On n-normed spaces and n-inner product spaces

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The notion of n-normed spaces was introduced by S. Gähler in the 1960's. Later on, the concept of n-inner product spaces was developed by A. Misiak at the end of the 1980's.

If in a normed space one speaks about the length of a vector, then in an n-normed space we discuss about the volume of the parallelepiped spanned by a set of n-vectors.

In this talk, we shall review some results (mostly that my collaborators and I obtained) on these two spaces, particularly those dealing with the topological and geometrical aspects on the spaces.

Recent results on various notions of orthogonality and angles in n-normed spaces will also be presented.

Who's Who, Among Others . . .

S. Gähler and collaborators, incl. C. Diminnie and A. White (1960's-1970's)
K. Iseki (1975-1976)
S.N. Lai and A.K. Singh (1978)
A. Khan and A. Siddiqui (1982)
G. Godini (1985)
A. Misiak (1989)
Y.J. Cho and collaborators, incl. S.S. Kim and N.J. Huang (1990's)
M.S. Khan dan M.D. Khan (1993)
D.R. Jain and R. Chugh (1995)
H. Mazaheri (2007-now)
M. Saha and collaborators (2008-now)
H. Gunawan and collaborators, incl. Mashadi and O. Neswan (2000-now)

Definition of n-normed spaces

Let $n \in \mathbb{N}$ and X be a real vector space of dimension $d \geq n$. (Here d can be infinite.) A real-valued function $\|\cdot, \dots, \cdot\|$ on X^n satisfying the following four properties

(N1) $\|x_1, \dots, x_n\| = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent;

(N2) $\|x_1, \dots, x_n\|$ is invariant under permutation;

(N3) $\|\alpha x_1, \dots, x_n\| = |\alpha| \, \|x_1, \dots, x_n\|$ for $\alpha \in \mathbb{R}$;

(N4) $\|x_1 + x_1', x_2, \dots, x_n\| \leq \|x_1, x_2, \dots, x_n\| + \|x_1', x_2, \dots, x_n\|$,

is called an n-norm on X. The pair $(X, \|\cdot, \dots, \cdot\|)$ is called an n-normed space.

Notes

In an n-normed space $(X, \|\cdot, \dots, \cdot\|)$, we have

$\|x_1, \dots, x_n\| \geq 0$ for every $x_1, \dots, x_n \in X$

and

$\|x_1, x_2, \dots, x_n\| = \|x_1 + \alpha_2 x_2 + \cdots + \alpha_n x_n, x_2, \dots, x_n\|$ for every $x_1, \dots, x_n \in X$ and $\alpha_2, \dots, \alpha_n \in \mathbb{R}$.

Standard example

If X is a real inner product space of dimension $d \geq n$, then the following formula defines an n-norm on X:

$$\|x_1, \ldots, x_n\|_S := \begin{vmatrix} \langle x_1, x_1 \rangle & \cdots & \langle x_1, x_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle & \cdots & \langle x_n, x_n \rangle \end{vmatrix}^{1/2},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on X.

Geometrically, $\|x_1, \ldots, x_n\|_S$ represents the volume of the n-dimensional parallelepiped spanned by $x_1, \ldots, x_n$ in X.

General example

For an arbitrary normed space X, one may define an n-norm on X by the formula

$$\|x_1, \ldots, x_n\|^* := \sup_{\substack{f_i \in X',\ \|f_i\| \le 1 \\ i=1,\ldots,n}} \begin{vmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n) \end{vmatrix}.$$

Here X' denotes the dual of X, which consists of bounded linear functionals on X.

Definition of n-inner product spaces

A real-valued function $\langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle$ on X^{n+1} satisfying the following five properties

(I1) $\langle x_1, x_1 | x_2, \ldots, x_n \rangle \geq 0$; and $\langle x_1, x_1 | x_2, \ldots, x_n \rangle = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent;

(I2) $\langle x_1, x_1 | x_2, \ldots, x_n \rangle = \langle x_{i_1}, x_{i_1} | x_{i_2}, \ldots, x_{i_n} \rangle$ for every permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$;

(I3) $\langle x_0, x_1 | x_2, \ldots, x_n \rangle = \langle x_1, x_0 | x_2, \ldots, x_n \rangle$;

(I4) $\langle \alpha x_0, x_1 | x_2, \ldots, x_n \rangle = \alpha \langle x_0, x_1 | x_2, \ldots, x_n \rangle$ for every $\alpha \in \mathbb{R}$;

(I5)

$$\langle x_0 + x_0', x_1 | x_2, \ldots, x_n \rangle = \langle x_0, x_1 | x_2, \ldots, x_n \rangle + \langle x_0', x_1 | x_2, \ldots, x_n \rangle,$$

is called an n-inner product on X. The pair $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle$ is called an n-inner product space.

