ c_{M1} \\ c_{12} \\ \cdots \\ c_{M2} \\ \cdots \\ \cdots \\ c_{1N} \\ \cdots \\ c_{MN} \end{bmatrix}$$

yang secara sederhana dapat dinyatakan sebagai: **A** . (a_{mn}) = (c_{mn}), di mana **A** adalah matriks berorde $(M.N)\times(M.N)$.

Dengan demikian agar supaya fungsi $u(x,y)$ yang melalui titik-titik tadi terdefinisi (ada), maka persamaan linear di atas harus mempunyai solusi, artinya bahwa determinan dari matriks **A** tidak boleh sama dengan nol. Artinya bahwa syarat cukup bagi eksistensi fungsi $u(x,y)$ adalah Det **A** $\neq 0$.

Untuk menunjukkan hal tersebut, terlebih dahulu matriks **A** di ”dekomposisi” menjadi:

$$\mathbf{A}_{\,(\mathbf{M.N})\,\times\,(\mathbf{M.N})} = \mathbf{A}^{x}_{\,(\mathbf{M.N})\,\times\,(\mathbf{M.N})} \;\bullet\; \mathbf{A}^{y}_{\,(\mathbf{M.N})\,\times\,(\mathbf{M.N})} \qquad \ldots(1)$$

dimana $\mathbf{A}^{x}_{\,(\mathbf{M.N})\,\times\,(\mathbf{M.N})}$ =

$$\left[\begin{array}{ccccc|ccccc|ccccc|c|ccccc}
\sin\pi x_1 & \sin 2\pi x_1 & \sin 3\pi x_1 & \cdots & \sin M\pi x_1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\sin\pi x_2 & \sin 2\pi x_2 & \sin 3\pi x_2 & \cdots & \sin M\pi x_2 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\sin\pi x_3 & \sin 2\pi x_3 & \sin 3\pi x_3 & \cdots & \sin M\pi x_3 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\sin\pi x_M & \sin 2\pi x_M & \sin 3\pi x_M & \cdots & \sin M\pi x_M & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\hline
0 & 0 & 0 & \cdots & 0 & \sin\pi x_1 & \sin 2\pi x_1 & \sin 3\pi x_1 & \cdots & \sin M\pi x_1 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & \sin\pi x_2 & \sin 2\pi x_2 & \sin 3\pi x_2 & \cdots & \sin M\pi x_2 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & \sin\pi x_3 & \sin 2\pi x_3 & \sin 3\pi x_3 & \cdots & \sin M\pi x_3 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \sin\pi x_M & \sin 2\pi x_M & \sin 3\pi x_M & \cdots & \sin M\pi x_M & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\hline
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \sin\pi x_1 & \sin 2\pi x_1 & \sin 3\pi x_1 & \cdots & \sin M\pi x_1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \sin\pi x_2 & \sin 2\pi x_2 & \sin 3\pi x_2 & \cdots & \sin M\pi x_2 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \sin\pi x_3 & \sin 2\pi x_3 & \sin 3\pi x_3 & \cdots & \sin M\pi x_3 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \sin\pi x_M & \sin 2\pi x_M & \sin 3\pi x_M & \cdots & \sin M\pi x_M & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\hline
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & \sin\pi x_1 & \sin 2\pi x_1 & \sin 3\pi x_1 & \cdots & \sin M\pi x_1 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & \sin\pi x_2 & \sin 2\pi x_2 & \sin 3\pi x_2 & \cdots & \sin M\pi x_2 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & \sin\pi x_3 & \sin 2\pi x_3 & \sin 3\pi x_3 & \cdots & \sin M\pi x_3 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \cdots & \sin\pi x_M & \sin 2\pi x_M & \sin 3\pi x_M & \cdots & \sin M\pi x_M
\end{array}\right]$$

