On the Product of $N$ Chebyshev Systems

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Introduction

The Product of 2 Chebyshev Systems

The Kronecker Product

The Product of $N$ Chebyshev Systems

References
Find a cts fn $u : [0, 1]^2 \rightarrow \mathbb{R}$ which minimizes an energy functional

$$E_\alpha(u) := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (m^2 + n^2)^\alpha a_{m,n}^2$$

and vanishes on the boundary and satisfies the interior conditions:

$$u(x_i, y_j) = c_{ij}, \quad i = 1, \ldots, M, \quad j = 1, \ldots, N,$$

where $0 < x_1 < \cdots < x_M < 1, \ 0 < y_1 < \cdots < y_N < 1$. 
Last Talk in Brunei, 2009 [1]

Since we are looking for a function $u(x, y)$ which vanishes on the boundary, we write $u(x, y)$ as a double sine series, that is,

$$u(x, y) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{m,n} \sin m\pi x \sin n\pi y,$$

where $a_{m,n}$'s are the coefficients that we need to find.

The solution is obtained iteratively, where the initial approximation is obtained by solving the system of equations

$$\sum_{m=1}^{M} \sum_{n=1}^{N} a_{m,n} \sin m\pi x_i \sin n\pi y_j = c_{ij}, \ i = 1, \ldots, M, \ j = 1, \ldots, N.$$
Last Talk in Brunei, 2009 [1]

The surface like in the following picture

may be obtained as the solution to our minimization problem, for some value of $\alpha$. 

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Let $A$ be a compact Hausdorff topological space that contains at least $m$ points.

A set of continuous, complex or real valued, functions \( \{\phi_1, \ldots, \phi_m\} \) on $A$ is called a **Chebyshev System** on $A$ if it satisfies the following condition $[2]$: For arbitrary $m$ distinct points $x_1, \cdots, x_m$ in $A$, we have

\[
\det[\phi_j(x_i)]_{m \times m} \neq 0,
\]

where $[\phi_j(x_i)]_{m \times m}$ is a matrix of order $m \times m$ (with $\phi_j(x_i)$ being the element on $i^{th}$-row and $j^{th}$-column).
For each \( n = 1, 2, \ldots, N \), let \( \Phi_n = \{ \phi_{n1}, \phi_{n2}, \ldots, \phi_{nm_n} \} \) be a Chebyshev system on Hausdorff topological space \( A_n \).

Then we are interested in how the tensor product \( \Phi \) of the \( N \) Chebyshev systems \( \Phi_n \)’s may be used to interpolate data on the Cartesian product \( A := A_1 \times A_2 \times \cdots \times A_N \).

We are aware that, in general, the product \( \Phi \) is not a Chebyshev System on \( A \) [5]. However, given certain set of data on \( A \), we may interpolate them using functions generated by \( \Phi \).
Our entry point is that the tensor product $\Phi$ of two Chebyshev Systems $\Phi_1 := \{\phi_{11}, \phi_{12}, \cdots, \phi_{1m_1}\}$ on $A_1$ and $\Phi_2 := \{\phi_{21}, \phi_{22}, \cdots, \phi_{2m_2}\}$ on $A_2$, that is, the set of functions

$$\Phi_{ij}(x_1, x_2) := \phi_{1i}(x_1)\phi_{2j}(x_2), \ i = 1, \ldots, m_1; \ j = 1, \ldots, m_2,$$

can interpolate data $\{(x_{1i}, x_{2j}, c_{ij}) : i = 1, \ldots, m_1; \ j = 1, \ldots, m_2\}$ on $A_1 \times A_2 \times F$, where $F = \mathbb{C}$ or $\mathbb{R}$ [4].
Example

For example, on the unit square $[0, 1]^2$, the set of functions

$$1, x, x^2, y, xy, x^2y, y^2, xy^2, x^2y^2,$$

may be viewed as the tensor product of the Chebyshev System

$$\{1, x, x^2\}$$

on $[0, 1]$ with itself.

This set can interpolate data \{$(x_i, y_j, c_{ij}) : i, j = 1, 2, 3$\}. 


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On the Product of $N$ Chebyshev Systems
The set \( \{(x_i, y_j) : i, j = 1, 2, 3\} \) forms a \( 3 \times 3 \) ‘grid’ on \([0, 1]^2\):
In general, suppose we are given an $m_1 \times m_2$ grid of points on $A_1 \times A_2$ which is the Cartesian product of $\{x_{11}, x_{12}, \ldots, x_{1m_1}\}$ and $\{x_{21}, x_{22}, \ldots, x_{2m_2}\}$, and arbitrary real numbers $c_{ij}$, $i = 1, \ldots, m_1$, $j = 1, \ldots, m_2$. Then, the interpolation problem

$$\sum_{i=1}^{m_1} \sum_{j=1}^{m_2} a_{ij} \Phi_{ij}(x_{1i}, x_{2j}) = c_{ij}, \ i = 1, \ldots, m_1; \ j = 1, \ldots, m_2,$$

has a unique solution, which is generated by $\Phi$.

