The Product of Chebyshev Systems and Its Applications

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In this talk we study the product of N Chebyshev systems on (compact Hausdorff) topological spaces $X_1, \ldots, X_N$.

We offer a formula that gives an interesting relationship between the determinant of the product at a grid of points on $X_1 \times \cdots \times X_N$ and the determinants of each system at the associated points.

We then use the formula to develop a procedure to solve some interpolation problems on $X_1 \times \cdots \times X_N$ and present some concrete examples in the 2-D case.

Related works can be found in [1, 2, 6, 7, 8].

Let X be a compact Hausdorff topological space that contains at least m points [4]. A set of continuous, complex or real functions $\Phi := \{\phi_1, \dots, \phi_m\}$ on X is called a **Chebyshev system** on X if it satisfies the following condition: for any m distinct points $x_1, \dots, x_m$ on X, we have

$$D_\Phi(x_1, \dots, x_m) := \det[\phi_j(x_i)] \neq 0$$

where $[\phi_j(x_i)]$ is an $m \times m$ matrix whose element on i^{th}-row and j^{th}-column is $\phi_j(x_i)$.

For example, $\{\sin x, \dots, \sin mx\}$ is a Chebyshev system on $X := [0, \pi]$, which forms an m-dimensional space of (real-valued) functions of the form $\alpha_1 \sin x + \dots + \alpha_m \sin mx$, where $\alpha_1, \dots, \alpha_m \in \mathbb{R}$.

Chebyshev systems are used to solve interpolation problems [3].

Given m points $x_1, \ldots, x_m$ on X and m (real or complex) numbers $c_1, \ldots, c_m$, there exists a unique function ϕ in the linear span of Φ that interpolates $(x_1, c_1), \ldots, (x_m, c_m)$.

In this paper, we shall discuss about the product of N Chebyshev systems $\Phi_1, \ldots, \Phi_N$ on topological spaces $X_1, \ldots, X_N$.

To be precise, if the set $\Phi_n := \{\phi_{n1}, \ldots, \phi_{nm_n}\}$ is a Chebyshev system on X_n $(n = 1, \ldots, N)$, then we talk about the set of functions of N variables which has the form

$$\phi_{j_1,\ldots,j_N}(x_1, \ldots, x_N) := \phi_{1j_1}(x_1) \cdots \phi_{Nj_N}(x_N) = \prod_{n=1}^{N} \phi_{nj_n}(x_n).$$

Our entry point is that the tensor product Φ of two Chebyshev Systems $\Phi_1 := \{\phi_{11}, \phi_{12}, \cdots, \phi_{1m_1}\}$ on X_1 and $\Phi_2 := \{\phi_{21}, \phi_{22}, \cdots, \phi_{2m_2}\}$ on X_2, that is, the set of functions

$$\Phi_{jl}(x_1, x_2) := \phi_{1j}(x_1)\phi_{2l}(x_2),\ i = j, \dots, m_1;\ l = 1, \dots, m_2,$$

can interpolate data $\{(x_{1i}, x_{2k}, c_{ik}) : i = 1, \dots, m_1; k = 1, \dots, m_2\}$ on $X_1 \times X_2 \times \mathbb{F}$, where $\mathbb{F} = \mathbb{C}$ or $\mathbb{R}$ [6].

Example

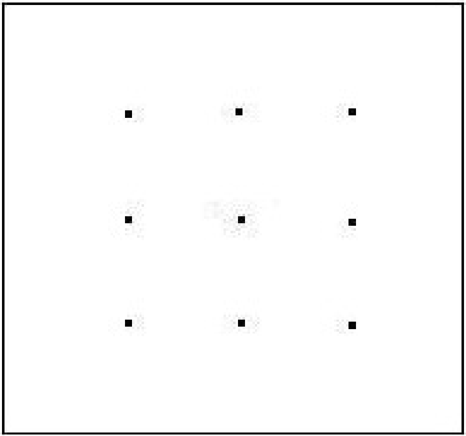
For example, on the unit square $[0,1]^2$, the set of functions

$$1, x, x^2, y, xy, x^2y, y^2, xy^2, x^2y^2,$$

may be viewed as the tensor product of the Chebyshev System $\{1, x, x^2\}$ on $[0,1]$ with itself.

