On Orthogonality in n-Normed Spaces

Hendra Gunawan
ITB Bandung
http://personal.fmipa.itb.ac.id/hgunawan/
Analysis and Geometry Group
Bandung Institute of Technology
Bandung, Indonesia

SEACMA 2010
Surabaya, 6 November 2010

Outline

Abstract

We discuss several notions of orthogonality in n-normed spaces. For each definition of orthogonality, we test whether it is equivalent to the usual one, especially when the space is also equipped with an inner product.

INTRODUCTION

Let $n \geq 2$ be a nonnegative integer and X be a vector space of dimension $d \geq n$ (including $d = \infty$). A real-valued function $\|\cdot, \ldots, \cdot\|$ on $X^n = X \times \cdots X$ (n times) satisfying the following four properties:

(1.1) $\|x_1, \ldots, x_n\| = 0$ if and only if $x_1, \ldots, x_n$ are linearly dependent;
(1.2) $\|x_1, \ldots, x_n\|$ is invariant under permutation;
(1.3) $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \, \|x_1, x_2, \ldots, x_n\|$ for any $\alpha \in \mathbb{R}$;
(1.4) $\|x + x', x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|x', x_2, \ldots, x_n\|$,

is called an n-*norm* on X, and the pair $(X, \|\cdot, \ldots, \cdot\|)$ is called an n-normed space.

Next, a real-valued function $\langle \cdot, \cdot | \cdot, \dots, \cdot \rangle$ on X^{n+1} satisfying the following five properties:

(1.5) $\langle z_1, z_1 | z_2, \dots, z_n \rangle \geq 0$; $\langle z_1, z_1 | z_2, \dots, z_n \rangle = 0$ if and only if $z_1, z_2, \dots, z_n$ are linearly dependent;
(1.6) $\langle z_1, z_1 | z_2, \dots, z_n \rangle = \langle z_{i_1}, z_{i_1} | z_{i_2}, \dots, z_{i_n} \rangle$ for every permutation $(i_1, \dots, i_n)$ of $(1, \dots, n)$;
(1.7) $\langle x, y | z_2, \dots, z_n \rangle = \langle y, x | z_2, \dots, z_n \rangle$;
(1.8) $\langle \alpha x, y | z_2, \dots, z_n \rangle = \alpha \langle x, y | z_2, \dots, z_n \rangle$, $\alpha \in \mathbb{R}$;
(1.9) $\langle x + x', y | z_2, \dots, z_n \rangle = \langle x, y | z_2, \dots, z_n \rangle + \langle x', y | z_2, \dots, z_n \rangle$,

is called an n-*inner product* on X, and the pair $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$ is called an n-inner product space.

One may observe that any inner product space $(X, \langle\cdot,\cdot\rangle)$ can be equipped with *the standard n-norm*

$$\|x_1, \ldots, x_n\| := \sqrt{\det\big(\langle x_i, x_j\rangle\big)}.$$

Note that this n-norm can be derived from *the standard n-inner product*

$$\langle x, y|z_2, \ldots, z_n\rangle := \begin{vmatrix} \langle x, y\rangle & \langle x, z_2\rangle & \ldots & \langle x, z_n\rangle \\ \langle z_2, y\rangle & \langle z_2, z_2\rangle & \ldots & \langle z_2, z_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle z_n, y\rangle & \langle z_n, z_2\rangle & \ldots & \langle z_n, z_n\rangle \end{vmatrix}.$$

A *standard n-normed space* (or a *standard n-inner product space*) is an inner product space equipped with the standard n-norm (or the standard n-inner product).

Remark. The determinan $\det\big(\langle x_i, x_j\rangle\big)$ is known as *the Gramian* of $x_1, \ldots, x_n$. Geometrically, $\|x_1, \ldots, x_n\|$ represents the volume of the n-dimensional parallelepiped spanned by $x_1, \ldots, x_n$ in X.

For further discussion about n-normed spaces and n-inner product spaces, we refer the reader to [8, 10, 15, 16].) For earlier works on 2-normed spaces and 2-inner product spaces, see [6, 4, 5].

