Approximate eigensolution of locally modified regular structures using a substructuring technique

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ABSTRACT

A structure is called regular if its model can be considered as a product graph. A great computational advantage of regular structures, especially in the context of free vibration analysis, is their capacity for decomposition. In the present paper, the idea of decomposition is extended to the analysis of practical structural systems emerging as slight perturbations of the regular ones. A free interface dynamic substructuring method is employed to reformulate the eigenproblem as an assemblage of a base model and the modification part. The proposed method is very efficient in estimating a few lower eigenpairs of large-scale modified regular structures.

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1. Introduction

Symmetric and regular structures commonly occur in engineering design because of ease of construction, esthetic appeal and their optimal load-carrying capabilities. A structure is said to possess symmetry if through one or more symmetry operations its configuration becomes physically indistinguishable from the initial configuration. A structure is called regular if its model can be considered as a product graph (see [1] for definition of a product graph). Physically, a regular structure is assembled from identical components repeated in a special pattern, governed by its graph model.

Symmetry and regularity can be exploited in structural mechanics problems to simplify the computations through decomposition of the structural models [2,3]. In practice it is not uncommon to encounter structural models that despite the dominance of regularity in their forms, are locally altered through some local changes to the original regular model. These changes may involve the addition or elimination of some members, or modification of structural properties of some members, or the application of some irregular supports. Let us call the original unmodified regular model, the base model. Matrices associated with the base models, such as the stiffness and mass matrices, exhibit decomposable canonical forms [4]. The aforementioned modifications may be viewed as a small rank perturbation of stiffness and mass matrices associated with the base model. Decomposition methods have been proposed for special types of perturbations in [5,6]. However the perturbations in a general setting prohibit the direct application of decomposition methods to modified matrices.

In this paper we present a numerical method for the solution of the eigenproblem associated with a modified regular structure. The proposed method is based on a free interface sub-structuring technique such that it provides for the information obtained from the base model to be directly taken into account. The governing eigenproblem is reduced using modal truncation approximations, while preserving the sparsity of the matrices.

The generalized eigenvalue problem for free vibration of a modified regular structure may be expressed as follows:

\[(K + E \Delta k E^T)u = \lambda (M + E \Delta m E^T)u\] (1)

where \(K\) and \(M\) of order \(N\) are the stiffness and mass matrices of the base model, \(\Delta k\) and \(\Delta m\) of order \(m\) are the modified parts and \(\lambda\) is the square of the natural frequency. \(E\) is a Boolean matrix of association between the set of \(m\) modified DOFs and \(N\) base DOFs. Matrices associated with the base model can be transformed into block-diagonal forms using different techniques, such as group theoretic methods [7–10] or canonical forms and product graphs [11–13]. Regardless of the method being used, the procedure is the same in which an orthogonal matrix \(T\) is constructed such that:

\[
K^{(D)} = T^TH\quad \text{and} \quad M^{(D)} = T^MT
\] (2)

each have the same block-diagonal form. Using the transformation (2), the eigenvalue problem of the base model is reduced to several smaller decoupled eigenvalue problems, producing drastic simplification and saving in the eigen-computations.

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With a closer look at problem (1) its connection with two other extensively investigated eigenproblems in structural mechanics is realized. The first one is structural dynamic reanalysis problem, which deals with efficient use of natural frequencies and modes obtained from a previous analysis to derive the response of the new modified structure, without expensive additional computations. Approximate methods for eigenvalue reanalysis based on Taylor’s series expansion, have been proposed [14,15]. However these methods are not suitable for significant modifications. Rank one modification of the eigenproblem has been the subject of research for quite some time [16,17]. However these methods rely on the complete eigensolution of the original system, which is prohibitively expensive for large-scale systems. Carey et al. [18] proposed a block Lanczos method to calculate a few lowest eigenvalues based on the information obtained from the solution of a few eigenvalues for the original system. They employed a clever idea of using starting Lanczos vectors which span the column space of the matrix E in (1). The modification problem (1) that we are going to address in this paper however differs from the reanalysis problem due to the block diagonalizable structure of K and M, as mentioned before. This is an occasion that offers great opportunities in eigensolution and linear solution of the corresponding equations, and will be efficiently employed in the proposed method of this paper. Nevertheless, the method developed here is directly applicable to reanalysis problems as well.