dan $\mathbf{A}^{y}{}_{(\mathbf{M.N})\,\times\,(\mathbf{M.N})}$ =

$$\left[\begin{array}{ccccc|ccccc|ccccc|c|ccccc}
\sin\pi y_1 & 0 & 0 & \cdots & 0 & \sin 2\pi y_1 & 0 & 0 & \cdots & 0 & \sin 3\pi y_1 & 0 & 0 & \cdots & 0 & \cdots & \sin N\pi y_1 & 0 & 0 & \cdots & 0 \\
0 & \sin\pi y_1 & 0 & \cdots & 0 & 0 & \sin 2\pi y_1 & 0 & \cdots & 0 & 0 & \sin 3\pi y_1 & 0 & \cdots & 0 & \cdots & 0 & \sin N\pi y_1 & 0 & \cdots & 0 \\
0 & 0 & \sin\pi y_1 & \cdots & 0 & 0 & 0 & \sin 2\pi y_1 & \cdots & 0 & 0 & 0 & \sin 3\pi y_1 & \cdots & 0 & \cdots & 0 & 0 & \sin N\pi y_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \sin\pi y_1 & 0 & 0 & 0 & \cdots & \sin 2\pi y_1 & 0 & 0 & 0 & \cdots & \sin 3\pi y_1 & \cdots & 0 & 0 & 0 & \cdots & \sin N\pi y_1 \\ \hline
\sin\pi y_2 & 0 & 0 & \cdots & 0 & \sin 2\pi y_2 & 0 & 0 & \cdots & 0 & \sin 3\pi y_2 & 0 & 0 & \cdots & 0 & \cdots & \sin N\pi y_2 & 0 & 0 & \cdots & 0 \\
0 & \sin\pi y_2 & 0 & \cdots & 0 & 0 & \sin 2\pi y_2 & 0 & \cdots & 0 & 0 & \sin 3\pi y_2 & 0 & \cdots & 0 & \cdots & 0 & \sin N\pi y_2 & 0 & \cdots & 0 \\
0 & 0 & \sin\pi y_2 & \cdots & 0 & 0 & 0 & \sin 2\pi y_2 & \cdots & 0 & 0 & 0 & \sin 3\pi y_2 & \cdots & 0 & \cdots & 0 & 0 & \sin N\pi y_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \sin\pi y_2 & 0 & 0 & 0 & \cdots & \sin 2\pi y_2 & 0 & 0 & 0 & \cdots & \sin 3\pi y_2 & \cdots & 0 & 0 & 0 & \cdots & \sin N\pi y_2 \\ \hline
\sin\pi y_3 & 0 & 0 & \cdots & 0 & \sin 2\pi y_3 & 0 & 0 & \cdots & 0 & \sin 3\pi y_3 & 0 & 0 & \cdots & 0 & \cdots & \sin N\pi y_3 & 0 & 0 & \cdots & 0 \\
0 & \sin\pi y_3 & 0 & \cdots & 0 & 0 & \sin 2\pi y_3 & 0 & \cdots & 0 & 0 & \sin 3\pi y_3 & 0 & \cdots & 0 & \cdots & 0 & \sin N\pi y_3 & 0 & \cdots & 0 \\
0 & 0 & \sin\pi y_3 & \cdots & 0 & 0 & 0 & \sin 2\pi y_3 & \cdots & 0 & 0 & 0 & \sin 3\pi y_3 & \cdots & 0 & \cdots & 0 & 0 & \sin N\pi y_3 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \sin\pi y_3 & 0 & 0 & 0 & \cdots & \sin 2\pi y_3 & 0 & 0 & 0 & \cdots & \sin 3\pi y_3 & \cdots & 0 & 0 & 0 & \cdots & \sin N\pi y_3 \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline
\sin\pi y_N & 0 & 0 & \cdots & 0 & \sin 2\pi y_N & 0 & 0 & \cdots & 0 & \sin 3\pi y_N & 0 & 0 & \cdots & 0 & \cdots & \sin N\pi y_N & 0 & 0 & \cdots & 0 \\
0 & \sin\pi y_N & 0 & \cdots & 0 & 0 & \sin 2\pi y_N & 0 & \cdots & 0 & 0 & \sin 3\pi y_N & 0 & \cdots & 0 & \cdots & 0 & \sin N\pi y_N & 0 & \cdots & 0 \\
0 & 0 & \sin\pi y_N & \cdots & 0 & 0 & 0 & \sin 2\pi y_N & \cdots & 0 & 0 & 0 & \sin 3\pi y_N & \cdots & 0 & \cdots & 0 & 0 & \sin N\pi y_N & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \sin\pi y_N & 0 & 0 & 0 & \cdots & \sin 2\pi y_N & 0 & 0 & 0 & \cdots & \sin 3\pi y_N & \cdots & 0 & 0 & 0 & \cdots & \sin N\pi y_N
\end{array}\right]$$

Untuk membuktikan bahwa kedua determinan matriks tidak nol, terlebih dahulu akan disajikan dua proposisi berikut.