In this paper, we shall discuss some notions of orthogonality in n-normed spaces. We recall that several notions of orthogonality in a normed space have been developed. For example, the following definitions of Pythagorean, isosceles, and the Birkhoff-James orthogonality in a (real) normed space $(X, \|\cdot\|)$ are known:

P-orthogonality: x is P-orthogonal to y (denoted by $x \perp_{\rm P} y$) if only if

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

I-orthogonality: x is I-orthogonal to y (denoted by $x \perp_{\rm I} y$) if only if

$$\|x+y\| = \|x-y\|.$$

BJ-orthogonality: x is BJ-orthogonal to y (denoted by $x \perp_{\rm BJ} y$) if only if

$$\|x+\alpha y\| \geq \|x\| \text{ for every } \alpha \in \mathbb{R}.$$

If X is actually equipped with an inner product $\langle \cdot, \cdot \rangle$, then one may observe that $x \perp_{\rm P} y$, $x \perp_{\rm I} y$, and $x \perp_{\rm BJ} y$ are all equivalent to the condition that $\langle x, y \rangle = 0$, for which we have the usual orthogonality $x \perp y$. However, in a normed space which is not an inner product space, the three types of orthogonality do not imply one another. See, for example, [14] and the references therein, for basic properties of these concepts of orthogonality.

The above notions of orthogonality have been extended to 2-normed spaces by several researchers (see, for example, [2, 13, 3]), and can be extended further to n-normed spaces. The purpose of this paper is to study each notion and decide which one is most plausible.

Throughout this paper, X will always denote a real vector space, unless otherwise stated.

Let us begin with the case $n = 2$ for convenience. As the notions of orthogonality in normed spaces are inspired by that in inner product spaces, the notions of orthogonality in 2-normed spaces are also connected to that in 2-inner product spaces. In [11], it is shown that the 'standard' definition of orthogonality in a 2-inner product space $(X, \langle \cdot, \cdot | \cdot \rangle)$, where $\dim(X) \geq 3$, is the following:

Definition 2.1 (G-orthogonality in 2-inner product spaces)
x is *G-orthogonal to* y, denoted by $x \perp_G y$, if and only if there exists a subspace V of X with $\text{codim}(V) = 1$ such that $\langle x, y | z \rangle = 0$ for all $z \in V$.

Notice that if X is a standard 2-inner product space, that is, when X is actually equipped with an inner product $\langle\cdot,\cdot\rangle$ and the 2-inner product

$$\langle x, y|z\rangle := \begin{vmatrix} \langle x, y\rangle & \langle x, z\rangle \\ \langle z, y\rangle & \langle z, z\rangle \end{vmatrix},$$

then we have $x \perp_G y$ if and only if $x \perp y$ (see [11]).
Accordingly, we may define P-, I-, and BJ-orthogonality in a 2-normed space $(X, \|\cdot,\cdot\|)$ of dimension 3 or higher as follows:

Definition 2.2 (P-, I-, and BJ-orthogonality in 2-normed spaces)
(a) $x \perp_{\rm P} y$ if only if there exists a subspace V of X with codim$(V) = 1$ such that

$$\|x+y,z\|^2 = \|x,z\|^2 + \|y,z\|^2 \quad \text{for every } z \in V;$$

(b) $x \perp_{\rm I} y$ if only if there exists a subspace V of X with codim$(V) = 1$ such that

$$\|x+y,z\| = \|x-y,z\| \quad \text{for every } z \in V;$$

(c) $x \perp_{\rm BJ} y$ if only if there exists a subspace V of X with codim$(V) = 1$ such that

$$\|x+\alpha y,z\| \geq \|x,z\| \quad \text{for every } z \in V \text{and } \alpha \in \mathbb{R}.$$

Let us now extend these notions of orthogonality to n-normed spaces and show that in the standard case they are equivalent to the usual one.

Definition 2.3 (G-orthogonality) Let $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$ be an n-inner product space of dimension $n+1$ or higher. For $x, y \in X$, we say that x is *G-orthogonal to* y and write $x \perp^G y$ if and only if there exists a subspace V of X with $\operatorname{codim}(V) = 1$ such that $\langle x, y | z_2, \dots, z_n \rangle = 0$ for every $z_2, \dots, z_n \in V$.