Another problem of structural mechanics related to the subject matter is the dynamic substructuring [19]. There are basically two methods available in literature for the eigenvalue problem of substructuring. Kron’s method is one of them; where the problem reduces to solving a nonlinear eigenvalue problem involving the Kron matrix [20,21]. Sehmi [22] applied the Lanczos algorithm to Kron’s method and showed that operation count decreases dramatically compared to the classic solutions of Kron’s scalar equation. However a remarkable drawback of Kron substructuring is the need to calculate all eigenpairs of each substructure primarily. This is very time consuming, since only the first a few eigensolutions are generally of interest for most applications. Weng et al. [23] improved the Kron’s substructuring method using modal truncation approximation. They incorporated the first-order and the second-order residual flexibility approximations of the higher modes. The other well-known method of dynamic substructuring is component mode synthesis (CMS), where a nonlinear eigenvalue problem is avoided by restricting the solution to a certain subspace [24,25]. CMS methods are classified as free interface, fixed interface and hybrid methods. MacNeal introduced the residual flexibility to include the static effects of higher normal modes [26]. Rozenblum applied an associated isostatic subset method to calculate the residual modes for positive semidefinite stiffness matrices [27]. Rixen proposed a dual Craig–Bampton formulation of the CMS method [28].

Most of the structural applications require just a few lower eigen-frequencies and their corresponding modal vectors. This information can be obtained much more easily and quickly for the base model due to block-diagonalizable structure of its matrices. Hence it is natural to express the dynamic behavior of a modified model in terms of eigenmodes of its base model. Higher eigenmodes can be estimated by static modes deduced from residual flexibility of the base model. Again the decomposable structure of the base model is of central importance in constructing such approximations. In the next section we provide a sub-structure assembled formulation of problem (1), and describe how a truncated modal basis together with static modes deduced from residual flexibility of the base model can be used to reduce the governing eigenproblem into a much smaller problem while maintaining the sparsity.

2. Free interface substructure formulation for the modified regular structure

Let us assume that, with respect to eigenvalue problem (1) the perturbations ΔK and Δm correspond to a substructure interacting with the base model of K and M. This substructure is not required to be a meaningful substructure as illustrated in Example 2. Partitioning the model into these two components and using internal forces f on the interface DOFs between the two substructures, the linear dynamic behavior of each part is governed by the local eigen-equations:

\[ Ku - λMu + Ef = 0 \]  
(3)

and

\[ Δkv - λΔmv - f = 0 \]  
(4)

The compatibility between the two substructures is enforced by:

\[ E^T u - v = 0 \]  
(5)

Putting them together Eqs. (3)–(5) can be written in block form

\[
\begin{bmatrix}
K & 0 & E \\
0 & ΔK & -I \\
E^T & -I & 0
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
f
\end{bmatrix}
= \begin{bmatrix}
M & Δm \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
u \\
v
\end{bmatrix}
= 0
\]

Vector f is also known as Lagrange multipliers. The assembled system (6) is of order N + 2m, and has for general solution N eigenvalues λ.