Proposisi-1:

$$\text{Det } \mathbf{A}^{x}{}_{(\mathbf{M.N})\,\times\,(\mathbf{M.N})} = \begin{vmatrix} \sin\pi x_1 & \sin 2\pi x_1 & \sin 3\pi x_1 & \cdots & \sin M\pi x_1 \\ \sin\pi x_2 & \sin 2\pi x_2 & \sin 3\pi x_2 & \cdots & \sin M\pi x_2 \\ \sin\pi x_3 & \sin 2\pi x_3 & \sin 3\pi x_3 & \cdots & \sin M\pi x_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sin\pi x_M & \sin 2\pi x_M & \sin 3\pi x_M & \cdots & \sin M\pi x_M \end{vmatrix}^{N} \qquad ...(2)$$

Bukti:

$\mathbf{A}^x$ adalah matriks berorde $(M.N) \times (M.N)$ yang terdiri dari N buah matriks partisi yang masing-masing berorde $(M.N)$ dan membentuk matriks diagonal pada $\mathbf{A}^x$.
Melalui operasi baris elementer [6], iterasi pertamanya adalah:
Baris : $B_{[i.M+2]} - \frac{sin\pi x_2}{sin\pi x_1}.B_{i.M+1}$; $B_{[i.M+3]} - \frac{sin\pi x_3}{sin\pi x_1}.B_{i.M+1}$; ... ; $B_{[i.M+M]} - \frac{sin\pi x_M}{sin\pi x_1}.B_{i.M+1}$.

dengan i = 0 , 1 , ..., (N-1).M

Selanjutnya dengan pola yang sama, iterasi diteruskan sampai iterasi ke-(N-1). Hasilnya, matriks $\mathbf{A}^x$ akan berubah menjadi matriks segi tiga atas dengan nilai determinan sama dengan perkalian unsur-unsur diagonalnya. Karena matriks partisinya sama, sebanyak N buah, maka nilai determinan tersebut sama dengan perkalian dari determinan matriks parisi masing-masing sebanyak N. Jadi (2) terbukti. ∎

Proposisi-2:

$$\text{Det } \mathbf{A}^{y}_{(\mathbf{M}\times\mathbf{N})\times(\mathbf{M}\times\mathbf{N})} = \begin{vmatrix} \sin \pi y_1 & \sin 2\pi y_1 & \sin 3\pi y_1 & \ldots & \sin N\pi y_1 \\ \sin \pi y_2 & \sin 2\pi y_2 & \sin 3\pi y_2 & \ldots & \sin N\pi y_2 \\ \sin \pi y_3 & \sin 2\pi y_3 & \sin 3\pi y_3 & \ldots & \sin N\pi y_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sin \pi y_N & \sin 2\pi y_N & \sin 3\pi y_N & \ldots & \sin N\pi y_N \end{vmatrix}^{M} \qquad \ldots(3)$$

Bukti:

Langkah pertama akan dihitung nilai dari $\begin{vmatrix} \sin \pi y_1 & \sin 2\pi y_1 & \sin 3\pi y_1 & \ldots & \sin N\pi y_1 \\ \sin \pi y_2 & \sin 2\pi y_2 & \sin 3\pi y_2 & \ldots & \sin N\pi y_2 \\ \sin \pi y_3 & \sin 2\pi y_3 & \sin 3\pi y_3 & \ldots & \sin N\pi y_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sin \pi y_N & \sin 2\pi y_N & \sin 3\pi y_N & \ldots & \sin N\pi y_N \end{vmatrix}$.