Definition 2.4 (P-, I-, and BJ-orthogonality) Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space of dimension $n+1$ or higher. For $x, y \in X$, we define

(a) $x \perp_{\mathrm{P}} y$ if only if there exists a subspace V of X with codim$(V) = 1$ such that

$$\|x+y, z_2, \ldots, z_n\|^2 = \|x, z_2, \ldots, z_n\|^2 + \|y, z_2, \ldots, z_n\|^2, \quad z_2, \ldots, z_n \in V$$

(b) $x \perp_{\mathrm{I}} y$ if only if there exists a subspace V of X with codim$(V) = 1$ such that

$$\|x + y, z_2, \ldots, z_n\| = \|x - y, z_2, \ldots, z_n\|, \quad z_2, \ldots, z_n \in V;$$

(c) $x \perp_{\mathrm{BJ}} y$ if only if there exists a subspace V of X with codim$(V) = 1$ such that

$$\|x + \alpha y, z_2, \ldots, z_n\| \geq \|x, z_2, \ldots, z_n\|, \quad z_2, \ldots, z_n \in V \text{ and } \alpha \in \mathbb{R}.$$

Note that in an n-inner product space $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$, P-, I-, and BJ-orthogonality are equivalent to G-orthogonality. The following theorem states that in a standard n-inner product space, G-orthogonality is equivalent to the usual orthogonality (with respect to the inner product).

Theorem 2.5 (Gunawan *et al.*) *Let $(X, \langle \cdot, \cdot | \cdot, \ldots, \cdot \rangle)$ be a standard n-inner product space of dimension $n+1$ or higher. Then, $x \perp^G y$ if and only if $x \perp y$.*

Proof I

Suppose that $x \perp y$, that is, $\langle x, y\rangle = 0$. Then we can choose $V \subseteq X$ such that $V^{\perp} = \mathrm{span}\{x\}$. Clearly V is a subspace of X with $\mathrm{codim}(V) = 1$. Now, for every $z_2, z_3, \ldots, z_n \in V$, we have

$$\langle x, y|z_2, \ldots, z_n\rangle = \begin{vmatrix} 0 & 0 & \ldots & 0 \\ \langle z_2, y\rangle & \langle z_2, z_2\rangle & \ldots & \langle z_2, z_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle z_n, y\rangle & \langle z_n, z_2\rangle & \ldots & \langle z_n, z_n\rangle \end{vmatrix} = 0.$$

(Alternatively, one may choose $V \subseteq X$ such that $V^{\perp} = \mathrm{span}\{y\}$ and get the same result.) This shows that $x \perp^G y$.
Conversely, suppose that $x \not\perp y$, that is, $\langle x, y\rangle \neq 0$. Here x and y cannot be zero. To show that $x \not\perp_G y$, let V be a subspace of X

Proof II

with $\text{codim}(V) = 1$. We claim that there must exist $z_2, \dots, z_n \in V$ such that $\langle x, y | z_2, \dots, z_n \rangle \neq 0$.
To prove the claim, we consider several cases. The first case below is clear.

Case 1. x and y are linearly dependent. Here $y = kx$, for some $k \neq 0$, so that

$$\langle x, y | z_2, \dots, z_n \rangle = k \langle x, x | z_2, \dots, z_n \rangle = k \|x, z_2, \dots, z_n\|^2.$$

Now choose $z_2, \dots, z_n \in V$ such that $x, z_2, \dots, z_n$ are linearly independent. This is possible since $\dim(X) \geq n + 1$ and $\text{codim}(V) = 1$. Hence, we have
$\langle x, y | z_2, \dots, z_n \rangle = k \|x, z_2, \dots, z_n\|^2 \neq 0$.

Proof III

Case 2. x and y are linearly independent. In this case, we consider the following subcases.

Case 2a. $x \in V$ and $y \in V$. Here we may choose $z_2 = x + y \in V$ and nonzero vectors $z_3, \ldots, z_n \in V$ such that $z_3 \perp \text{span}\{x, y\}$ and $z_i \perp \text{span}\{x, y, z_3, \ldots, z_{i-1}\}$ for $i = 4, \ldots, n$. (Again, this is possible since $\dim(X) \geq (n+1)$ and $\text{codim}(V) = 1$.) Hence, we have

$$\langle x, y | z_2, \ldots, z_n \rangle = \begin{vmatrix} \langle x, y \rangle & \langle x, x+y \rangle & 0 & \cdots & 0 \\ \langle x+y, y \rangle & \langle x+y, x+y \rangle & 0 & \cdots & 0 \\ 0 & 0 & \|z_3\|^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \|z_n\|^2 \end{vmatrix}$$

$$= - \begin{vmatrix} \langle x, x \rangle & \langle x, y \rangle \\ \langle y, x \rangle & \langle y, y \rangle \end{vmatrix} \|z_3\|^2 \cdots \|z_n\|^2 \neq 0.$$

Proof IV

(Note that the 2×2 determinant in the last expression is the Gramian of x and y, which is the square of the area of the parallelogram spanned by x and y. This determinant is positive definite because x and y are linearly independent.)