3. Modal truncation

Let \( Φ \) be the matrix of M-orthonormal eigenvectors and \( Λ \) the diagonal matrix of eigenvalues associated with the eigenproblem of the base model. Hence the following relations hold:

\[ Φ^T K Φ = Λ \quad \text{and} \quad Φ^T M Φ = I \]  
(7)

Let us define a cutoff eigenvalue \( λ_c \) and suppose that we are interested in those eigenvalues \( λ \) of Eq. (6), that

\[ λ \ll λ_c \]  
(8)

Now, partition \( Λ \) into lower and higher eigenvalues based on cutoff value \( λ_c \), as

\[ Λ = \begin{bmatrix}
A_g \\
A_h
\end{bmatrix} \]  
(9)

And let the corresponding partitioning of \( Φ \) be

\[ Φ = \begin{bmatrix}
Φ_g \\
Φ_h
\end{bmatrix} \]  
(10)

Using modal coordinates \( q \) defined by

\[ u = Φ_g q_g + Φ_h q_h \]  
(11)

and pre-multiplying the problem (3) by \( Φ_g^T \), it can be deduced that

\[ Λ_g q_g - λ_c q_g + Φ_g^T E f = 0 \]  
(12)

Premultiplying Eq. (12) by \( A_g^{-1} \) and putting \( λ A_g^{-1} \approx 0 \) due to the assumption (8), we arrive at

\[ q_g = -A_g^{-1} Φ_g^T E f \]  
(13)

Substituting into (11), we obtain the following approximation for vector \( u \)

\[ u = Φ_q q_g - G_{res} E f \]  
(14)

where

\[ G_{res} = Φ_h A_g^{-1} Φ_h^T \]  
(15)

is the residual flexibility of the base model.
In summary we construct the following approximation of the coordinate vectors and Lagrange multipliers for the reduction of eigenproblem (6):

\[
\begin{bmatrix}
  u \\
  v \\
  f
\end{bmatrix} = T
\begin{bmatrix}
  q_g \\
  v \\
  f
\end{bmatrix}
= \begin{bmatrix}
  \Phi_g & 0 & -G_{res}E \\
  0 & 1 & 0 \\
  0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
  q_g \\
  v \\
  f
\end{bmatrix}
\]  (16)

4. Reduced eigenproblem

Using the approximation (16), we will reduce the assembled system (6). First, notice the following properties of the residual flexibility matrix

\[
G_{res} = G_{res} - G_{res}K_{Gres} = G_{res}
\]

\[
\Phi_g^T K_{Gres} = 0, \quad \Phi_g^T M_{Gres} = 0
\]  (17)

The reduced eigenproblem of the modified regular structure is then obtained by using transformation (16), as

\[
\begin{bmatrix}
  q_g \\
  v \\
  f
\end{bmatrix} = 0
\]

\[
\begin{bmatrix}
  \tilde{K} & -\lambda \tilde{M} \\
  \tilde{M} & 0
\end{bmatrix}
\begin{bmatrix}
  q_g \\
  v \\
  f
\end{bmatrix} = 0
\]  (18)

with the reduced matrices

\[
\tilde{K} = \begin{bmatrix}
  \Lambda_g & 0 & 0 \\
  0 & \Delta k & -1 \\
  0 & 0 & \Phi_g^T F_{res}
\end{bmatrix}
\]  (19)

where

\[
F_{res} = -E_{Gres}E, \quad M_{res} = E_{Gres}M_{Gres}E
\]  (20)

\[
\Lambda_g and \Phi_g correspond to the calculated eigenvalues and eigenmodes of the base structure also called the master modes. They include zero eigenvalues and the corresponding rigid body modes of the base model if available. However the calculations differ in constructing the residual flexibility matrix, with or without the rigid body modes.

5. Evaluation of the residual flexibility matrix

In general case, stiffness matrix \(K\) of the base model may be positive semi-definite. First, we discuss the evaluation of \(G_{res}\) when \(K\) is positive definite.