Melalui operasi baris elementer pada iterasi-1 dengan baris $B_2 - \frac{sin\pi y_2}{sin\pi y_1}.B_1$;

$B_3 - \frac{sin\pi y_3}{sin\pi y_1}.B_1$; ... ; $B_i - \frac{sin\pi y_2}{sin\pi y_1}.B_1$, i = 2 , 3 , . . . , N, hasilnya adalah

$$= \text{Det} \begin{pmatrix} \sin\pi y_1 & \sin 2\pi y_1 & \sin 3\pi y_1 & \sin 4\pi y_1 & \ldots & \sin N\pi y_1 \\ 0 & A_{22}^{(1)} & A_{23}^{(1)} & A_{24}^{(1)} & \ldots & A_{2N}^{(1)} \\ 0 & A_{32}^{(1)} & A_{33}^{(1)} & A_{34}^{(1)} & \ldots & A_{3N}^{(1)} \\ 0 & A_{42}^{(1)} & A_{43}^{(1)} & A_{44}^{(1)} & \ldots & A_{4N}^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & A_{N2}^{(1)} & A_{N3}^{(1)} & A_{N4}^{(1)} & \ldots & A_{NN}^{(1)} \end{pmatrix}$$

di mana: $A_{22}^{(1)} = sin2\pi y_2 - \frac{sin\pi y_2}{sin\pi y_1}.sin2\pi y_1$; $A_{32}^{(1)} = sin2\pi y_3 - \frac{sin\pi y_3}{sin\pi y_1}.sin2\pi y_1$; dst . . .,

yang secara umum: $A_{ij}^{(1)} = sinj\pi y_i - \frac{sin\pi y_i}{sin\pi y_1}.sinj\pi y_1$

Selanjutnya setelah dilakukan iterasi-2 di mana

baris $B_3 - \frac{A_{32}^{(1)}}{A_{22}^{(1)}}.B_2$; $B_4 - \frac{A_{42}^{(1)}}{A_{22}^{(1)}}.B_2$; ... ; $B_i - \frac{A_{i2}^{(1)}}{A_{22}^{(1)}}.B_2$, i = 3 , 4 , . . . , N dan

iterasi-3 di mana

baris $B_4 - \frac{A_{43}^{(2)}}{A_{33}^{(2)}}.B_3$; ... ; $B_i - \frac{A_{i3}^{(2)}}{A_{33}^{(2)}}.B_3$, i = 4 , . . . , N, maka hasilnya menjadi

$$= \text{Det}\begin{pmatrix} \sin\pi y_1 & \sin 2\pi y_1 & \sin 3\pi y_1 & \sin 4\pi y_1 & \cdots & \sin N\pi y_1 \\ 0 & A_{22}^{(1)} & A_{23}^{(1)} & A_{24}^{(1)} & \cdots & A_{2N}^{(1)} \\ 0 & 0 & A_{33}^{(2)} & A_{34}^{(2)} & \cdots & A_{3N}^{(2)} \\ 0 & 0 & 0 & A_{44}^{(3)} & \cdots & A_{4N}^{(3)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & A_{N4}^{(3)} & \cdots & A_{NN}^{(3)} \end{pmatrix}$$

dengan $A_{ij}^{(2)} = A_{ij}^{(1)} - \frac{A_{i2}^{(1)}}{A_{22}^{(1)}} . A_{2j}^{(1)}$ dan $A_{ij}^{(3)} = A_{ij}^{(2)} - \frac{A_{i3}^{(2)}}{A_{33}^{(2)}} . A_{3j}^{(2)}$

Apabila proses iterasi diteruskan, maka setelah iterasi ke-(N-1), diperoleh:

$$= \text{Det}\begin{pmatrix} \sin\pi y_1 & \sin 2\pi y_1 & \sin 3\pi y_1 & \sin 4\pi y_1 & \cdots & \sin N\pi y_1 \\ 0 & A_{22}^{(1)} & A_{23}^{(1)} & A_{24}^{(1)} & \cdots & A_{2N}^{(1)} \\ 0 & 0 & A_{33}^{(2)} & A_{34}^{(2)} & \cdots & A_{3N}^{(2)} \\ 0 & 0 & 0 & A_{44}^{(3)} & \cdots & A_{4N}^{(3)} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & A_{NN}^{(N-1)} \end{pmatrix}$$

$$= \sin\pi y_1 \, . \, A_{22}^{(1)} \, . \, A_{33}^{(2)} \, . A_{44}^{(3)} \ldots A_{NN}^{(N-1)}$$

$$= \prod_{i=1}^{N}\left(A_{ii}^{(i-1)}\right) \qquad \ldots(4)$$

di mana : $A_{11}^{(0)} = sin\pi y_1$, $A_{22}^{(1)} = sin2\pi y_2 - \frac{sin\pi y_2}{sin\pi y_1} . sin2\pi y_1$, . . .

secara umum: $A_{ij}^{(k)} = A_{ij}^{(k-1)} - \frac{A_{ik}^{(k-1)}}{A_{ii}^{(k-1)}} . A_{kj}^{(k-1)}$.