Case 2b. $x \in V$ and $y \notin V$. Here we may choose a nonzero vector $z_2 \in V$ that is orthogonal to x and nonzero vectors $z_3, \dots, z_n \in V$ such that $z_i \perp \mathrm{span}\{x, z_2, \dots, z_{i-1}\}$ for $i = 3, \dots, n$. Hence, we have

$$\langle x, y | z_2, \dots, z_n \rangle = \begin{vmatrix} \langle x, y \rangle & 0 & 0 & \cdots & 0 \\ \langle z_2, y \rangle & \|z_2\|^2 & 0 & \cdots & 0 \\ \langle z_3, y \rangle & 0 & \|z_3\|^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \langle z_n, y \rangle & 0 & 0 & \cdots & \|z_n\|^2 \end{vmatrix}$$

$$= \langle x, y \rangle \|z_2\|^2 \cdots \|z_n\|^2 \neq 0.$$

Proof V

Case 2c. $x \notin V$ and $y \in V$. Here we may choose a nonzero vector $z_2 \in V$ that is orthogonal to y and nonzero vectors $z_3, \ldots, z_n \in V$ such that $z_i \perp \mathrm{span}\{y, z_2, \ldots, z_{i-1}\}$ for $i = 3, \ldots, n$. Hence, we have

$$\langle x, y | z_2, \ldots, z_n \rangle = \begin{vmatrix} \langle x, y \rangle & \langle x, z_2 \rangle & \langle x, z_3 \rangle & \cdots & \langle x, z_n \rangle \\ 0 & \|z_2\|^2 & 0 & \cdots & 0 \\ 0 & 0 & \|z_3\|^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \|z_n\|^2 \end{vmatrix}$$
$$= \langle x, y \rangle \|z_2\|^2 \cdots \|z_n\|^2 \neq 0.$$

Case 2d. $x \notin V$ and $y \notin V$. Here we can write $x = \alpha u + x'$ and $y = \beta u + y'$ where u is a fixed nonzero vector perpendicular to V,

Proof VI

α and β are nonzero scalars, and $x', y' \in V$. Next choose $z_2 = \beta x - \alpha y = \beta x' - \alpha y' \in V$ and nonzero vectors $z_3, \dots, z_n \in V$ such that $z_3 \perp \mathrm{span}\{x', y'\}$, and $z_i \perp \mathrm{span}\{x', y', z_3, \dots, z_{i-1}\}$ for $i = 4, \dots, n$. Observe that for $i = 3, \cdots, n$, we have

$$\langle x, z_i \rangle = \langle \alpha u + x', z_i \rangle = \alpha \langle u, z_i \rangle + \langle x', z_i \rangle = 0,$$

$$\langle z_i, y \rangle = \langle z_i, \beta u + y' \rangle = \beta \langle z_i, u \rangle + \langle z_i, y' \rangle = 0,$$

and

$$\langle z_i, z_2 \rangle = \langle z_2, z_i \rangle = \langle \beta x' - \alpha y', z_i \rangle = 0.$$

Proof VII

Hence

$$\langle x, y | z_2, \ldots, z_n \rangle$$

$$= \begin{vmatrix} \langle x, y \rangle & \langle x, \beta x - \alpha y \rangle & 0 & \cdots & 0 \\ \langle \beta x - \alpha y, y \rangle & \langle \beta x - \alpha y, \beta x - \alpha y \rangle & 0 & \cdots & 0 \\ 0 & 0 & \|z_3\|^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \|z_n\|^2 \end{vmatrix}$$

$$= \alpha\beta \begin{vmatrix} \langle x, x \rangle & \langle x, y \rangle \\ \langle y, x \rangle & \langle y, y \rangle \end{vmatrix} \|z_2\|^2 \cdots \|z_n\|^2 \neq 0.$$

So, in any case, there must exist $z_2, \ldots, z_n \in V$ such that $\langle x, y | z_2, \ldots, z_n \rangle \neq 0$. This proves our claim, and hence proves the theorem.

