J. Indones. Math. Soc.
Vol. xx, No. xx (20xx), pp. xx–xx.

EQUIVALENCE OF n-NORMS ON THE SPACE OF p-SUMMABLE SEQUENCES

Anwar Mutaqin[1] and Hendra Gunawan[5]

[1]**Department of Mathematics Education, Universitas Sultan Ageng Tirtayasa, Serang, Indonesia, anwarmutaqin@gmail.com**
[2]**Department of Mathematics, Institut Teknologi Bandung, Bandung, Indonesia, hgunawan@math.itb.ac.id**

Abstract. We study the relation between two known n-norms on ℓ^p, the space of p-summable sequences. One n-norm is derived from Gähler's formula [3], while the other is due to Gunawan [6]. We show in particular that the convergence in one n-norm implies that in the other. The key is to show that the convergence in each of these n-norms is equivalent to that in the usual norm on ℓ^p.
Key words: n-normed spaces, p-summable sequence spaces, n-norm equivalence.

Abstrak. Dalam makalah ini dipelajari kaitan antara dua norm-n di ℓ^p, ruang barisan *summable-p*. Norm-n pertama diperoleh dari rumus Gähler [3], sementara norm-n kedua diperkenalkan oleh Gunawan [6]. Ditunjukkan antara lain bahwa kekonvergenan dalam norm-n yang satu mengakibatkan kekonvergenan dalam norm-n lainnya. Kuncinya adalah bahwa kekonvergenan dalam masing-masing norm-n tersebut setara dengan kekonvergenan dalam norm biasa di ℓ^p.
Kata kunci: ruang norm-n, ruang barisan *summable-p*, kesetaraan norm-n

1. Introduction

In [6], Gunawan introduced an n-norm on ℓ^p $(1 \le p \le \infty)$, the space of p-summable sequences (of real numbers), given by the formula

$$\|x_1, \dots, x_n\|_p := \left[\frac{1}{n!} \sum_{j_1} \cdots \sum_{j_n} \operatorname{abs} \begin{vmatrix} x_{1j_1} & \cdots & x_{nj_1} \\ \vdots & \ddots & \vdots \\ x_{1j_n} & \cdots & x_{nj_n} \end{vmatrix}^p \right]^{1/p}$$

2000 Mathematics Subject Classification:
Received: dd-mm-yyyy, accepted: dd-mm-yyyy.

for $1 \le p < \infty$, and

$$\|x_1, \ldots, x_n\|_\infty = \sup_{j_1} \sup_{j_2} \cdots \sup_{j_n} \left\{ \text{abs} \begin{vmatrix} x_{1j_1} & \cdots & x_{1j_n} \\ \vdots & \ddots & \vdots \\ x_{nj_1} & \cdots & x_{nj_n} \end{vmatrix} \right\},$$

where $x_i = (x_{ij}),\ i = 1, \ldots, n$. For $p = 2$, the above formula may be rewritten as

$$\|x_1, \ldots, x_n\|_2 = \begin{vmatrix} \langle x_1, x_1 \rangle & \cdots & \langle x_1, x_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, x_1 \rangle & \cdots & \langle x_n, x_n \rangle \end{vmatrix}^{1/2},$$

where $\langle x_i, x_j \rangle$ denotes the usual inner product on ℓ^2. Here $\|x_1, \ldots, x_n\|_2$ represents the volume of the n-dimensional parallelepiped spanned by $x_1, \ldots, x_n$ in ℓ^2.

In general, an n-norm on a real vector space X is a mapping $\|\cdot, \ldots, \cdot\| : X^n \to \mathbb{R}$ which satisfies the following four conditions:

(N1) $\|x_1, \ldots, x_n\| = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent;

(N2) $\|x_1, \ldots, x_n\|$ is invariant under permutation;

(N3) $\|\alpha x_1, \ldots, x_n\| = |\alpha| \, \|x_1, \ldots, x_n\|$ for $\alpha \in \mathbb{R}$;

(N4) $\|x_1 + x_1', x_2, \ldots, x_n\| \le \|x_1, x_2, \ldots, x_n\| + \|x_1', x_2, \ldots, x_n\|$.

The theory of n-normed spaces was developed by Gähler in 1969 and 1970 [3, 4, 5]. The special case where $n = 2$ was studied earlier, also by Gähler, in 1964 [2]. Related work may be found in [1]. For more recent works, see [7, 8, 10].