This set can interpolate data $\{(x_i, y_k, c_{ik}) : i, k = 1, 2, 3\}$.

The set $\{(x_i, y_k) : i, k = 1, 2, 3\}$ forms a 3×3 'grid' on $[0, 1]^2$:



In general, suppose we are given an $m_1 \times m_2$ grid of points on $X_1 \times X_2$ which is the Cartesian product of $\{x_{11}, x_{12}, \ldots, x_{1m_1}\}$ and $\{x_{21}, x_{22}, \ldots, x_{2m_2}\}$, and arbitrary real numbers c_{ik}, $i = 1, \ldots, m_1$, $k = 1, \ldots, m_2$. Then, the interpolation problem

$$\sum_{j=1}^{m_1} \sum_{l=1}^{m_2} a_{jl} \Phi_{jl}(x_{1i}, x_{2k}) = c_{ik},\ i = 1, \ldots, m_1;\ k = 1, \ldots, m_2,$$

has a unique solution, which is generated by Φ.

Moreover, given some data on a subset of the grid, we can always find a function from Φ that interpolates the related nodes.

The previous result is due to the fact that the **Kronecker product** $M := M_1 \otimes M_2$, where $M_1 := [\phi_{1j}(x_{1i})]_{m_1 \times m_1}$ and $M_2 := [\phi_{2l}(x_{2k})]_{m_2 \times m_2}$, is non-singular — since we have [5]

$$\det M = (\det M_1)^{m_2} (\det M_2)^{m_1}. \quad (1)$$

Note. The Kronecker Product of M_1 and M_2 is given by the formula

$$M_1 \otimes M_2 = \begin{bmatrix} \phi_{11}(x_{11}) M_2 & \phi_{12}(x_{11}) M_2 & \cdots & \phi_{1m_1}(x_{11}) M_2 \\ \phi_{11}(x_{12}) M_2 & \phi_{12}(x_{12}) M_2 & \cdots & \phi_{1m_1}(x_{12}) M_2 \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{11}(x_{1m_1}) M_2 & \phi_{12}(x_{1m_1}) M_2 & \cdots & \phi_{1m_1}(x_{1m_1}) M_2 \end{bmatrix}$$

The matrix is a block matrix of size $m_1 m_2 \times m_1 m_2$.

The Kronecker product of two matrices $M_1 := [a_{ij}]$ and $M_2 := [a_{kl}]$ is given by the formula

$$M_1 \otimes M_2 := \begin{bmatrix} a_{11}M_2 & a_{12}M_2 & \cdots & a_{1n_1}M_2 \\ a_{21}M_2 & a_{22}M_2 & \cdots & a_{2n_1}M_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m_11}M_2 & a_{m_12}M_2 & \cdots & a_{m_1n_1}M_2 \end{bmatrix}.$$

The Kronecker product is an associative operation on matrices, that is, if K, L and M are the three matrices, then

$$(K \otimes L) \otimes M = K \otimes (L \otimes M).$$

Hence we may write $K \otimes L \otimes M$ for $(K \otimes L) \otimes M$ or $K \otimes (L \otimes M)$.

The following theorem generalizes the formula (1):

Theorem

Let $N \geq 2$ be an integer. For $n = 1, 2, \ldots, N$, let M_n be non-singular matrices of size $m_n \times m_n$. Then we have

$$\det(M_1 \otimes M_2 \otimes \cdots \otimes M_N) = \prod_{n=1}^{N} (\det M_n)^{\frac{P}{m_n}} \tag{2}$$

where $P = \prod_{n=1}^{N} m_n$.

Remark. It is interesting to note that, when applied to Chebyshev systems, our result connects the three types of products: the Cartesian product (of the domains), the tensor product (of the functions), and the Kronecker product (of the matrices).

Proof.