If $\langle \cdot, \cdot | \cdot, \dots, \cdot \rangle$ is an n-inner product on X, then we have **the Cauchy-Schwarz inequality**

$$\langle x_0, x_1 | x_2, \dots, x_n \rangle^2 \leq \langle x_0, x_0 | x_2, \dots, x_n \rangle \langle x_1, x_1 | x_2, \dots, x_n \rangle.$$

Moreover, the following function

$$\| x_1, x_2, \dots, x_n \| := \langle x_1, x_1 | x_2, \dots, x_n \rangle^{1/2}$$

defines an n-norm on X.

Standard example

If X is a real inner product space of dimension $d \geq n$, then the following formula defines an n-inner product on X:

$$\langle x_0, x_1 | x_2, \ldots, x_n \rangle_S := \begin{vmatrix} \langle x_0, x_1 \rangle & \langle x_0, x_2 \rangle & \cdots & \langle x_0, x_n \rangle \\ \langle x_2, x_1 \rangle & \langle x_2, x_2 \rangle & \cdots & \langle x_2, x_n \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle & \langle x_n, x_2 \rangle & \cdots & \langle x_n, x_n \rangle \end{vmatrix},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on X.

Geometrically, $\langle x_0, x_1 | x_2, \ldots, x_n \rangle_S$ has something to do with the angle between two n-dimensional parallelepipeds in X, one spanned by $x_0, x_2, \ldots, x_n$ and the other spanned by $x_1, x_2, \ldots, x_n$. (Here the two parallelepipeds intersect on the $(n-1)$ dimensional base spanned by $x_2, \ldots, x_n$.)

Theorem (GM, 2001): *Every n-normed space is a normed space*

Proof. Take a linearly independent set $\{a_1, \ldots, a_n\}$ in X. Then one may check that the following function $\|\cdot\|$ defines a norm on X:

$$\|x\| := \sum\nolimits_* \|x, a_{i_2}, \ldots, a_{i_n}\|,$$

where the sum $\sum_*$ is taken over all subsets $\{i_2, \ldots, i_n\}$ of $\{1, \ldots, n\}$. □

Note: We call the above norm the *derived* norm.

Convergent sequences in an n-normed space

A sequence $x(k)$ in an n-normed space $(X, \|\cdot, \dots, \cdot\|)$ is said *to converge* to a limit $x \in X$ (in the n-norm) whenever

$$\lim_{k\to\infty} \|x(k) - x, x_2, \dots, x_n\| = 0$$

for every $x_2, \dots, x_n \in X$.

Theorem (GM, 2001): *If $x(k)$ converges to x in the n-norm, then $x(k)$ also converges to x in the derived norm.*

Note: In the standard case, the converse is also true.

Cauchy sequences in n-normed spaces

A sequence $x(k)$ in an n-normed space X is *Cauchy* (with respect to the n-norm) if for every $x_2, \ldots, x_n \in X$, the sequence $(\|x(k), x_2, \ldots, x_n\|)$ forms a Cauchy sequence in $\mathbb{R}$.

An n-normed space X in which every Cauchy sequence in X is convergent to a limit in X is said to be *complete* and X is called an n-Banach space.

Theorem (GM, 2001): *A standard n-normed space is an n-Banach space if and only if it is a Banach space with respect to the induced norm* $\| \cdot \| := \langle \cdot, \cdot \rangle^{1/2}$.

Theorem (GM, 2001): *Let X be a standard dimensional n-Banach space, and T be a contractive mapping of X into itself, that is, there exists a constant $C \in (0,1)$ such that*

$$\|Tx_0 - Tx_1, x_2, \ldots, x_n\| \leq C\|x_0 - x_1, x_2, \ldots, x_n\|$$

for all $x_0, x_1, \ldots, x_n \in X$. Then T has a unique fixed point in X.

The space of p-summable sequences

For $1 \le p < \infty$, the following function $\|\cdot, \dots, \cdot\|$ defines an n-norm on $(\ell^p)^n$:

$$\|x_1, \dots, x_n\|_p := \left[\sum_{j_1} \cdots \sum_{j_n} \left| \det(x_{ij_k}) \right|^p \right]^{1/p}.$$

For $p = \infty$, the n-norm is defined by the formula

$$\|x_1, \dots, x_n\|_\infty := \sup_{j_1} \cdots \sup_{j_n} \left| \det(x_{ij_k}) \right|.$$

One may check that for $p = 2$, the formula coincides with the standard n-norm.