Langkah selanjutnya adalah membuktikan bahwa

$$\text{Det } \mathbf{A}^y{}_{(\mathbf{M} \times \mathbf{N}) \times (\mathbf{M} \times \mathbf{N})} = \prod_{i=1}^{N}\left(A_{ii}^{(i-1)}\right) .$$

Melalui operasi baris elementer, setelah iterasi-1 dengan:

baris $B_{[(i-1)M + 1]} - \frac{sin\pi y_i}{sin\pi y_1} . B_1$, $B_{[(i-1)M + 2]} - \frac{sin\pi y_i}{sin\pi y_1} . B_2$ dst...,

yang secara umum: $B_{[(i-1)M + j]} - \frac{sin\pi y_i}{sin\pi y_1} . B_j$, i = 2 , 3 , . . . , N-1 ; j = 1 , 2 , . . . , M , maka hasilnya det $\mathbf{A}^y$ sama dengan determinan matriks:

$$
\left[\begin{array}{ccccc|ccccc|ccccc|c|ccccc}
\sin\pi y_1 & 0 & 0 & \cdots & 0 & \sin 2\pi y_1 & 0 & 0 & \cdots & 0 & \sin 3\pi y_1 & 0 & 0 & \cdots & 0 & \cdots & \sin N\pi y_1 & 0 & 0 & \cdots & 0 \\
0 & \sin\pi y_1 & 0 & \cdots & 0 & 0 & \sin 2\pi y_1 & 0 & \cdots & 0 & 0 & \sin 3\pi y_1 & 0 & \cdots & 0 & \cdots & 0 & \sin N\pi y_1 & 0 & \cdots & 0 \\
0 & 0 & \sin\pi y_1 & \cdots & 0 & 0 & 0 & \sin 2\pi y_1 & \cdots & 0 & 0 & 0 & \sin 3\pi y_1 & \cdots & 0 & \cdots & 0 & 0 & \sin N\pi y_1 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & \sin\pi y_1 & 0 & 0 & 0 & \cdots & \sin 2\pi y_1 & 0 & 0 & 0 & \cdots & \sin 3\pi y_1 & \cdots & 0 & 0 & 0 & \cdots & \sin N\pi y_1 \\ \hline
0 & 0 & 0 & \cdots & 0 & A^{(1)}_{22} & 0 & 0 & \cdots & 0 & A^{(1)}_{23} & 0 & 0 & \cdots & 0 & \cdots & A^{(1)}_{2N} & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & A^{(1)}_{22} & 0 & \cdots & 0 & 0 & A^{(1)}_{23} & 0 & \cdots & 0 & \cdots & 0 & A^{(1)}_{2N} & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & A^{(1)}_{22} & \cdots & 0 & 0 & 0 & A^{(1)}_{23} & \cdots & 0 & \cdots & 0 & 0 & A^{(1)}_{2N} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & A^{(1)}_{22} & 0 & 0 & 0 & \cdots & A^{(1)}_{23} & \cdots & 0 & 0 & 0 & \cdots & A^{(1)}_{2N} \\ \hline
0 & 0 & 0 & \cdots & 0 & A^{(1)}_{32} & 0 & 0 & \cdots & 0 & A^{(1)}_{33} & 0 & 0 & \cdots & 0 & \cdots & A^{(1)}_{3N} & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & A^{(1)}_{32} & 0 & \cdots & 0 & 0 & A^{(1)}_{33} & 0 & \cdots & 0 & \cdots & 0 & A^{(1)}_{3N} & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & A^{(1)}_{32} & \cdots & 0 & 0 & 0 & A^{(1)}_{33} & \cdots & 0 & \cdots & 0 & 0 & A^{(1)}_{3N} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & A^{(1)}_{32} & 0 & 0 & 0 & \cdots & A^{(1)}_{33} & \cdots & 0 & 0 & 0 & \cdots & A^{(1)}_{3N} \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline
0 & 0 & 0 & \cdots & 0 & A^{(1)}_{N2} & 0 & 0 & \cdots & 0 & A^{(1)}_{N3} & 0 & 0 & \cdots & 0 & \cdots & A^{(1)}_{NN} & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & A^{(1)}_{N2} & 0 & \cdots & 0 & 0 & A^{(1)}_{N3} & 0 & \cdots & 0 & \cdots & 0 & A^{(1)}_{NN} & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & A^{(1)}_{N2} & \cdots & 0 & 0 & 0 & A^{(1)}_{N3} & \cdots & 0 & \cdots & 0 & 0 & A^{(1)}_{NN} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & A^{(1)}_{N2} & 0 & 0 & 0 & \cdots & A^{(1)}_{N3} & \cdots & 0 & 0 & 0 & \cdots & A^{(1)}_{NN}
\end{array}\right]
$$

