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ON VARIOUS CONCEPTS OF ORTHOGONALITY IN 2-NORMED SPACES

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ABSTRACT

In this paper we review various concepts of orthogonality in 2-normed spaces. Their definitions are presented and some of its consequences are discussed. In particular, we test if each definition gives the same orthogonality as the usual one when the space is equipped with an inner product. Finally, we focus on the so-called b-orthogonality and present some results on it.

Keywords: 2-normed space; orthogonality

1. Introduction

One important concept in normed spaces theory, especially in inner product spaces, is orthogonality between two vectors. In some cases, observation on its main properties gives us some valuable information regarding these spaces (Alonso & Benitez 1988,1989). This leads many researchers to devote their works on it.

In a (real) normed space, there are at least three well-known formulations for orthogonality, namely

i. Phytagorean orthogonality:

$$x \perp^P y \;\Leftrightarrow\; \|x+y\|^2 = \|x\|^2 + \|y\|^2$$

ii. Isosceled orthogonality:

$$x \perp^I y \;\Leftrightarrow\; \|x+y\| = \|x-y\|$$

iii. Birkhoff-James orthogonality:

$$x \perp^{BJ} y \;\Leftrightarrow\; \|x+\alpha y\| \geq \|x\| \text{ for all } \alpha \in \mathbb{R}$$

It can be verified that in an inner product space, the three formulations are equivalent to the usual orthogonality, that is $\langle x, y\rangle = 0$. Meanwhile, in general normed spaces one may not imply another.

Some researchers have introduced and investigated the concept of orthogonality in 2-normed spaces. A (real) 2-normed space, initially introduced in (Gähler, 1964), is a (real) vector space X equipped by *2-norm* $\|\cdot,\cdot\|$ which satisfies the following conditions

(1) $\|x, y\| = 0$ if only if x, y are linearly dependent.

(2) $\|x, y\| = \|y, x\|$ for all $x, y \in X$.

(3) $\|\alpha x, y\| = |\alpha|\|x, y\|$ for all $\alpha \in \mathbb{R},\ x, y \in X$.

(4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for all $x, y, z \in X$.

Note that we have $\|x + \alpha y, y\| = \|x, y\|$ for any $\alpha \in \mathbb{R}$.

Let $(X, \langle\cdot,\cdot\rangle)$ be an inner product space with $\dim(X) \geq 2$. The function $\|\cdot,\cdot\|_S$ on $X \times X$ defined by

$$\|x, y\|_S := \begin{vmatrix} \langle x, x\rangle & \langle x, y\rangle \\ \langle y, x\rangle & \langle y, y\rangle \end{vmatrix}^{\frac{1}{2}} \tag{1}$$

satisfies all conditions of 2-norm above. We call this the *standard 2-norm* on X. Observe that the value $\|x, y\|_S$ is nothing but the area of parallelogram spanned by x and y.

In general, any 2-norm $\|\cdot,\cdot\|$ can be interpreted as the area of the parallelogram spanned by the associated vectors.

The concept of 2-normed spaces is related to that of 2-inner product spaces. Let X be a vector space, and $\langle\cdot,\cdot|\cdot\rangle$ be a function on $X \times X \times X$ satisfying

(1) $\langle x, x|z\rangle \geq 0$ for every $x, z \in X$; $\langle x, x|z\rangle = 0$ if only if x, z are linearly dependent.

(2) $\langle x, y|z\rangle = \langle y, x|z\rangle$ for all $x, y, z \in X$.

(3) $\langle \alpha x, y|z\rangle = \alpha\langle x, y|z\rangle$ for all $\alpha \in \mathbb{R},\ x, y, z \in X$.

(4) $\langle x_1 + x_2, y|z\rangle = \langle x_1, y|z\rangle + \langle x_2, y|z\rangle$ for all $x_1, x_2, y, z \in X$.

Then, $\langle\cdot,\cdot|\cdot\rangle$ is called *2-inner product* on X and $(X, \langle\cdot,\cdot|\cdot\rangle)$ forms a *2-inner product space*.

If $(X, \langle\cdot,\cdot\rangle)$ is an inner product space we may define

$$\langle x, y|z\rangle = \begin{vmatrix} \langle x, y\rangle & \langle x, z\rangle \\ \langle z, y\rangle & \langle z, z\rangle \end{vmatrix}. \tag{2}$$

This is the so-called the *standard 2-inner product* on X. One can see that $\langle x, x|z\rangle^{\frac{1}{2}}$ is the standard 2-norm mentioned before.