Theorem (G, 2001): *A sequence in* ℓ^p *is convergent in the* n*-norm* $\|\cdot,\ldots,\cdot\|_p$ *if and only if it is convergent in the usual norm* $\|\cdot\|_p$*. Similarly, a sequence in* ℓ^p *is Cauchy w.r.t. the* n*-norm if and only if it is Cauchy w.r.t.* $\|\cdot\|_p$*.*

Theorem (G, 2001): *The space* ℓ^p *is an* n*-Banach space w.r.t. the* n*-norm* $\|\cdot,\ldots,\cdot\|_p$*.*

A fixed point theorem

Theorem (G, 2001): *Suppose that T is a self-mapping of ℓ^p and there exists a constant $C \in (0,1)$ such that*

$$\|Tx_0 - Tx_1, x_2, \ldots, x_n\|_p \le C\|x_0 - x_1, x_2, \ldots, x_n\|_p$$

for all $x_0, x_1, \ldots, x_n \in \ell^p$. Then T has a unique fixed point in X.

Proof. We can show that T is contractive w.r.t. the usual norm $\|\cdot\|_p$ on ℓ^p. Since ℓ^p is complete, T must have a unique fixed point. □

Theorem (G, 2000a): *The C-S inequality for the standard n-inner product is equivalent to*

$$\begin{vmatrix} \langle x_0, x_0\rangle & \langle x_0, x_1\rangle & \cdots & \langle x_0, x_n\rangle \\ \langle x_1, x_1\rangle & \langle x_1, x_2\rangle & \cdots & \langle x_1, x_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_n, x_1\rangle & \langle x_n, x_2\rangle & \cdots & \langle x_n, x_n\rangle \end{vmatrix} \geq 0.$$

(For $n = 1$, the C-S inequality is equivalent to

$$\begin{vmatrix} \langle x_0, x_0\rangle & \langle x_0, x_1\rangle \\ \langle x_1, x_0\rangle & \langle x_1, x_1\rangle \end{vmatrix} \geq 0.)$$

Note: The determinant is the Gramian of size $(n+1) \times (n+1)$ w.r.t. the vectors $x_0, x_1, \ldots, x_n$. From the above result, we see that the equality holds if and only if $x_0, x_1, \ldots, x_n$ are linearly dependent.

Theorem (G, 2002b): *Every n-inner product space is an inner product space.*

Proof. Let $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ be an n-inner product space. Take a linearly independent set $\{a_1, \ldots, a_n\}$ in X. Then

$$\langle x, y \rangle := \sum\nolimits_* \langle x, y | a_{i_2}, \ldots, a_{i_n} \rangle$$

defines an inner product on X. (As before, the sum $\sum_*$ is taken over all subsets $\{i_2, \ldots, i_n\}$ of $\{1, \ldots, n\}$.) □

Generalized Cauchy-Schwarz Inequality

Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space, $x_1, \dots, x_n, y_1, \dots, y_n \in X$. Then we have

$$\begin{vmatrix} \langle x_1, y_1 \rangle & \cdots & \langle x_1, y_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, y_1 \rangle & \cdots & \langle x_n, y_n \rangle \end{vmatrix}^2 \leq \begin{vmatrix} \langle x_1, x_1 \rangle & \cdots & \langle x_1, x_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle & \cdots & \langle x_n, x_n \rangle \end{vmatrix} \times \begin{vmatrix} \langle y_1, y_1 \rangle & \cdots & \langle y_1, y_n \rangle \\ \vdots & \ddots & \vdots \\ \langle y_n, y_1 \rangle & \cdots & \langle y_n, y_n \rangle \end{vmatrix}.$$

In short, $\det^2(\langle x_i, y_j \rangle) \leq \det(\langle x_i, x_j \rangle) \cdot \det(\langle y_i, y_j \rangle)$.

As in inner product spaces, we have

$$\cos^2\theta := \frac{\det^2(\langle x_i, y_j\rangle)}{\det(\langle x_i, x_j\rangle)\cdot\det(\langle y_i, y_j\rangle)}$$

defines the angle θ between the two subspaces spanned by $x_1, \dots, x_n$ and $y_1, \dots, y_n$.

In general, the angle between the subspace spanned by $x_1, \dots, x_m$ and the subspace spanned by $y_1, \dots, y_n$ where $m \leq n$ is formulated in (GNS, 2005).

P-, I- and B-orthogonality

In normed spaces, the notions of Pythagorean (P), Isosceled (I) and Birkhoff-James (BJ) orthogonality are known.

(a) *P-orthogonality*: $x \perp_P y$ iff $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

(b) *I-orthogonality*: $x \perp_I y$ iff $\|x+y\| = \|x-y\|$.

(c) *BJ-orthogonality*: $x \perp_{BJ} y$ iff $\|x+\alpha y\| \geq \|x\|$ for every $\alpha \in \mathbb{R}$.

Note: In an inner product space $(X, \langle \cdot, \cdot \rangle)$, the three orthogonality conditions are equivalent to $\langle x, y \rangle = 0$.