Moreover, given some data on a subset of the grid, we can always find a function from $\Phi$ that interpolates the related nodes.
It is due to the fact that the **Kronecker product** \( M := M_1 \otimes M_2 \), where \( M_1 := [\phi_{1k}(x_{1i})]_{m_1 \times m_1} \) and \( M_2 := [\phi_{2l}(x_{2j})]_{m_2 \times m_2} \), is non-singular — since we have [3]

\[
\text{det } M = (\text{det } M_1)^{m_2}(\text{det } M_2)^{m_1}.
\]

Note. The Kronecker Product is given by the formula

\[
M_1 \otimes M_2 = \begin{bmatrix}
\phi_{11}(x_{11})M_2 & \phi_{11}(x_{12})M_2 & \ldots & \phi_{11}(x_{1m_1})M_2 \\
\phi_{12}(x_{11})M_2 & \phi_{12}(x_{12})M_2 & \ldots & \phi_{12}(x_{1m_1})M_2 \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{1m_1}(x_{11})M_2 & \phi_{1m_1}(x_{12})M_2 & \ldots & \phi_{1m_1}(x_{1m_1})M_2
\end{bmatrix}
\]

The matrix is a block matrix of size \( m_1m_2 \times m_1m_2 \).
Example

For example, suppose we want to interpolate \((\frac{1}{3}, \frac{1}{3}, 1), (\frac{2}{3}, \frac{1}{3}, \frac{1}{2}), (1, \frac{1}{3}, 1), (\frac{2}{3}, \frac{2}{3}, 1), \text{ and } (\frac{2}{3}, 1, 1)\) using the tensor product of the Chebyshev System \(\{1, x, x^2\}\) on \([0, 1]\) with itself. The data are given on a subset of a \(3 \times 3\) grid:
We know that there will be a function of the form

\[ u(x, y) = a_{11} + a_{12}x + a_{13}x^2 + a_{21}y + a_{22}xy + a_{23}x^2y + a_{31}y^2 + a_{32}xy^2 + a_{33}x^2y^2 \]

that interpolates the given nodes. Substituting the values from the given nodes and reducing into a row echelon form, we get

\[
\begin{bmatrix}
1 & 0 & 0 & 0 & -\frac{2}{9} & -\frac{4}{27} & 0 & -\frac{2}{27} & -\frac{4}{81} & \frac{7}{2} \\
0 & 1 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{9} & 0 & -6 \\
0 & 0 & 1 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{9} & \frac{9}{2} \\
0 & 0 & 0 & 1 & \frac{2}{3} & \frac{4}{9} & 0 & 0 & 0 & -\frac{17}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{2}{3} & \frac{4}{9} & \frac{15}{4}
\end{bmatrix}
\]
There are many functions that interpolate the given nodes, one of them is:

\[
u(x, y) = \frac{7}{2} - 6x + \frac{9}{2} x^2 - \frac{17}{4} y + \frac{15}{4} y^2.
\]
Recall that if $M_1$ and $M_2$ are non-singular matrices of size $m_1 \times m_1$ and $m_2 \times m_2$ respectively, then the Kronecker product of $M_1$ and $M_2$, namely $M = M_1 \otimes A_2$, is non-singular too. This follows from the formula (1):

$$\det M = (\det M_1)^{m_2} (\det M_2)^{m_1}.$$

The Kronecker product is an associative operation on matrices, that is, if $K, L$ and $M$ are the three matrices, then

$$(K \otimes L) \otimes M = K \otimes (L \otimes M).$$

Hence we may write $K \otimes L \otimes M$ for $(K \otimes L) \otimes M$ or $K \otimes (L \otimes M)$. 

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The following theorem generalizes the formula (1):

**Theorem**

Let $N \geq 2$ be an integer. For $n = 1, 2, \ldots, N$, let $M_n$ be non-singular matrices of size $m_n \times m_n$. Then we have

$$\det(M_1 \otimes M_2 \otimes \cdots \otimes M_N) = \prod_{n=1}^{N} (\det M_n) \frac{P}{m_n}$$

(2)

where $P = \prod_{n=1}^{N} m_n$. 

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Proof.

The theorem is proved by mathematical induction. We know that the formula is true for $N = 2$. Suppose it is true for $N \geq 2$. Then, we have

$$\det(M_1 \otimes \cdots \otimes M_N \otimes M_{N+1})$$

$$= \det((M_1 \otimes \cdots \otimes M_N) \otimes M_{N+1})$$

$$= \{\det(M_1 \otimes \cdots \otimes M_N)\}^{m_N+1} (\det M_{N+1})^{P_N}$$

$$= \prod_{n=1}^{N} (\det M_n) \frac{P_N m_{N+1}^{m_N+1}}{m_n} (\det M_{N+1}) \frac{P_N m_{N+1}}{m_{N+1}}$$

$$= \prod_{n=1}^{N+1} (\det M_n) \frac{P_{N+1} m_{N+1}}{m_{N+1}}$$,

where $P_N = \prod_{n=1}^{N} m_n$. 

\[\square\]
Consequently, we have:

**Corollary**

For \( n = 1, 2, \ldots, N \), let \( \Phi_n = \{\phi_{n1}, \phi_{n2}, \ldots, \phi_{nm_n}\} \) be a Chebyshev System on \( A_n \). Then the tensor product \( \Phi \) of the \( \Phi_n \)'s, namely the set of functions of the form

\[
\left\{ \prod_{n=1}^{N} \phi_{nj_n}(x_n): j_n = 1, \ldots, m_n; n = 1, \ldots, N \right\}
\]

can interpolate data on (arbitrary subsets of) any \( m_1 \times \cdots \times m_N \) grid in the Cartesian product \( A_1 \times \cdots \times A_N \).
Remark

It is interesting to note that our result connects the three types of products: the Cartesian product (of the domains), the tensor product (of the functions), and the Kronecker product (of the matrices).

A full paper on this topic is being written and will be submitted to a suitable journal when it is ready.