In general, any 2-inner product space is a 2-normed space.

Inspired by phytagorean, isosceled, and Birkhoff-James orthogonality, Khan and Siddiqui defined orthogonality in 2-normed spaces which was later refined in (Godini, 1985) as follows

(A1) $x \perp^P y \ \Leftrightarrow\ \|x, z\|^2 + \|y, z\|^2 = \|x + y, z\|^2$ for all $z \notin \mathrm{span}\{x, y\}$.

(A2) $x \perp^I y \ \Leftrightarrow\ \|x - y, z\| = \|x + y, z\|$ for all $z \notin \mathrm{span}\{x, y\}$.

(A3) $x \perp^{BJ} y \ \Leftrightarrow\ \|x, z\| \leq \|x + \alpha y, z\|$ for all $\alpha \in \mathbb{R},\ z \notin \mathrm{span}\{x, y\}$.

Now, let $(X, \langle \cdot, \cdot | \rangle)$ be any 2-inner product space, and $x, y \in X$ be fixed. Assume that $x \perp^{BJ} y$. Using $\|x, z\| = \langle x, x|z\rangle^{\frac{1}{2}}$, one may observe that

$$\|x, z\| \leq \|x + \alpha y, z\| forall \alpha \in \mathbb{R},\ z \notin \text{span}\{x, y\} \tag{3}$$

is equivalent to

$$0 \leq 2\alpha\langle x, y|z\rangle + \alpha^2\langle y, y|z\rangle, \tag{4}$$

for which we obtain $\langle x, y|z\rangle = 0$ for all $z \notin \text{span}\{x, y\}$. So, we conclude that A3 is equivalent to the condition

$$\langle x, y|z\rangle = 0 \quad \text{for every } z \notin \ \text{span } \{x, y\}. \tag{5}$$

One may check that A1, A2 are also equivalent to this new condition.

We see that Khan and Siddiqui's definition is analogous to that in normed spaces, with an addition of the universal quantifiers for z. In this regard, Gunawan *et al.* studied the definition and showed that one cannot find two nonzero vectors that are orthogonal in the standard case (Gunawan, 2006). Moreover, they revised the notion of orthogonality as follows, by restricting the underlying set for z.

(B1) $x \perp^P y \ \Leftrightarrow$ there exists a subspace $V \subseteq X$ with $codim(V) = 1$ such that

$$\|x, z\|^2 + \|y, z\|^2 = \|x + y, z\|^2 \text{ for all } z \in V.$$

(B2) $x \perp^I y \ \Leftrightarrow$ there exists a subspace $V \subseteq X$ with $codim(V) = 1$ such that

$$\|x - y, z\| = \|x + y, z\| \text{ for all } z \in V.$$

(B3) $x \perp^{BJ} y \ \Leftrightarrow$ there exists a subspace $V \subseteq X$ with $codim(V) = 1$ such that

$$\|x, z\| \leq \|x + \alpha y, z\| \text{ for all } \alpha \in \ \mathbb{R},\ z \in V.$$

For further results on this revisited version, the reader can refer to (Gunawan, 2006).

In the next section, we shall discuss the concept of b-orthogonality and present some results on it.

2. Results

In 2007, Mazaheri introduced the concept of b-orthogonality in 2-normed spaces as follows (Mazaheri & Nezhad, 2007).

(C) $x \perp^b y \ \Leftrightarrow$ there exists $b \in X$ such that $0 < \|x, b\| \leq \|x + \alpha y, b\|$ for all $\alpha \in \ \mathbb{R}$.

Here, an existential quantifier is used instead of a universal one. Obviously, the required condition here is weaker than that of Gunawan's version. We just need one vector b satisfying the required inequality.

Our first observation on b-orthogonality is that it is not equivalent to usual orthogonality in standard case. Let $X = \mathbb{R}^3$ be equipped with standard 2-norm. For $b = (1, 1, 0)$, by simple evaluation, we find $x = (2, 0, 0), y = (1, 1, 1)$ are b-orthogonal. Meanwhile, x and y are not orthogonal in $\mathbb{R}^3$ in the usual sense. This fact tells us that b-orthogonality is not equivalent to the usual orthogonality when X is a Euclidean space.