5.1. Positive definite stiffness matrix

In this case, \(K\) is nonsingular and using the following result

\[
K^{-1} = \Phi A^{-1} \Phi^T = \Phi S \Lambda_s^{-1} \Phi_s^T + \Phi A_e^{-1} \Phi_e^T
\]  (21)

We conclude that

\[
G_{res} = K^{-1} - \Phi S \Lambda_s^{-1} \Phi_s^T
\]  (22)

The inversion \(K^{-1}\) is constructed using the block-diagonal transform (2) as

\[
K^{-1} = T[K^{(b)}]^{-1} T^T
\]  (23)

5.2. Positive semi-definite stiffness matrix

In this case there are rigid-body modes present in the calculated modal matrix \(\Phi_s\), and they should be suppressed first to obtain the elastic part of the response, from which the residual flexibility is then calculated.

Let us partition the \(M\)-orthonormal modal matrix \(\Phi\) as follows:

\[
\Phi = [\Phi_e, \Phi_r] = [\Phi_e, \Phi_e, \Phi_s] = [\Phi_s, \Phi_e]
\]  (24)

where \(\Phi_s\) is the matrix of the rigid body modes, \(\Phi_e\), the matrix of calculated eigenvectors of nonzero eigenvalues, and \(\Phi_e = [\Phi_e, \Phi_e]\) the elastic eigenvectors.

The response of the following system

\[
Kx = F_0
\]  (25)

is composed of the a rigid body response \(x_r\) and an elastic response \(x_e\). The elastic response can be expressed in terms of eigenvalues and eigenmodes as

\[
x_e = [\Phi_e \Lambda_e^{-1} \Phi_e^T F_0 = [\Phi_e \Lambda_e^{-1} \Phi_e^T + \Phi_s \Lambda_s^{-1} \Phi_s^T] F_0 = G_{res} F_0
\]  (26)

Hence, the residual flexibility matrix \(G_{res}\) can be obtained, provided that we have already calculated the elastic flexibility matrix \(G_e\). The relation is

\[
G_{res} = G_e - \Phi_s \Lambda_s^{-1} \Phi_s^T
\]  (27)

The elastic flexibility matrix \(G_e\) is calculated using an inertia relief procedure to remove the rigid body modes. Let \(R\) be the orthogonal projector onto the complement space of \(\Phi_s\), defined by

\[
R = (I - \Phi_s \Phi_s^T M)
\]  (28)

It can be shown [26,27] that

\[
G_e = R G_e R^T
\]  (29)

where \(G_e\) is the elastic flexibility matrix relative to a set of imposed constraints. \(G_e\) may be obtained by taking the stiffness matrix \(K\) which is singular, deleting rows and columns corresponding to constrained DOFs, inverting the resulting matrix and expanding back to the original size by adding zeros. If we use the block-diagonal transform \(K^{(b)}\) of (2) instead of \(K\) for inversion, then we will have

\[
G_e = R G_e^{(b)} T^T R^T
\]  (30)

where \(G_e^{(b)}\) is obtained form \(K^{(b)}\) in an analogous way to that of \(C_e\). It should be noted that in the actual implementations, the inverse matrices are not computed explicitly, and the calculations are performed much more efficiently using LU decomposition with partial pivoting.

6. Numerical experiments

Example 1. Consider the mass–spring system depicted in Fig. 1(a). The system is composed of 30 point masses connected together with identical springs of stiffness \(k\). Two end points are connected to the ground with springs of stiffness \(3k\). Each point has a single degree of freedom in the \(x\)-direction with a concentrated mass \(m\), except for the two end points with mass \(5m\).

The system is decomposed into a regular model in Fig. 1(b) which serves as our base model, and a small modification part as shown in Fig. 1(c). The interface forces \(f\), acting between the two subsystems are also depicted in these figures.