di mana: $A_{22}^{(1)} = sin2\pi y_2 - \frac{sin\pi y_2}{sin\pi y_1}.sin2\pi y_1$; $A_{32}^{(1)} = sin2\pi y_3 - \frac{sin\pi y_3}{sin\pi y_1}.sin2\pi y_1$; dst . . .,

yang secara umum: $A_{ij}^{(1)} = sinj\pi y_i - \frac{sin\pi y_i}{sin\pi y_1}.sinj\pi y_1$

Setelah iterasi-2 dengan baris $B_{[2M+1]} - \frac{A_{32}^{(1)}}{A_{22}^{(1)}}.B_{M+1}$, $B_{[2M+2]} - \frac{A_{32}^{(1)}}{A_{22}^{(1)}}.B_{M+2}$ dst...

yang secara umum: $B_{[(i-1)M+j]} - \frac{A_{i2}^{(1)}}{A_{22}^{(1)}}.B_{M+j}$; i = 3 , 4 , . . . , N-1 ; j = 1 , 2 , . . . , M, hasilnya adalah sama dengan determinan matriks:

$$
\left[\begin{array}{ccccc|ccccc|ccccc|ccccc|c}
\sin\pi y_1 & 0 & 0 & \cdots & 0 & \sin 2\pi y_1 & 0 & 0 & \cdots & 0 & \sin 3\pi y_1 & 0 & 0 & \cdots & 0 & \sin 4\pi y_1 & 0 & 0 & \cdots & 0 & \cdots \\
0 & \sin\pi y_1 & 0 & \cdots & 0 & 0 & \sin 2\pi y_1 & 0 & \cdots & 0 & 0 & \sin 3\pi y_1 & 0 & \cdots & 0 & 0 & \sin 4\pi y_1 & 0 & \cdots & 0 & \cdots \\
0 & 0 & \sin\pi y_1 & \cdots & 0 & 0 & 0 & \sin 2\pi y_1 & \cdots & 0 & 0 & 0 & \sin 3\pi y_1 & \cdots & 0 & 0 & 0 & \sin 4\pi y_1 & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\
0 & 0 & 0 & \cdots & \sin\pi y_1 & 0 & 0 & 0 & \cdots & \sin 2\pi y_1 & 0 & 0 & 0 & \cdots & \sin 3\pi y_1 & 0 & 0 & 0 & \cdots & \sin 4\pi y_1 & \cdots \\ \hline
0 & 0 & 0 & \cdots & 0 & A^{(1)}_{22} & 0 & 0 & \cdots & 0 & A^{(2)}_{23} & 0 & 0 & \cdots & 0 & A^{(2)}_{24} & 0 & 0 & \cdots & 0 & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & A^{(1)}_{22} & 0 & \cdots & 0 & 0 & A^{(2)}_{23} & 0 & \cdots & 0 & 0 & A^{(2)}_{24} & 0 & \cdots & 0 & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & A^{(1)}_{22} & \cdots & 0 & 0 & 0 & A^{(2)}_{23} & \cdots & 0 & 0 & 0 & A^{(2)}_{24} & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & A^{(1)}_{22} & 0 & 0 & 0 & \cdots & A^{(2)}_{23} & 0 & 0 & 0 & \cdots & A^{(2)}_{24} & \cdots \\ \hline
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & A^{(2)}_{33} & 0 & 0 & \cdots & 0 & A^{(2)}_{34} & 0 & 0 & \cdots & 0 & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & A^{(2)}_{33} & 0 & \cdots & 0 & 0 & A^{(2)}_{34} & 0 & \cdots & 0 & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & A^{(2)}_{33} & \cdots & 0 & 0 & 0 & A^{(2)}_{34} & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & A^{(2)}_{33} & 0 & 0 & 0 & \cdots & A^{(2)}_{34} & \cdots \\ \hline
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & A^{(2)}_{43} & 0 & 0 & \cdots & 0 & A^{(2)}_{44} & 0 & 0 & \cdots & 0 & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & A^{(2)}_{43} & 0 & \cdots & 0 & 0 & A^{(2)}_{44} & 0 & \cdots & 0 & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & A^{(2)}_{43} & \cdots & 0 & 0 & 0 & A^{(2)}_{44} & \cdots & 0 & \cdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots \\
0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & A^{(2)}_{43} & \cdots & 0 & 0 & 0 & A^{(2)}_{44} & \cdots \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right]
$$