The theorem is proved by mathematical induction. We know that the formula is true for $N = 2$. Suppose it is true for $N \geq 2$. Then, we have

$$\begin{aligned}
\det(M_1 \otimes \cdots \otimes M_N \otimes M_{N+1}) \\
&= \det\big((M_1 \otimes \cdots \otimes M_N) \otimes M_{N+1}\big) \\
&= \{\det(M_1 \otimes \cdots \otimes M_N)\}^{m_{N+1}} (\det M_{N+1})^{P_N} \\
&= \prod_{n=1}^{N} (\det M_n)^{\frac{P_N m_{N+1}}{m_n}} (\det M_{N+1})^{\frac{P_N m_{N+1}}{m_{N+1}}} \\
&= \prod_{n=1}^{N+1} (\det M_n)^{\frac{P_{N+1}}{m_n}} ,
\end{aligned}$$

where $P_N = \prod_{n=1}^{N} m_n$. □

Consequently, we have:

Corollary

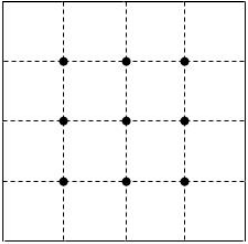
For $n = 1, 2, \ldots, N$, let $\Phi_n = \{\phi_{n1}, \phi_{n2}, \ldots, \phi_{nm_n}\}$ be a Chebyshev System on X_n. Then the tensor product Φ of the Φ_n's, namely the set of functions of the form

$$\left\{ \prod_{n=1}^{N} \phi_{nj_n}(x_n) \, : \, j_n = 1, \ldots, m_n; n = 1, \ldots, N \right\}$$

can interpolate data on (arbitrary subsets of) any $m_1 \times \cdots \times m_N$ grid in the Cartesian product $X_1 \times \cdots \times X_N$.

Example 1 I

Suppose we want to interpolate $(\frac{1}{4},\frac{1}{4},\frac{1}{2}),(\frac{1}{4},\frac{1}{2},1)$, $(\frac{1}{4},\frac{3}{4},2)$, $(\frac{1}{2},\frac{1}{4},1)$, $(\frac{1}{2},\frac{1}{2},2)$, $(\frac{1}{2},\frac{3}{4},1)$, $(\frac{3}{4},\frac{1}{4},\frac{1}{2})$, $(\frac{3}{4},\frac{1}{2},1),(\frac{3}{4},\frac{3}{4},1)$ using the product of two Chebyshev system on [0,1]. The points are given as a 3×3 grid:


.

Example 1 II

Since we want to interpolate points on 3×3 grid on $[0,1] \times [0,1]$, we use two Chebyshev systems that consist of 3 functions on $[0,1]$.

If we use $\{\sin \pi x, \sin 2\pi x, \sin 3\pi x\}$ and $\{1, \cos \pi y, \cos 2\pi y\}$ as Chebyshev systems, then the general interpolant has the form

$$\begin{aligned} U(x,y) = & a_{11} \sin \pi x + a_{12} \sin \pi x \cos \pi y + a_{13} \sin \pi x \cos 2\pi y \\ & + a_{21} \sin 2\pi x + a_{22} \sin 2\pi x \cos \pi y + a_{23} \sin 2\pi x \cos 2\pi y \\ & + a_{31} \sin 3\pi x + a_{32} \sin 3\pi x \cos \pi y + a_{33} \sin 3\pi x \cos 2\pi y. \end{aligned}$$

Example 1 III

Substituting the values from the given points and reducing the matrix into a row echelon form, we get

$$\left[\begin{array}{ccccccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2}+\frac{1}{2} \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2} \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2} \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{-\sqrt{2}}{4} \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{\sqrt{2}}{2}-\frac{1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{-1}{2} \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2}
\end{array}\right],$$

Example 1 IV

so

$$U(x,y) = \left(\frac{\sqrt{2}}{2} + \frac{1}{2}\right)\sin\pi x - \frac{1}{2}\sin\pi x\cos\pi y - \frac{1}{2}\sin\pi x\cos 2\pi y$$
$$+ \frac{1}{4}\sin 2\pi x - \frac{\sqrt{2}}{4}\sin 2\pi x\cos\pi y + \frac{1}{4}\sin 2\pi x\cos 2\pi y$$
$$+ \left(\frac{\sqrt{2}}{2} - \frac{1}{2}\right)\sin 3\pi x - \frac{1}{2}\sin 3\pi x\cos\pi y + \frac{1}{2}\sin 3\pi x\cos 2\pi y$$

interpolates the given points.