The mass matrix of the base system is a diagonal matrix with entries equal to \(m\). The stiffness matrix has the form

\[
K = k
\]

\[
\begin{bmatrix}
  2 & -1 & 0 & \cdots & 0 \\
  -1 & 2 & -1 & \cdots & \vdots \\
  0 & \ddots & \ddots & \ddots & \ddots \\
  \vdots & \ddots & \ddots & \ddots & \ddots \\
  0 & \cdots & -1 & 2 & -1 \\
  0 & \cdots & 0 & -1 & 2
\end{bmatrix}
\]  (31)
This is a matrix of Toeplitz type and its eigenvalues and eigenvectors can be given by

\[
\lambda_j = k \left( 2 - 2 \cos \left( \frac{j \pi}{31} \right) \right) \quad \text{and} \quad \varphi_j = \begin{bmatrix} \sin(1j\pi/31) \\
\sin(2j\pi/31) \\
\vdots \\
\sin(3j\pi/31) \\
\sin(30j\pi/31) \end{bmatrix} \quad \text{(for } j = 1, 2, \ldots, 30) \tag{32}
\]

Stiffness, mass and association matrices corresponding to modification subsystem are as follows:

\[
\Delta K = k \begin{bmatrix} 2 & 0 \\
0 & 2 \end{bmatrix}, \quad \Delta m = m \begin{bmatrix} 4 & 0 \\
0 & 4 \end{bmatrix}, \quad F' = \begin{bmatrix} 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 1 \end{bmatrix}_{2 \times 30}
\]

Utilizing only eight eigenvalues and eigenmodes from the base system, i.e.

\[
\Lambda_k = \begin{bmatrix} \lambda_1 \\
\vdots \\
\lambda_8 \end{bmatrix} \quad \text{and} \quad \Phi_k = [\varphi_1, \ldots, \varphi_8]
\]

we calculate the residual modes, \( G_k = E_k \), using an LU decomposition of \( K \). Next, we use this information to construct the reduced system.

Comparison of the results for Example 1.

<table>
<thead>
<tr>
<th>Index</th>
<th>( \frac{\lambda_i}{\lambda_m} )</th>
<th>Relative error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.011196179375082</td>
<td>0.000242308</td>
</tr>
<tr>
<td>2</td>
<td>0.04456671315552</td>
<td>0.002520582</td>
</tr>
<tr>
<td>3</td>
<td>0.09943251737941</td>
<td>0.018074404</td>
</tr>
<tr>
<td>4</td>
<td>0.174489252521154</td>
<td>0.002520582</td>
</tr>
<tr>
<td>5</td>
<td>0.267571530235334</td>
<td>0.018074404</td>
</tr>
<tr>
<td>6</td>
<td>0.373992592569361</td>
<td>0.002520582</td>
</tr>
</tbody>
</table>

It can be seen from the Table 1 that lower eigenvalues are estimated more accurately, and the relative error increases with the index of eigenvalues. Note that the relative error of an estimated eigenvalue is proportional to the ratio of that eigenvalue to the cutoff value. The cutoff value is the lower eigenvalue in the set \( \Lambda_k \). In this example the cutoff value is the 9th eigenvalue of the base system (\( \lambda_9 = 0.775788034904674 \)), because we used only 8 of them as master eigenvalues. Hence the eigenvalues that are well below this value are approximated more accurately. With reference to Table 1 it is realized that the maximum relative error is less than 0.1% which is a sufficient accuracy for usual engineering applications.

Example 2. This example demonstrates that the modification part need not be a meaningful substructure. Consider a 120 story shear building depicted in Fig. 2(a). Each floor has a total mass \( m \) concentrated at the floor level. The second to the last stories are identical each of lateral stiffness \( k \). The first story has a lateral stiffness 0.42k.

In Fig. 2(b) necessary springs are attached to the building at the first and the last floor levels in order to create a base structure with a regular stiffness matrix similar in pattern to that of (31) but of order \( 120 \times 120 \). The mass matrix is a diagonal matrix with diagonal elements \( m \). Fig. 2(c) shows the modifications part. It is observed that springs with negative stiffness are introduced. Although a negative stiffness spring is not a meaningful substructure, but it does not affect the algorithm because the formulations are developed based on the algebraic equilibrium equations partitioned using the Lagrange multipliers and no further assumptions are made regarding the modification matrices.