Let $(X, \|\cdot, \dots, \cdot\|)$ be a standard n-normed space of dimension n. With respect to the G-orthogonality defined in the previous section, one may observe that two arbitrary vectors x and y are G-orthogonal to each other. Indeed, given x and y, just take $V = \mathrm{span}\{x, u_2, \dots, u_{n-1}\}$ where $x, u_2, \dots, u_{n-1}$ are linearly independent. Now, for every $z_2, \dots, z_n \in V$, the set $\{x, z_2, \dots, z_n\}$ is linearly dependent. Then, supposing that z_n is a linear combination of $x, z_2, \dots, z_{n-1}$, we get

$$\langle x, y|z_2, \dots, z_n\rangle = \begin{vmatrix} \langle x, y\rangle & \langle x, z_2\rangle & \dots & \langle x, z_n\rangle \\ \langle z_2, y\rangle & \langle z_2, z_2\rangle & \dots & \langle z_2, z_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle z_n, y\rangle & \langle z_n, z_2\rangle & \dots & \langle z_n, z_n\rangle \end{vmatrix}$$

$$= \begin{vmatrix} \langle x, y\rangle & \langle x, z_2\rangle & \dots & \langle x, z_n\rangle \\ \langle z_2, y\rangle & \langle z_2, z_2\rangle & \dots & \langle z_2, z_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 0 \end{vmatrix} = 0.$$

(The same also happens if we take $V = \mathrm{span}\{y, u_2, \dots, u_{n-1}\}$ where $y, u_2, \dots, u_{n-1}$ are linearly independent.)
As a matter of fact, the concept of orthogonality in n-normed spaces of dimension n needs to be formulated in a different way.

As suggested by [10], a norm can be derived from the n-norm and so we can define P-, I-, and BJ-orthogonality using this norm. In the standard case, the derived norm can be obtained from the n-norm in such a way that it coincides with the existing one (see [9]). Thus, the three types of orthogonality which are defined by using the derived norm will coincide with the usual orthogonality (with respect to the existing inner product).

Acknowledgement

This work is initiated a few years ago, in collaboration with E. Kikianty, Mashadi, S. Gemawati, and I. Sihwaningrum [12]. The current research is joint with S.M. Gozali and is funded by ITB Research Program 2010. The ultimate goal of the research is to show that the role of the space V of codimension 1 in our definition cannot be weakened. In particular, we are studying a different type of orthogonality which uses an existential quantor instead of a universal quantor for the vectors $z_2, \dots, z_n$. Of course this definition is weaker, but on what cost.

REFERENCES I

1. Y.J. Cho, C.R. Diminnie, S. Gähler, R.W. Freese, and E.Z. Andalafte, "Isosceles orthogonal triple in linear 2-normed spaces", *Math. Nachr.* **157** (1992), 225–234.
2. Y.J. Cho and S.S. Kim, "Gateaux derivatives and 2-inner product spaces", *Glas. Mat. Ser. III* **27(47)** (1983), 197–203.
3. C.R. Diminnie, "A new orthogonality relation for normed linear spaces", *Math. Nachr.* **114** (1983), 197–203.
4. C.R. Diminnie, S. Gähler, and A. White, "2-inner product spaces", *Demonstratio Math.* **6** (1973), 525–536.
5. C.R. Diminnie, S. Gähler, and A. White, "2-inner product spaces. II", *Demonstratio Math.* **10** (1977), 169–188.

REFERENCES II

6. S. Gähler, "Lineare 2-normierte räume", *Math. Nachr.* **28** (1964), 1–43.
7. G. Godini, MR0743643 (85g:46028), MathSciNet: *Mathematical Reviews* on the Web.
8. H. Gunawan, "On n-inner products, n-norms, and the Cauchy-Schwarz inequality", *Sci. Math. Jpn.* **55** (2002), 53–60.
9. H. Gunawan, "Inner products on n-inner product spaces", *Soochow J. Math.* **28** (2002), 289–298.
10. H. Gunawan and Mashadi, "On n-normed spaces", *Int. J. Math. Math. Sci.* **27** (2001), 631–639.

REFERENCES III

11. H. Gunawan, Mashadi, S. Gemawati, Nursupiamin and I. Sihwaningrum, "Orthogonality in 2-normed spaces revisited", submitted.
12. H. Gunawan, E. Kikianty, Mashadi, S. Gemawati, I. Sihwaningrum, "Orthogonality in n-normed spaces", unpublished.
13. A. Khan and A. Siddiqui, "B-orthogonality in 2-normed space", *Bull. Calcutta Math. Soc.* **74** (1982), 216–222.
14. J.R. Partington, "Orthogonality in normed spaces", *Bull. Austral. Math. Soc.* **33** (1986), 449–455.
15. A. Misiak, "n-inner product spaces", *Math. Nachr.* **140** (1989), 299–319.

REFERENCES IV

16. A. Misiak, "Orthogonality and orthonormality in n-inner product spaces", *Math. Nachr.* **143** (1989), 249-261.