On the Product of N Chebyshev Systems

Lukita Ambarwati[1,2] and Hendra Gunawan[1]
[1]ITB Bandung, [2]Universitas Negeri Jakarta
[1]http://personal.fmipa.itb.ac.id/hgunawan/
Analysis and Geometry Group
Bandung Institute of Technology
Bandung, Indonesia

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Outline

Last Talk in Brunei, 2009 [1]

Find a cts fn $u : [0,1]^2 \to \mathbb{R}$ which minimizes an energy functional

$$E_\alpha(u) := \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (m^2 + n^2)^\alpha a_{m,n}^2$$

and vanishes on the boundary and satisfies the interior conditions:

$$u(x_i, y_j) = c_{ij},\ i = 1, \ldots, M,\ j = 1, \ldots, N,$$

where $0 < x_1 < \cdots < x_M < 1,\ 0 < y_1 < \cdots < y_N < 1$.

Last Talk in Brunei, 2009 [1]

Since we are looking for a function $u(x,y)$ which vanishes on the boundary, we write $u(x,y)$ as a double sine series, that is,

$$u(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n} \sin m\pi x \sin n\pi y,$$

where $a_{m,n}$'s are the coefficients that we need to find.

The solution is obtained iteratively, where the initial approximation is obtained by solving the system of equations

$$\sum_{m=1}^{M}\sum_{n=1}^{N} a_{m,n} \sin m\pi x_i \sin n\pi y_j = c_{ij},\ i = 1, \dots, M,\ j = 1, \dots, N.$$

Last Talk in Brunei, 2009 [1]

The surface like in the following picture

may be obtained as the solution to our minimization problem, for some value of α.

Let A be a compact Hausdorff topological space that contains at least m points.

A set of continuous, complex or real valued, functions $\{\phi_1, \ldots, \phi_m\}$ on A is called a **Chebyshev System** on A if it satisfies the following condition [2]: For arbitrary m distinct points $x_1, \cdots, x_m$ in A, we have

$$\det[\phi_j(x_i)]_{m\times m} \neq 0,$$

where $[\phi_j(x_i)]_{m\times m}$ is a matrix of order $m \times m$ (with $\phi_j(x_i)$ being the element on i^{th}-row and j^{th}-column).

For each $n = 1, 2, \ldots, N$, let $\Phi_n = \{\phi_{n1}, \phi_{n2}, \ldots, \phi_{nm_n}\}$ be a Chebyshev system on Hausdorff topological space A_n.

Then we are interested in how the **tensor product** Φ of the N Chebyshev systems Φ_n's may be used to interpolate data on the **Cartesian product** $A := A_1 \times A_2 \times \cdots \times A_N$.

We are aware that, in general, the product Φ is not a Chebyshev System on A [5]. However, given certain set of data on A, we may interpolate them using functions generated by Φ.

Our entry point is that the tensor product Φ of two Chebyshev Systems $\Phi_1 := \{\phi_{11}, \phi_{12}, \cdots, \phi_{1m_1}\}$ on A_1 and $\Phi_2 := \{\phi_{21}, \phi_{22}, \cdots, \phi_{2m_2}\}$ on A_2, that is, the set of functions

$$\Phi_{ij}(x_1, x_2) := \phi_{1i}(x_1)\phi_{2j}(x_2),\ i = 1, \ldots, m_1;\ j = 1, \ldots, m_2,$$

can interpolate data $\{(x_{1i}, x_{2j}, c_{ij}) : i = 1, \ldots, m_1; j = 1, \ldots, m_2\}$ on $A_1 \times A_2 \times \mathbb{F}$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ [4].