If X is equipped with a norm $\|\cdot\|$, then according to Gähler, one may define an n-norm on X (assuming that X is at least n-dimensional) by the formula

$$\|x_1, \ldots, x_n\|^* := \sup_{\substack{f_i \in X',\ \|f_i\| \le 1 \\ i = 1, \ldots, n}} \begin{vmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n) \end{vmatrix}.$$

Here X' denotes the dual of X, which consists of bounded linear functionals on X.

For $X = \ell^p$ $(1 \le p < \infty)$, we know that $X' = \ell^{p'}$ with $\frac{1}{p} + \frac{1}{p'} = 1$. In this case the above formula reduces to

$$\|x_1, \ldots, x_n\|_p^* := \sup_{\substack{z_i \in \ell^{p'},\ \|z_i\|_{p'} \le 1 \\ i = 1, \ldots, n}} \begin{vmatrix} \sum x_{1j} z_{1j} & \cdots & \sum x_{1j} z_{nj} \\ \vdots & \ddots & \vdots \\ \sum x_{nj} z_{1j} & \cdots & \sum x_{nj} z_{nj} \end{vmatrix},$$

where $\|\cdot\|_{p'}$ denotes the usual norm on $\ell^{p'}$ and each of the sums is taken over $j \in \mathbb{N}$. Thus, on ℓ^p, we have two definitions of n-norms, one is due to Gunawan and the other is derived from Gähler's formula. For $p = 2$, one may verify that the two n-norms are identical.

The purpose of this paper is to study the relation between the two n-norms on ℓ^p for $1 \le p < \infty$. In particular, we shall show that the two n-norms are weakly equivalent, that is, the convergence in one n-norm implies that in the other. Here a sequence $(x(m))$ in an n-normed space $(X, \|\cdot, \dots, \cdot\|)$ is said to *converge* to $x \in X$ if $\|x(m) - x, x_2, \dots, x_n\| \to 0$ as $m \to \infty$, for every $x_2, \dots, x_n \in X$.

For convenience, we prove the result for $n = 2$ first, and then extend it to any $n \geq 2$.

2. Main Results

Recall that Gunawan's definition of 2-norm on ℓ^p $(1 \leq p \leq \infty)$ is given by

$$\|x, y\|_p = \left[\frac{1}{2} \sum_j \sum_k \operatorname{abs} \begin{vmatrix} x_j & x_k \\ y_j & y_k \end{vmatrix}^p \right]^{1/p}$$

if $1 \leq p < \infty$, and

$$\|x, y\|_\infty = \sup_j \sup_k \left\{ \operatorname{abs} \begin{vmatrix} x_j & x_k \\ y_j & y_k \end{vmatrix} \right\}.$$

Meanwhile, Gähler's definition is given by

$$\|x, y\|_p^* = \sup_{z, w \in \ell^{p'},\ \|z\|_{p'}, \|w\|_{p'} \leq 1} \begin{vmatrix} \sum x_j z_j & \sum x_j w_j \\ \sum y_j z_j & \sum y_j w_j \end{vmatrix}.$$

By the same trick as in [6], one may obtain

$$\|x, y\|_p^* = \sup_{z, w \in \ell^{p'},\ \|z\|_{p'}, \|w\|_{p'} \leq 1} \frac{1}{2} \sum_j \sum_k \begin{vmatrix} x_j & x_k \\ y_j & y_k \end{vmatrix} \begin{vmatrix} z_j & z_k \\ w_j & w_k \end{vmatrix}.$$

From the last expression, we have the following fact.

Fact 2.1. The inequality $\|x, y\|_p^* \leq 2^{1/p} \|x, y\|_p$ holds for every $x, y \in \ell^p$.