Table 2 presents the first six natural periods of the structure obtained using the present method and a direct method, assuming \( m = 5 \) ton, and \( k = 200 \times 10^3 \) kN/m.

Only 16 modes of the base structure are employed in the calculations using the proposed method. The corresponding cutoff value is \( (T_1 = 0.071757536) \). The lower modes obtained here have periods well over the cutoff value and hence the relative errors in Table 2 are very small. Also presented in this table are the mode shape errors. The mode shape error is a measure of the angle between two vectors. It is calculated using the following relations:

\[
\cos^2 \theta = \frac{u^T u}{u^T u}
\]

(33)

\[
\text{Mode shape error} = 1 - \cos^2 \theta
\]

(34)

In which \( u \) and \( \bar{u} \) are the mode shapes calculated using the direct method and the proposed method, respectively, and \( \theta \) is the angle between the two modes.
With reference to Table 2, one can deduce that the obtained mode shapes are very satisfactory.

**Example 3.** Consider a double layer dome as shown in Fig. 3(a). The top layer is the Cartesian product $C_n \otimes P_m$ and the bottom layer is the Cartesian product $C_n \otimes P_{m-1}$. Each node of the bottom layer is connected to four adjacent nodes in the top layer. Here $n = 18$, and $m = 13$. Fig. 3(b) shows a cross section of the dome in elevation and some dimensions in metric units. The top and bottom arcs are divided into equal length segments by cross members. The structure is built of pin jointed bar elements having cross section area $a = 6.41 \text{ cm}^2$ and elastic modulus is $E = 2.1 \times 10^{11} \text{ N/m}^2$. For simplicity a concentrated mass of 80 kg in each node is assumed. Each node of the structures has three translational DOFs.

The eigenproblem associated with free vibration of the base model is transformed from global Cartesian coordinates into a local symmetry adapted coordinates system in which stiffness and mass matrices become block-diagonalizable and are easily solved to obtain the first 60 eigenfrequencies and eigenmodes constituting the master modes of the base structure. Table 3 presents the results for the first 16 nonzero natural frequencies (the 6 zero frequencies associated with rigid body modes are discarded), and the corresponding natural periods. Calculations are performed at double precision arithmetic and on a computer with Intel® Core™2 Duo CPU 2.33 GHz and 2 GB of RAM, which was running Microsoft Windows XP professional Service Pack 3.

Now consider the addition of some supports to the base structure as shown in Fig. 4. We use the penalty method to impose the constraints. In the penalty method, matrix $K$ is updated to $K + E \Delta k E^T$, with $\Delta k$ being a diagonal matrix of penalty numbers. The penalty numbers are taken as $10^6$ times the largest diagonal coefficient in the structure stiffness matrix. The results of the analysis using the proposed method with 60 master modes and also a direct sparse eigensolver of the MATLAB software are presented in Table 4 for the first 16 natural frequencies, and the corresponding natural periods. Note that the supported structure has no rigid body modes due to sufficient restraining of the structure.
Comparing the results in Tables 3 and 4, it can be verified that the fundamental period of the supported structure is decreased compared to that of the base structure, due to contribution of the supports to total structural stiffness. With reference to Table 4, one can observe that relative errors range from negligible to 0.1% for the last estimated natural period, indicating that the accuracy of the obtained results is satisfactory. The mode shapes are verified by the angles between them using Eq. (34). For simple eigenvalues, Eq. (33) is used to measure the angle between the corresponding mode shapes. For multiple eigenvalues, it is possible for the algorithm to calculate a different basis for the associated eigenspace. Hence, the subspaces spanned by the set of orthonormal eigenvectors associated with the multiple eigenvalue, should be tested against each other. For this purpose, the angle between two equi-dimensional subspaces with orthonormal bases $\mathbf{U}$ and $\hat{\mathbf{U}}$ is calculated from the following Eq. [29]:

$$\cos^2 \theta = \det^2 (\mathbf{M})$$

(35)

where

$$\mathbf{M} = \hat{\mathbf{U}}^T \mathbf{U}$$

(36)

In these relations, $\cos^2 \theta \approx 1$ is an indication of similarity between the two eigenspaces. Referring to Table 4 it can be deduced that the mode shapes are obtained with reasonable approximations.