Example

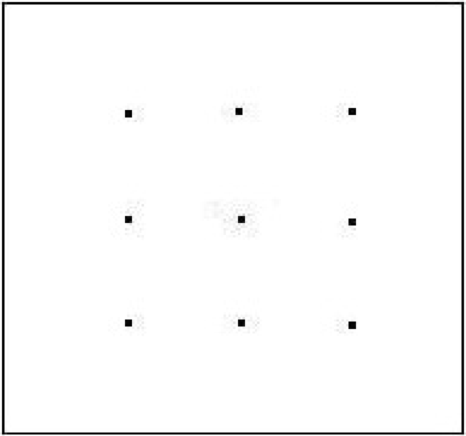
For example, on the unit square $[0,1]^2$, the set of functions

$$1, x, x^2, y, xy, x^2y, y^2, xy^2, x^2y^2,$$

may be viewed as the tensor product of the Chebyshev System $\{1, x, x^2\}$ on $[0,1]$ with itself.

This set can interpolate data $\{(x_i, y_j, c_{ij}) : i, j = 1, 2, 3\}$.

The set $\{(x_i, y_j) : i, j = 1, 2, 3\}$ forms a 3×3 'grid' on $[0, 1]^2$:



In general, suppose we are given an $m_1 \times m_2$ grid of points on $A_1 \times A_2$ which is the Cartesian product of $\{x_{11}, x_{12}, \ldots, x_{1m_1}\}$ and $\{x_{21}, x_{22}, \ldots, x_{2m_2}\}$, and arbitrary real numbers c_{ij}, $i = 1, \ldots, m_1$, $j = 1, \ldots, m_2$. Then, the interpolation problem

$$\sum_{i=1}^{m_1} \sum_{j=1}^{m_2} a_{ij} \Phi_{ij}(x_{1i}, x_{2j}) = c_{ij}, \ i = 1, \ldots, m_1; \ j = 1, \ldots, m_2,$$

has a unique solution, which is generated by Φ.

Moreover, given some data on a subset of the grid, we can always find a function from Φ that interpolates the related nodes.

It is due to the fact that the **Kronecker product** $M := M_1 \otimes M_2$, where $M_1 := [\phi_{1k}(x_{1i})]_{m_1 \times m_1}$ and $M_2 := [\phi_{2l}(x_{2j})]_{m_2 \times m_2}$, is non-singular — since we have [3]

$$\det M = (\det M_1)^{m_2} (\det M_2)^{m_1}. \quad (1)$$

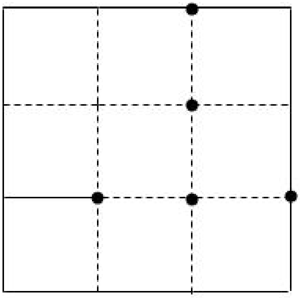
Note. The Kronecker Product is given by the formula

$$M_1 \otimes M_2 = \begin{bmatrix} \phi_{11}(x_{11})M_2 & \phi_{11}(x_{12})M_2 & \dots & \phi_{11}(x_{1m_1})M_2 \\ \phi_{12}(x_{11})M_2 & \phi_{12}(x_{12})M_2 & \dots & \phi_{12}(x_{1m_1})M_2 \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{1m_1}(x_{11})M_2 & \phi_{1m_1}(x_{12})M_2 & \dots & \phi_{1m_1}(x_{1m_1})M_2 \end{bmatrix}$$

The matrix is a block matrix of size $m_1 m_2 \times m_1 m_2$.

Example

For example, suppose we want to interpolate $(\frac{1}{3}, \frac{1}{3}, 1)$, $(\frac{2}{3}, \frac{1}{3}, \frac{1}{2})$, $(1, \frac{1}{3}, 1)$, $(\frac{2}{3}, \frac{2}{3}, 1)$, and $(\frac{2}{3}, 1, 1)$ using the tensor product of the Chebyshev System $\{1, x, x^2\}$ on $[0, 1]$ with itself. The data are given on a subset of a 3×3 grid:



We know that there will be a function of the form

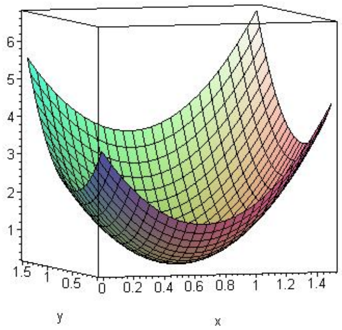
$$u(x,y) = a_{11} + a_{12}x + a_{13}x^2 + a_{21}y + a_{22}xy + a_{23}x^2y + a_{31}y^2 + a_{32}xy^2 + a_{33}x^2y^2$$

that interpolates the given nodes. Substituting the values from the given nodes and reducing into a row echelon form, we get

$$\left[\begin{array}{ccccccccc|c} 1 & 0 & 0 & 0 & -\frac{2}{9} & -\frac{4}{27} & 0 & -\frac{2}{27} & -\frac{4}{81} & \frac{7}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{9} & 0 & -6 \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{9} & \frac{9}{2} \\ 0 & 0 & 0 & 1 & \frac{2}{3} & \frac{4}{9} & 0 & 0 & 0 & -\frac{17}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{2}{3} & \frac{4}{9} & \frac{15}{4} \end{array}\right]$$

There are many functions that interpolate the given nodes, one of them is:

$$u(x,y) = \frac{7}{2} - 6x + \frac{9}{2}x^2 - \frac{17}{4}y + \frac{15}{4}y^2.$$



Recall that if M_1 and M_2 are non-singular matrices of size $m_1 \times m_1$ and $m_2 \times m_2$ respectively, then the Kronecker product of M_1 and M_2, namely $M = M_1 \otimes A_2$, is non-singular too. This follows from the formula (1):

$$\det M = (\det M_1)^{m_2} (\det M_2)^{m_1}.$$

The Kronecker product is an associative operation on matrices, that is, if K, L and M are the three matrices, then

$$(K \otimes L) \otimes M = K \otimes (L \otimes M).$$

Hence we may write $K \otimes L \otimes M$ for $(K \otimes L) \otimes M$ or $K \otimes (L \otimes M)$.

The following theorem generalizes the formula (1):

Theorem

Let $N \geq 2$ be an integer. For $n = 1, 2, \ldots, N$, let M_n be non-singular matrices of size $m_n \times m_n$. Then we have

$$\det(M_1 \otimes M_2 \otimes \cdots \otimes M_N) = \prod_{n=1}^{N} (\det M_n)^{\frac{P}{m_n}} \qquad (2)$$

where $P = \prod_{n=1}^{N} m_n$.

Proof.

The theorem is proved by mathematical induction. We know that the formula is true for $N = 2$. Suppose it is true for $N \geq 2$. Then, we have

$$\begin{aligned}
\det(M_1 \otimes \cdots \otimes M_N \otimes M_{N+1}) \\
&= \det\big((M_1 \otimes \cdots \otimes M_N) \otimes M_{N+1}\big) \\
&= \{\det(M_1 \otimes \cdots \otimes M_N)\}^{m_{N+1}} (\det M_{N+1})^{P_N} \\
&= \prod_{n=1}^{N} (\det M_n)^{\frac{P_N m_{N+1}}{m_n}} (\det M_{N+1})^{\frac{P_N m_{N+1}}{m_{N+1}}} \\
&= \prod_{n=1}^{N+1} (\det M_n)^{\frac{P_{N+1}}{m_n}},
\end{aligned}$$

where $P_N = \prod_{n=1}^{N} m_n$. □

Consequently, we have:

Corollary

For $n = 1, 2, \ldots, N$, let $\Phi_n = \{\phi_{n1}, \phi_{n2}, \ldots, \phi_{nm_n}\}$ be a Chebyshev System on A_n. Then the tensor product Φ of the Φ_n's, namely the set of functions of the form

$$\left\{ \prod_{n=1}^{N} \phi_{nj_n}(x_n) \, : \, j_n = 1, \ldots, m_n; n = 1, \ldots, N \right\}$$

can interpolate data on (arbitrary subsets of) any $m_1 \times \cdots \times m_N$ grid in the Cartesian product $A_1 \times \cdots \times A_N$.

Remark

It is interesting to note that our result connects the three types of products: the Cartesian product (of the domains), the tensor product (of the functions), and the Kronecker product (of the matrices).

A full paper on this topic is being written and will be submitted to a suitable journal when it is ready.

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