Example 1 V

If we use $\{1, \cos \pi x, \cos 2\pi x\}$ and $\{1, y, y^2\}$ as Chebyshev systems, then the general interpolant has the form

$$\begin{aligned} U(x,y) =& a_{11} + a_{12}y + a_{13}y^2 + a_{21}\cos\pi x + a_{22}y\cos\pi x + a_{23}y^2\cos\pi x \\ &+ a_{31}\cos 2\pi x + a_{32}y\cos 2\pi x + a_{33}y^2\cos 2\pi x. \end{aligned}$$

Substituting the values from the given points and solving the system of linear equations,

$$\begin{aligned} U(x,y) =& 2y + \frac{\sqrt{2}}{2}\cos\pi x - 3\sqrt{2}y\cos\pi x + 4\sqrt{2}y^2\cos\pi x \\ &+ 2\cos 2\pi x - 14y\cos 2\pi x + 16y^2\cos 2\pi x \end{aligned}$$

interpolates the given points.

Example 1 VI

Let us use $\{1, x, x^2\}$, and $\{\sin \pi y, \sin 2\pi y, \sin 2\pi y\}$ as Chebyshev systems. Similar to the above, we get

$$U(x,y) = \left(\frac{3\sqrt{2}}{4} - 1\right)\sin \pi y + \frac{-5}{2}\sin 2\pi y + \left(\frac{3\sqrt{2}}{4} + 1\right)\sin 3\pi y$$
$$+ \left(\frac{-\sqrt{2}}{2} + 8\right) x \sin \pi y + 9x \sin 2\pi y + \left(\frac{-\sqrt{2}}{2} - 8\right) x \sin 3\pi y$$
$$- 8x^2 \sin \pi y - 8x^2 \sin 2\pi y + 8x^2 \sin 3\pi y$$

as an interpolant.

The graph of the functions that interpolate the given points can be seen in Figure 1.

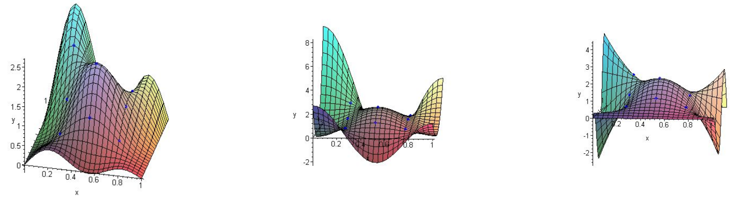
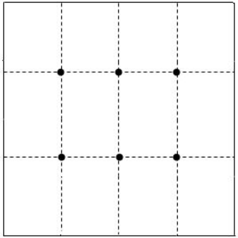


Figure: 1. The graph of the interpolants using the product of $\{\sin \pi x, \sin 2\pi x, \sin 3\pi x\}$ and $\{1, \cos \pi y, \cos 2\pi y\}$ (left); $\{1, \cos \pi x, \cos 2\pi x\}$ and $\{1, y, y^2\}$ (center); $\{1, x, x^2\}$ and $\{\sin \pi y, \sin 2\pi y, \sin 2\pi y\}$ (right).

Example 2 I

Suppose we want to interpolate $(\frac{1}{4}, \frac{1}{3}, 4), (\frac{1}{4}, \frac{2}{3}, 1), (\frac{1}{2}, \frac{1}{3}, 1), (\frac{1}{2}, \frac{2}{3}, 4), (\frac{3}{4}, \frac{1}{3}, 4), (\frac{3}{4}, \frac{2}{3}, 1)$ using the product of two Chebyshev system on [0,1]. The points are given as a 3×2 grid:



Example 2 II

Since we want to interpolate points on a 3×2 grid on $[0,1] \times [0,1]$, we use two Chebyshev systems that consist of 3 functions and 2 functions. Similar to the above example above, we have

$$U(x,y) = \left(\frac{5}{4} + \frac{5\sqrt{2}}{4}\right)\sin \pi x + \left(\frac{3\sqrt{2}}{2} - \frac{3}{2}\right)\sin \pi x \cos \pi y + \left(\frac{5\sqrt{2}}{4} - \frac{5}{4}\right)\sin 3\pi x + \left(\frac{3\sqrt{2}}{2} + \frac{3}{2}\right)\sin 3\pi x \cos \pi y$$

as an interpolant if we use $\{\sin \pi x, \sin 2\pi x, \sin 3\pi x\}$ and $\{1, \cos \pi y\}$ as Chebyshev systems.

Example 2 III

If we use $\{1, \cos \pi x, \cos 2\pi x\}$ and $\{1, y\}$ as Chebyshev systems, then we get

$$U(x,y) = 7 - 9y + 9\cos 2\pi x - 18y\cos 2\pi x$$

as an interpolant.

Meanwhile, if we use $\{1, x, x^2\}$ and $\{\sin \pi y, \sin 2\pi y\}$ as Chebyshev systems, then we get

$$U(x,y) = \frac{5\sqrt{3}}{3}\sin \pi y + 7\sqrt{3}\sin 2\pi y - 32\sqrt{3}x\sin 2\pi y + 32\sqrt{3}x^2\sin 2\pi y$$

as an interpolant.

Example 2 IV

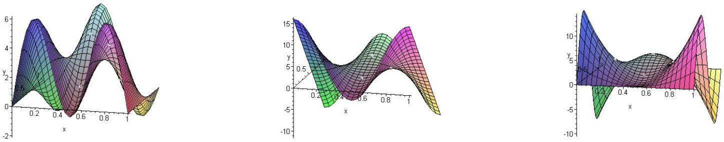


Figure: 2. The graph of the interpolants using the product of $\{\sin \pi x, \sin 2\pi x, \sin 3\pi x\}$ and $\{1, \cos \pi y\}$ (left), $\{1, \cos \pi x, \cos 2\pi x\}$ and $\{1, y\}$ (center); $\{1, x, x^2\}$ and $\{\sin \pi y, \sin 2\pi y\}$ (right).

Problem 2: Part of Grid

Let $G = \{(x_n, y_n, c_n) : n = 1, 2, \ldots, m\}$ be any set of points on $X_1 \times X_2 \times \mathbb{F}$. So there is a set of a grid form $H = \{(x_i, y_k, c_{ik}) : i = 1, 2, ..., m_1; k = 1, 2, ..., m_2\}$ such that H is a 'minimal' grid that contains G. This implies $m < m_1 \cdot m_2$. Let $\{\phi_1, \phi_2, ..., \phi_{m_1}\}$ and $\{\psi_1, \psi_2, ..., \psi_{m_2}\}$ be Chebyshev systems on X_1 and X_2 respectively. We can use

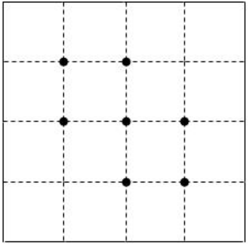
$$U(x,y) = \sum_{j=1}^{m_1} \sum_{l=1}^{m_2} a_{jl}\phi_j(x)\psi_l(y) \tag{3}$$

as an interpolant of G.

Substituting the points on G to (3), we obtain the system of linear equations with m equations and $m_1 \cdot m_2$ variables. Since $m < m_1 \cdot m_2$, the system has many solutions. This implies there are many possible values for a_{jl}'s such that (3) interpolates the given points.

Example 3 I

Suppose we want to interpolate $(\frac{1}{4}, \frac{1}{2}, 2), (\frac{1}{4}, \frac{3}{4}, 1), (\frac{1}{2}, \frac{1}{4}, 2),$ $(\frac{1}{2}, \frac{1}{2}, 3), (\frac{1}{2}, \frac{3}{4}, 2), (\frac{3}{4}, \frac{1}{4}, 1), (\frac{3}{4}, \frac{1}{2}, 2)$ using the product of two Chebyshev system on [0,1]. The points are given on a subset of a 3×3 grid:



Example 3 II

The minimal grid that contains the given points is a 3×3 grid on $[0,1] \times [0,1]$, so we use two Chebyshev systems that consist of 3 functions.