Proof. By Hölder's inequality for $\frac{1}{p} + \frac{1}{p'} = 1$, we have

$$\frac{1}{2} \sum_j \sum_k \begin{vmatrix} x_j & x_k \\ y_j & y_k \end{vmatrix} \begin{vmatrix} z_j & z_k \\ w_j & w_k \end{vmatrix} \leq \left[\frac{1}{2} \sum_j \sum_k \operatorname{abs} \begin{vmatrix} x_j & x_k \\ y_j & y_k \end{vmatrix}^p \right]^{1/p} \times \left[\frac{1}{2} \sum_j \sum_k \operatorname{abs} \begin{vmatrix} z_j & z_k \\ w_j & w_k \end{vmatrix}^{p'} \right]^{1/p'}$$

Now, observe that

$$\left[\sum_j\sum_k \text{abs}\begin{vmatrix} z_j & z_k \\ w_j & w_k \end{vmatrix}^{p'}\right]^{1/p'} \le \left[\sum_j\sum_k \big[|z_jw_k| + |z_kw_j|\big]^{p'}\right]^{1/p'}$$

$$\le \left[\sum_j\sum_k |z_jw_k|^{p'}\right]^{1/p'} + \left[\sum_j\sum_k |z_kw_j|^{p'}\right]^{1/p'}$$

$$= 2\,\|z\|_{p'}\|w\|_{p'}.$$

But for $\|z\|_{p'},\ \|w\|_{p'} \le 1$ we have

$$\left[\frac{1}{2}\sum_j\sum_k \text{abs}\begin{vmatrix} z_j & z_k \\ w_j & w_k \end{vmatrix}^{p'}\right]^{1/p'} \le 2^{1-(1/p')} = 2^{1/p}.$$

This proves the inequality.

Note that for $p = 1$, Hölder's inequality gives

$$\frac{1}{2}\sum_j\sum_k \left|\begin{vmatrix} x_j & x_k \\ y_j & y_k \end{vmatrix}\right| \left|\begin{vmatrix} z_j & z_k \\ w_j & w_k \end{vmatrix}\right| \le \|x,y\|_1 \cdot \|z,w\|_\infty .$$

But $\|z,w\|_\infty \le 2\,\|z\|_\infty\,\|w\|_\infty$ (see [6]), and so taking the supremum over $\|z\|_\infty$ and $\|w\|_\infty \le 1$, we get $\|x,y\|_1^* \le 2\|x,y\|_1$. □

Corollary 2.2 *If $(x(m))$ converges in $\|\cdot,\cdot\|_p$, then it also converges (to the same limit) in $\|\cdot,\cdot\|_p^*$.*

We shall show next that the convergence in $\|\cdot,\cdot\|_p^*$ also implies the convergence in $\|\cdot,\cdot\|_p$. We do so by showing that: (1) the convergence in $\|\cdot,\cdot\|_p^*$ implies that in $\|\cdot\|_p$, and (2) the convergence in $\|\cdot\|_p$ implies that in $\|\cdot,\cdot\|_p$.

The second implication is already proved in [6] (using the inequality $\|x,y\|_p \le 2^{1-(1/p)}\|x\|_p\|y\|_p$). Hence it remains only to show the first implication.

Theorem 2.3 *If $(x(m))$ converges in $\|\cdot,\cdot\|_p^*$, then it also converges (to the same limit) in $\|\cdot\|_p$.*

Proof. Let $(x(m))$ be a sequence in ℓ^p which converges to $x \in \ell^p$ in $\|\cdot,\cdot\|_p^*$. Then, for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for $m \ge N$ we have

$$\frac{1}{2}\sum_j\sum_k \left|\begin{vmatrix} x_j(m)-x_j & x_k(m)-x_k \\ y_j & y_k \end{vmatrix}\right| \left|\begin{vmatrix} z_j & z_k \\ w_j & w_k \end{vmatrix}\right| < \epsilon$$

for every $y \in \ell^p$ and $z, w \in \ell^{p'}$ with $\|z\|_{p'},\ \|w\|_{p'} \le 1$. [Notice here that, for each m, we have $x(m) = (x_j(m)) \in \ell^p$.] In particular, if we take $y := (1,0,0,\dots),\ z = (z_j)$

with $z_j := \frac{\mathrm{sgn}(x_j(m)-x_j)|x_j(m)-x_j|^{p-1}}{\|x(m)-x\|_p^{p-1}}$ and $w := (1,0,0,\dots)$, then we have

$$\sum_{j=2}^{\infty} \frac{|x_j(m)-x_j|^p}{\|x(m)-x\|_p^{p-1}} < \epsilon.$$

[Here we are handling only the case where $\|x(m)-x\|_p \neq 0$.] Next, if we take $y := (0,1,0,\dots)$, $z = (z_1,0,0,\dots)$ with $z_1 := \frac{\mathrm{sgn}(x_1(m)-x_1)|x_1(m)-x_1|^{p-1}}{\|x(m)-x\|_p^{p-1}}$ and $w := (0,1,0,\dots)$, then we have

$$\frac{|x_1(m)-x_1|^p}{\|x(m)-x\|_p^{p-1}} < \epsilon.$$

Adding up, we get

$$\|x(m)-x\|_p = \sum_{j=1}^{\infty} \frac{|x_j(m)-x_j|^p}{\|x(m)-x\|_p^{p-1}} < 2\epsilon.$$