The following result suggests us that the criteria of b-orthogonality is nothing but the criteria of linear independence of vectors (Gozali & Gunawan, 2009).

Theorem 2.1. *Let X be any 2-normed space and $x, y \in X \setminus \{0\}$. Then, $x \perp^b y$ if and only if $\|x, y\| \neq 0$.*

Proof. Let $x, y \in X$ such that $\|x, y\| \neq 0$. It is clear that by choosing $b = y$ we get $\|x + \alpha y, b\| = \|x, b\|$ for all $\alpha \in \mathbb{R}$. Thus, we have $x \perp^b y$.

Conversely, assume that $x \in X$ and $y = kx$ for some $k \in \mathbb{R} \setminus \{0\}$. Then, for any $b \in X$ with $\|x, b\| \neq 0$ we have $x + \alpha y, b\| = |1 + k\alpha| \, \|x, b\| < \|x, b\|$ for some α. Therefore, $x \not\perp^b y$. □

Now, let us give additional requirement that b, y must linearly independent. It is clear that with this new condition, the necessary part of Theorem 1 still holds. It is interesting to check its sufficient part.

First, let us check for two-dimensional case. Suppose that $b = cx + dy$ for some nonzero scalars c, d. Using the properties of 2-norm, we have

$$\|x + \alpha y, b\| = \|x, (d - \alpha c)y\|, \quad \text{whenever} \quad \alpha \neq d/c. \tag{6}$$

So, we can find $\alpha \neq d/c$ such that

$$\|x, (d - \alpha c)y\| < \|x, b\| = \|x, dy\|. \tag{7}$$

We conclude that in a two-dimensional space, any two vectors are not b-orthogonal.

For three-dimensional (or higher) case, we shall try to obtain $b \notin \mathrm{span}\{x, y\}$ satisfying the condition above. We observe that even with this requirement, b-orthogonality is still loose in the standard 2-normed space.

Let $(X, \langle \cdot, \cdot | \cdot \rangle)$ be a standard 2-inner product with $\dim(X) \geq 3$. As discussed before, we have standard 2-norm $\|x, y\|_S = \langle x, x | y \rangle^{\frac{1}{2}}$. Here, the following fact holds.

Proposition 2.2. *Let $x, y \in X$ be fixed. For any $b \in X$ we have*

$$\|x, b\|_S \leq \|x + \alpha y, b\|_S \text{ forall } \alpha \in \mathbb{R} \;\Leftrightarrow\; \langle x, y | b \rangle = 0. \tag{8}$$

This fact leads us to the following result.

Theorem 2.3. *Let X be equipped with standard 2-norm and $0 \neq x, y \in X$. If $\|x, y\| \neq 0$ then there exists $b \notin \mathrm{span}\{x, y\}$ such that $x \perp^b y$.*

The main idea of the proof is choosing normal vector z which is orthogonal to $\mathrm{span}\{x, y\}$, and then we set $b = x \pm y + \beta z$. Assuming $\|x\| = \|y\| = 1$, we get $\|b\|^2 = 2 \pm 2\langle x, y\rangle + \beta^2$. One may check that $\langle x, y|b\rangle = 0$ is equivalent to $\beta^2 = \pm\frac{1-\langle x,y\rangle^2}{\langle x,y\rangle}$. Therefore, we can choose an appropriate β to obtain b satisfying $\langle x, y|b\rangle = 0$.

3. Concluding Remark

We have seen three concepts of orthogonality in 2-normed spaces. The early one was that of Khan and Siddiqui which use a universal quantifier for which the definition is too tight. Later, Gunawan *et al.* revised it by weakening the associated quantifier. One important fact of Gunawan's version is its equality to the usual one in the standard case. Recently, Mazaheri introduced the so-called b-orthogonality which use an existential quantifier.

As we have discussed in previous section, the concept of b-orthogonality is to loose. The criteria of b-orthogonal is nothing but linear independence in general 2-normed spaces. Meanwhile, if we improve it by setting the additional requirement, we find a fact that any two vectors are not b-orthogonal in two-dimensional space. For higher dimensional space we have showed that b-orthogonality still loose in standard 2-normed spaces.

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