di mana $A_{33}^{(2)} = A_{33}^{(1)} - \frac{A_{32}^{(1)}}{A_{22}^{(1)}}.A_{23}^{(1)}$. Secara umum: $A_{ij}^{(2)} = A_{ij}^{(1)} - \frac{A_{i2}^{(1)}}{A_{22}^{(1)}}.A_{2j}^{(1)}$.

Setelah iterasi ke-(N-1), diperoleh matriks segi tiga atas dengan nilai determinan:

$$= (sin\pi y_1)^M \,.\left(A_{22}^{(1)}\right)^M .\ \left(A_{33}^{(2)}\right)^M \ \ldots \ \left(A_{NN}^{(N-1)}\right)^M = \left[\prod_{i=1}^{N}\left(A_{ii}^{(i-1)}\right)\right]^M$$

Dengan menggunakan (4), maka terbukti bahwa

$$\text{Det } \mathbf{A}^y = \begin{vmatrix} \sin \pi y_1 & \sin 2\pi y_1 & \sin 3\pi y_1 & \ldots & \sin N\pi y_1 \\ \sin \pi y_2 & \sin 2\pi y_2 & \sin 3\pi y_2 & \ldots & \sin N\pi y_2 \\ \sin \pi y_3 & \sin 2\pi y_3 & \sin 3\pi y_3 & \ldots & \sin N\pi y_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sin \pi y_N & \sin 2\pi y_N & \sin 3\pi y_N & \ldots & \sin N\pi y_N \end{vmatrix}^M \qquad \blacksquare$$

Teorema:

Jika **A** =

$$\begin{bmatrix} \sin\pi x_1.\sin\pi y_1 & \ldots & \sin M\pi x_1.\sin\pi y_1 & \sin\pi x_1.\sin 2\pi y_1 & \ldots & \sin M\pi x_1.\sin 2\pi y_1 & \ldots\ \ldots & \sin\pi x_1.\sin N\pi y_1 & \ldots & \sin M\pi x_1.\sin N\pi y_1 \\ \vdots & & & & & & & & & \vdots \\ \sin\pi x_M.\sin\pi y_1 & \ldots & \sin M\pi x_M.\sin\pi y_1 & \sin\pi x_M \sin 2\pi y_1 & \ldots & \sin M\pi x_M.\sin 2\pi y_1 & \ldots\ \ldots & \sin\pi x_M.\sin N\pi y_1 & \ldots & \sin M\pi x_M.\sin N\pi y_1 \\ \sin\pi x_1.\sin\pi y_2 & \ldots & \sin M\pi x_M.\sin\pi y_2 & \sin\pi x_M \sin 2\pi y_2 & \ldots & \sin M\pi x_M.\sin 2\pi y_2 & \ldots\ \ldots & \sin\pi x_M.\sin N\pi y_2 & \ldots & \sin M\pi x_M.\sin N\pi y_M \\ \vdots & & & & & & & & & \vdots \\ \sin\pi x_M.\sin\pi y_2 & \ldots & \sin M\pi x_M.\sin\pi y_2 & \sin\pi x_M \sin 2\pi y_2 & \ldots & & \ldots\ \ldots & & \ldots & \\ \vdots & & & & & & & & & \vdots \\ \vdots & & & & & & & & & \vdots \\ \sin\pi x_M.\sin\pi y_N & \ldots & \sin M\pi x_M.\sin\pi y_N & \sin\pi x_M \sin 2\pi y_N & \ldots & & \ldots\ \ldots & & \ldots & \\ \vdots & & & & & & & & \ddots & \vdots \\ \sin\pi x_M.\sin\pi y_N & \ldots & \sin M\pi x_M.\sin\pi y_N & \sin\pi x_M \sin 2\pi y_N & \ldots & & \ldots\ \ldots & & \ldots & \sin M\pi x_M.\sin N\pi y_N \end{bmatrix}$$