This shows that $(x(m))$ converges to x in $\|\cdot\|_p$. □

Corollary 2.4 *A sequence is convergent in* $\|\cdot,\cdot\|_p^*$ *if and only if it is convergent (to the same limit) in* $\|\cdot,\cdot\|_p$.

All these results can be extended to n-normed spaces for any $n \geq 2$. As an extension of Fact 2.1, we have:

Fact 2.5 The inequality $\|x_1,\dots,x_n\|_p^* \leq (n!)^{1/p}\|x_1,\dots,x_n\|_p$ holds for every $x_1,\dots,x_n \in \ell^p$.

Corollary 2.6 *If* $(x(m))$ *converges in* $\|\cdot,\dots,\cdot\|_p$, *then it converges (to the same limit) in* $\|\cdot,\dots,\cdot\|_p^*$.

Analogous to Theorem 2.3, we have:

Theorem 2.7 *If* $(x(m))$ *converges in* $\|\cdot,\dots,\cdot\|_p^*$, *then it also converges (to the same limit) in* $\|\cdot\|_p$.

Proof. Let $(x_1(m))$ be a sequence in ℓ^p which converges to $x_1 = (x_{11}, x_{12}, \dots) \in \ell^p$ in $\|\cdot,\dots,\cdot\|_p^*$. Then, for any $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that for $m \geq N$ we have

$$\frac{1}{n!}\sum_{j_1}\cdots\sum_{j_n}\left|\begin{vmatrix} x_{1j_1}(m)-x_{1j_1} & \cdots & x_{1j_n}(m)-x_{1j_n} \\ \vdots & \ddots & \vdots \\ x_{nj_1} & \cdots & x_{nj_n}\end{vmatrix}\begin{vmatrix} z_{1j_1} & \cdots & z_{1j_n} \\ \vdots & \ddots & \vdots \\ z_{nj_1} & \cdots & z_{nj_n}\end{vmatrix}\right| < \epsilon$$

for every $x_2,\dots,x_n \in \ell^p$ and $z_1,\dots,z_n \in \ell^p$ with $\|z_1\|,\dots,\|z_n\| \leq 1$. Now, take $x_k = z_k := (0,\dots,0,1,0,\dots)$ for every $k = 2,\dots,n$, where 1 is $(n+1-k)$-th term and $z_1 = (z_{11}, z_{12}, \dots) \in \ell^{p'}$ with $z_{1j} := \frac{\mathrm{sgn}(x_{1j}(m)-x_{1j})|x_{1j}(m)-x_{1j}|^{p-1}}{\|x_1(m)-x_1\|_p^{p-1}}$, then we have

$$\sum_{j_1=n}^{\infty} \frac{|x_{1j_1}(m) - x_{1j_1}|^p}{\|x_1(m) - x_1\|_p^{p-1}} < \epsilon.$$

Next, if we take $x_k = z_k := (0, \dots, 0, 1, 0, \dots)$ for every $k = 2, \dots, n$, where 1 is k-th term, and $z_1 := (z_{11}, 0, 0, \dots)$ with $z_{11} := \frac{\mathrm{sgn}(x_{11}(m)-x_{11})|x_{11}(m)-x_{11}|^{p-1}}{\|x_1(m)-x_1\|_p^{p-1}}$, then we have

$$\frac{|x_{11}(m) - x_{11}|^p}{\|x_1(m) - x_1\|_p^{p-1}} < \epsilon.$$

Similarly, if we alter the position of the entry 1 in x_k and z_k for $k = 2, \dots, n$, and change the nonzero entry of z_1 accordingly, then we can get

$$\frac{|x_{12}(m) - x_{12}|^p}{\|x_1(m) - x_1\|_p^{p-1}} < \epsilon$$

and so on until

$$\frac{\left|x_{1(n-1)}(m) - x_{1(n-1)}\right|^p}{\|x_1(m) - x_1\|_p^{p-1}} < \epsilon.$$

Adding up, we get

$$\|x_1(m) - x_1\|_p = \sum_{j_1=1}^{\infty} \frac{|x_{1j_1}(m) - x_{1j_1}|^p}{\|x_1(m) - x_1\|_p^{p-1}} < n\epsilon.$$

This shows that $(x(m))$ converges to x in $\|\cdot\|_p$. □

Corollary 2.8 *A sequence is convergent in* $\|\cdot, \dots, \cdot\|_p^*$ *if and only if it is convergent (to the same limit) in* $\|\cdot, \dots, \cdot\|_p$.