Comparing the run times for the two methods, it is obvious that in the case of this example (1350 DOFs) the proposed method presents a 169-fold reduction in the work load.

In order to further investigate the efficiency of the proposed method in the analysis of large scale structural models, more test cases are considered here for comparison. The base models are similar to Fig. 3(a) in general configurations, but are different in the number of DOFs and the dimensions. The support conditions are similar to those configured in Fig. 4.

Table 5 summarizes all the cases considered here for calculation of the first 16 frequencies and mode-shapes. Also presented in this table are the results for the two extreme natural periods of the obtained sequence calculated using the proposed method and the direct method. The proposed method was implemented using 60 master modes from the base models. Again, relative errors indicate that the estimated natural periods are within the acceptable tolerance. Fig. 5 demonstrates CPU times required to accomplish the calculations, using the proposed method and the direct method. Referring to this figure it can be argued that it is in average 63 times faster to use the proposed method to estimate a few frequencies and mode shapes of locally modified regular structures, if the results are considered acceptable within a 0.1% tolerance.

### Table 3

<table>
<thead>
<tr>
<th>Index</th>
<th>Frequency (rad/s)</th>
<th>Period (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.43111403210</td>
<td>0.268155667013</td>
</tr>
<tr>
<td>2</td>
<td>5.43111403210</td>
<td>0.268155667013</td>
</tr>
<tr>
<td>3</td>
<td>5.43111403210</td>
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</tr>
<tr>
<td>4</td>
<td>5.43111403210</td>
<td>0.268155667013</td>
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<tr>
<td>5</td>
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<td>6</td>
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</tr>
<tr>
<td>16</td>
<td>5.43111403210</td>
<td>0.268155667013</td>
</tr>
</tbody>
</table>

Fig. 3. A double layer dome.

Fig. 4. The dome with supports.

Comparing the results in Tables 3 and 4, it can be verified that the fundamental period of the supported structure is decreased compared to that of the base structure, due to contribution of the supports to total structural stiffness. With reference to Table 4, one can observe that relative errors range form a negligible to 0.1% for the last estimated natural period, indicating that the accuracy of the obtained results is satisfactory.
A local symmetry adapted coordinate system as discussed in the structure can be considered as a special product of a path composed of pin-jointed bar elements. The graph model of the method. A single layer dome is considered in Fig. 6(a) which is degree of modification on the performance of the proposed Example 4.

<table>
<thead>
<tr>
<th>Test cases considered.</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N ) (number of DOFs)</td>
</tr>
<tr>
<td>------------------------</td>
</tr>
<tr>
<td>405</td>
</tr>
<tr>
<td>900</td>
</tr>
<tr>
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<td>1998</td>
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<td>2646</td>
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<td>3960</td>
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| \( \frac{T_{ke}}{s} \) Present method | Direct method | Relative error (%) |
|------------------------|
| 405 | 0.034740329 | 0.034746611 | 0.0018084 |
| 900 | 0.045631364 | 0.045714109 | 0.0181004 |
| 1350 | 0.049730545 | 0.0497822 | 0.013561 |
| 1998 | 0.06150333 | 0.061557176 | 0.012006 |
| 2646 | 0.07302754 | 0.073081879 | 0.012006 |
| 3960 | 0.094488851 | 0.094539638 | 0.012006 |
| 5112 | 0.113689448 | 0.113690448 | 0.012006 |

Table 5

Fig. 5. Run time comparison for calculation of first 16 eigenpairs.