maka: Det **A** $\neq 0$.

Bukti:

Dari (1) diperoleh : det **A** = det $\mathbf{A}^x$ **.** det $\mathbf{A}^y$
Kemudian berdasarkan proposisi 1 dan 2 didapat:

$$\text{Det } \mathbf{A} = \begin{vmatrix} \sin\pi x_1 & \sin 2\pi x_1 & \sin 3\pi x_1 & \ldots & \sin M\pi x_1 \\ \sin\pi x_2 & \sin 2\pi x_2 & \sin 3\pi x_2 & \ldots & \sin M\pi x_2 \\ \sin\pi x_3 & \sin 2\pi x_3 & \sin 3\pi x_3 & \ldots & \sin M\pi x_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sin\pi x_M & \sin 2\pi x_M & \sin 3\pi x_M & \ldots & \sin M\pi x_M \end{vmatrix}^N \cdot \begin{vmatrix} \sin\pi y_1 & \sin 2\pi y_1 & \sin 3\pi y_1 & \ldots & \sin N\pi y_1 \\ \sin\pi y_2 & \sin 2\pi y_2 & \sin 3\pi y_2 & \ldots & \sin N\pi y_2 \\ \sin\pi y_3 & \sin 2\pi y_3 & \sin 3\pi y_3 & \ldots & \sin N\pi y_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sin\pi y_N & \sin 2\pi y_N & \sin 3\pi y_N & \ldots & \sin N\pi y_N \end{vmatrix}^M$$

Karena masing-masing det $\mathbf{A}^x$, dan det $\mathbf{A}^y \neq 0$, maka det **A** $\neq 0$. ■

3. Eksistensi fungsi $u(x,y)$ yang Melalui K Buah Titik Sembarang

Dalam bagian ini akan dibahas eksistensi fungsi $u(x,y)$ yang melalui K buah titik sembarang (x_i , y_i , c_i) dengan $0 < x_i < 1$, dan $0 < y_i < 1$ untuk i = 1, 2, . . . , K.
Pembahasan akan dibagi dalam dua kasus berbeda.

Kasus-1:
Apabila $x_i \neq x_j$ dan $y_i \neq y_j$ untuk setiap i dan j, maka pandanglah $u(x,y)$ yang melalui K^2 buah titik sehingga K buah titik di atas merupakan bagian di dalamnya. Berdasarkan masalah pada poin **2** di atas maka dijamin ada $u(x,y)$ yang melalui K^2 buah titik tadi, yang berarti juga melalui K buah titik asalnya.

Kasus-2:
Apabila $x_i = x_j$ untuk suatu i dan j, serta $y_s = y_t$ untuk suatu s dan t, maka x_i dan y_i dapat disusun ulang sedemikian sehingga $0 < x_1 < x_2 < \ldots < x_M < 1$ dan $0 < y_1 < y_2 < \ldots < y_N < 1$ dengan M dan N masing-masing lebih kecil dari K. Selajutnya pandanglah $u(x,y)$ yang melalui (M.N) buah titik sehingga K buah titik di atas merupakan bagian di dalamnya. Dengan demikian berdasarkan masalah pada poin **2** di atas maka dijamin ada $u(x,y)$ yang melalui (M.N) buah titik tadi, yang berarti juga melalui K buah titik asalnya.

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