Related to the above results, one may also prove that a sequence is Cauchy in $\|\cdot, \dots, \cdot\|_p^*$ if and only if it is Cauchy in $\|\cdot, \dots, \cdot\|_p$. [A sequence $(x(m))$ in an n-normed space $(X, \|\cdot, \dots, \cdot\|)$ is Cauchy if given $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $\|x(l) - x(m), x_2, \dots, x_n\| < \epsilon$ whenever $l, m \geq N$, for every $x_2, \dots, x_n \in X$.] Since $(\ell^p, \|\cdot, \dots, \cdot\|_p)$ is a Banach space [6], we conclude, by Theorem 2.7, that $(\ell^p, \|\cdot, \dots, \cdot\|_p^*)$ also forms an n-Banach space.

3. Concluding Remarks

As we have mentioned earlier, the case where $p = 2$ is of course special. Here, the two n-norms $\|\cdot, \dots, \cdot\|_2$ and $\|\cdot, \dots, \cdot\|_2^*$ are identical. Indeed, by using Cauchy-Schwarz inequality (see [9]), one may obtain

$$\|x_1, \dots, x_n\|_2^* = \sup_{\substack{z_i \in \ell^2,\ \|z_i\|_2 \leq 1 \\ i=1,\dots,n}} \begin{vmatrix} \langle x_1, z_1 \rangle & \cdots & \langle x_1, z_n \rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, z_1 \rangle & \cdots & \langle x_n, z_n \rangle \end{vmatrix} \leq \|x_1, \dots, x_n\|_2.$$

By taking $z_1, \dots, z_n$ to be the orthonormalized vectors obtained from $x_1, \dots, x_n$ through Gram-Schmidt process, one can show that the above upper bound is actually attained. Hence we have

$$\|x_1, \dots, x_n\|_2^* = \|x_1, \dots, x_n\|_2.$$

For $p \neq 2$, things are not so simple and we have difficulties in proving the strong equivalence between the two n-norms $\|\cdot, \dots, \cdot\|_p^*$ and $\|\cdot, \dots, \cdot\|_p$. The research on this problem, however, is still ongoing.

Acknowledgement. The research was carried out while the first author did his master thesis at Faculty of Mathematics and Natural Sciences, Institut Teknologi Bandung.

References

[1] C.R. Diminnie, S. Gähler, and A. White, "2-inner product spaces", *Demonstratio Math.* **6** (1973), 525–536.
[2] S. Gähler, "Lineare 2-normierte räume", *Math. Nachr.* **28** (1964), 1–43.
[3] S. Gähler, "Untersuchungen über verallgemeinerte m-metrische Räume. I", *Math. Nachr.* **40** (1969), 165 - 189.
[4] S. Gähler, "Untersuchungen über verallgemeinerte m-metrische Räume. II", *Math. Nachr.* **40** (1969), 229–264.
[5] S. Gähler, "Untersuchungen über verallgemeinerte m-metrische Räume. III", *Math. Nachr.* **41** (1970), 23–26.
[6] H. Gunawan, "The space of p-summable sequences and its natural n-norms", *Bull. Austral. Math. Soc.* **64** (2001), 137–147.
[7] H. Gunawan, "On n-inner products, n-norms, and the Cauchy-Schwarz inequality", *Sci. Math. Jpn.* **55** (2002), 53–60.
[8] H. Gunawan and Mashadi, "On n-normed spaces", *Int. J. Math. Math. Sci.* **27** (2001), 631–639.
[9] H. Gunawan, O. Neswan and W. Setya-Budhi, "A formula for angles between two subspaces of inner product spaces", *Beiträge Algebra Geom.* **46** (2005), 311–320.
[10] A. Misiak, "n-inner product spaces", *Math. Nachr.* **140** (1989), 299–319.