Example 4. This example is intended to demonstrate the effects of degree of modification on the performance of the proposed method. A single layer dome is considered in Fig. 6(a) which is composed of pin-jointed bar elements. The graph model of the structure can be considered as a special product of a path \( P_2 \) by a cycle \( C_5 \), augmented by a node at the apex. Each node of the structure has three translational DOFs. The bottom nodes are all constrained in three directions, and hence they have been eliminated from the graph model. This leads to a total of 2973 DOFs for the entire structure. The diameter of the dome is 40 m. The bar elements have a uniform cross section area of \( A = 6.41 \text{ cm}^2 \) and the elastic modulus is considered as \( E = 2.1 \times 10^{11} \text{ N/m}^2 \). For simplicity a concentrated mass of 80 kg in each node is assumed.

The stiffness and mass matrices associated with the base model depicted in Fig. 6(a) are transformed from the global coordinates to a local symmetry adapted coordinate system as discussed in the previous example. With such a transformation, the stiffness and mass matrices will take the pattern of generalized form III symmetry as illustrated in [5]. Several methods are available for the decomposition of matrices having generalized form III symmetry pattern. Here, the method of [6] is employed for decomposition of the base system. Then, the required number of natural frequencies and mode shapes are obtained from the solution of the decomposed subsystems. For this example, 60 natural modes are obtained \((\omega_{adj} = 3460 \text{ rad/s})\), so that the frequencies of the modified structure well below the cutoff value \((\omega_c \approx \omega_{adj})\) could be approximated with an acceptable accuracy.

Now, let us modify the base model by changing the cross-sectional area of some of its diagonal members. One such modified structure is depicted in Fig. 6(b). The bold lines in this figure indicate those members which are doubled in cross-sectional area. Let us define the modification ratio as the ratio of the number of DOFs affected by such modification to the total DOFs. For example the modification ratio of the structure in Fig. 6(b) is 99/2973 x 100 = 3.33%. The first 12 eigenfrequencies and mode shapes of the modified structure are obtained using both the proposed method and the direct method, for different modification ratios. Maximum relative error of the natural frequencies obtained from the proposed method, are represented in Fig. 7 versus the modification ratio. It is observed that employing the same number of master modes from the base model, the maximum relative error tends to increase with modification ratio. This behavior is attributed to the introduction of more residual flexibility modes with the increase of modified DOFs. These residual modes are static approximations for the higher eigenmodes from the base model, and in many cases they come close to be linearly dependent, resulting in numerical error and bad approximation of the eigenmodes.
With reference to Fig. 7, maximum relative error of the calculated natural frequencies exceeds 0.1% threshold for modification ratios greater than 20%. To improve the accuracy of the results, the number of master modes from the base model may be increased. On the other hand, the problem size increases with modification ratio, and the incorporation of extra modes, further adds to the size. The overall performance of the algorithm in terms of the CPU time is adversely affected by this size increment. Fig. 8 represents the time ratio of the proposed method to direct method versus the modification ratio. It can be concluded from this figure that for modification ratios below 20%, the proposed method requires a fraction of the time taken by the direct method to calculate the requested natural frequencies and mode shapes with reasonable accuracy. Also, the efficiency of the present method reduces by increasing the modifications. Therefore the proposed method is most suitable for those local modifications which do not alter the regularity of the structural models considerably.

7. Concluding remarks

The aim of this paper is to extend the computational advantages inherent in regular structural models to those models that are considered small perturbations of regular structures. Dynamic substructuring and modal approximations were used to reduce the size of the governing eigenproblem. The proposed method relies on the information obtained from regular base models which are computationally much more efficient due to their decomposable structures.

The proposed method is very efficient for estimating a few natural periods and mode-shapes of large scale modified regular structures and the accuracy of the obtained results is satisfactory in usual engineering applications.

Acknowledgement

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References