If we use $\{\sin \pi x, \sin 2\pi x, \sin 3\pi x\}$ and $\{1, \cos \pi y, \cos 2\pi y\}$ as the Chebyshev systems, then the general interpolant has the form

$$\begin{aligned} U(x,y) = & a_{11} \sin \pi x + a_{12} \sin \pi x \cos \pi y + a_{13} \sin \pi x \cos 2\pi y \\ & + a_{21} \sin 2\pi x + a_{22} \sin 2\pi x \cos \pi y + a_{23} \sin 2\pi x \cos 2\pi y \\ & + a_{31} \sin 3\pi x + a_{32} \sin 3\pi x \cos \pi y + a_{33} \sin 3\pi x \cos 2\pi y. \end{aligned}$$

Example 3 III

Substituting the values from the given points and reducing the matrix into a row echelon form, we get

$$\left[\begin{array}{ccccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2} & \sqrt{2}+\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & \sqrt{2}-1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & \frac{-\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{-1}{2} & \sqrt{2}-\frac{3}{2} \end{array}\right].$$

Example 3 IV

There are many functions that interpolate the given points, one of them is:

$$U(x,y) = \left(\frac{-1}{2}\sqrt{2} + \frac{1}{2}\right)\sin \pi x - \sin \pi x \cos 2\pi y + \left(\sqrt{2} - 1\right)\sin 2\pi x + \left(\sqrt{2} - \frac{3}{2}\right) 0.25 \sin 2\pi x \cos 2\pi y.$$

If we use $\{1, \cos \pi x, \cos 2\pi x\}$ and $\{1, y, y^2\}$ as the Chebyshev systems, then the general interpolant has the form

$$U(x,y) = a_{11} + a_{12}y + a_{13}y^2 + a_{21}\cos \pi x + a_{22}y\cos \pi x + a_{23}y^2\cos \pi x + a_{31}\cos 2\pi x + a_{32}y\cos 2\pi x + a_{33}y^2\cos 2\pi x.$$

Example 3 V

The same process applied to the above gives

$$U(x, y) = -2 + 16y - 16y^2 - \cos 2\pi x.$$

as one of the functions that interpolate the given points.

If we use $\{1, x, x^2\}$, and $\{\sin \pi y, \sin 2\pi y, \sin 2\pi y\}$ as the Chebyshev systems. Then, one of the functions that interpolate the given points is

$$U(x, y) = \left(\frac{-5}{2} + \sqrt{2}\right) \sin \pi y + \left(2 - \sqrt{2}\right) \sin 2\pi y + \left(\sqrt{2} - \frac{3}{2}\right) \sin 3\pi y$$
$$+ 16x \sin \pi y + \left(-4 + 2\sqrt{2}\right) x \sin 2\pi y - 16x^2 \sin \pi y.$$

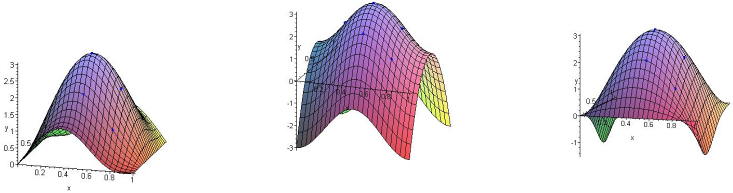


Figure: 3. The graph of the interpolants using the product of $\{\sin \pi x, \sin 2\pi x, \sin 3\pi x\}$ and $\{1, \cos \pi y, \cos 2\pi y\}$ (left); $\{1, \cos \pi x, \cos 2\pi x\}$ and $\{1, y, y^2\}$ (center); $\{1, x, x^2\}$ and $\{\sin \pi y, \sin 2\pi y, \sin 2\pi y\}$ (right).

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A full paper on this topic is being written and will be submitted to a suitable journal when it is ready.

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