Khan & Siddiqui's Definition

Let $(X, \|\cdot, \dots, \cdot\|)$ be an n-normed space of dimension $n+1$ or higher. For $x, y \in X$, Khan & Siddiqui (1982) defined

(a) $x \perp_{\mathrm{P}} y$ if only if
$\|x+y, z_2, \dots, z_n\|^2 = \|x, z_2, \dots, z_n\|^2 + \|y, z_2, \dots, z_n\|^2$ for every $z_2, \dots, z_n \neq 0$;
(b) $x \perp_{\mathrm{I}} y$ if only if $\|x+y, z_2, \dots, z_n\| = \|x-y, z_2, \dots, z_n\|$ for every $z_2, \dots, z_n \neq 0$;
(c) $x \perp_{\mathrm{BJ}} y$ if only if
$\|x+\alpha y, z_2, \dots, z_n\| \geq \|x, z_2, \dots, z_n\| \quad \text{for every } \alpha \in \mathbb{R}$ and for every $z_2, \dots, z_n \neq 0$.

This definition is too tight: there do not exist two nonzero vectors x and y satisfying the above conditions, even in the standard case.

Godini's Definition

Let $(X, \|\cdot, \dots, \cdot\|)$ be an n-normed space of dimension $n+1$ or higher. For $x, y \in X$, Godini (1985) defined

(a) $x \perp_{\rm P} y$ if only if $\|x+y, z_2, \dots, z_n\|^2 = \|x, z_2, \dots, z_n\|^2 + \|y, z_2, \dots, z_n\|^2$ for every $z_2, \dots, z_n \notin$ span$\{x, y\}$;
(b) $x \perp_{\rm I} y$ if only if $\|x+y, z_2, \dots, z_n\| = \|x-y, z_2, \dots, z_n\|$ for every $z_2, \dots, z_n \notin$ span$\{x, y\}$;
(c) $x \perp_{\rm BJ} y$ if only if $\|x+\alpha y, z_2, \dots, z_n\| \geq \|x, z_2, \dots, z_n\| \quad \text{for every } \alpha \in \mathbb{R}$ and for every $z_2, \dots, z_n \notin$ span$\{x, y\}$.

This definition is still void: there do not exist two nonzero vectors x and y satisfying the above conditions, even in the standard case.

Mazaheri's Definition

Let $(X, \|\cdot, \dots, \cdot\|)$ be an n-normed space of dimension $n+1$ or higher. For $x, y \in X$, Mazaheri (2007) defined

(a) $x \perp_{\mathrm{P}} y$ if only if there exist $b_2, \dots, b_n \in X$ with $\|x, b_2, \dots, b_n\| \neq 0$ such that $\|x+y, b_2, \dots, b_n\|^2 = \|x, b_2, \dots, b_n\|^2 + \|y, b_2, \dots, b_n\|^2$;
(b) $x \perp_{\mathrm{I}} y$ if only if there exist $b_2, \dots, b_n \in X$ with $\|x, b_2, \dots, b_n\| \neq 0$ such that $\|x+y, b_2, \dots, b_n\| = \|x-y, b_2, \dots, b_n\|$;
(c) $x \perp_{\mathrm{BJ}} y$ if only if there exist $b_2, \dots, b_n \in X$ with $\|x, b_2, \dots, b_n\| \neq 0$ such that $\|x+\alpha y, b_2, \dots, b_n\| \geq \|x, b_2, \dots, b_n\| \quad \text{for every } \alpha \in \mathbb{R}.$

This definition is too loose: in the standard case, any two linearly independent vectors are orthogonal.

Our definition

Let $(X, \|\cdot, \dots, \cdot\|)$ be an n-normed space of dimension $n+1$ or higher. For $x, y \in X$, we define

(a) $x \perp_{\mathrm{P}} y$ if only if there exists a subspace V of X with $\mathrm{codim}(V) = 1$ such that $\|x+y, z_2, \dots, z_n\|^2 = \|x, z_2, \dots, z_n\|^2 + \|y, z_2, \dots, z_n\|^2$ for every $z_2, \dots, z_n \in V$;
(b) $x \perp_{\mathrm{I}} y$ if only if there exists a subspace V of X with $\mathrm{codim}(V) = 1$ such that $\|x+y, z_2, \dots, z_n\| = \|x-y, z_2, \dots, z_n\|$ for every $z_2, \dots, z_n \in V$;
(c) $x \perp_{\mathrm{BJ}} y$ if only if there exists a subspace V of X with $\mathrm{codim}(V) = 1$ such that $\|x+\alpha y, z_2, \dots, z_n\| \geq \|x, z_2, \dots, z_n\|$ for every $\alpha \in \mathbb{R}$ and for every $z_2, \dots, z_n \in V$.

Important Fact: In the standard case the conditions are equivalent to $\langle x, y\rangle = 0$.

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