# ON THE PRODUCT OF $N$ CHEBYSHEV SYSTEMS

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ABSTRACT. In this paper we study the product of $N$ Chebyshev systems on topological spaces $X_1, \ldots, X_N$. We offer a formula that gives an interesting relationship between the determinant of the product at a grid of points on $X_1 \times \cdots \times X_N$ and the determinants of each system at the associated points. We then use the formula to develop a procedure to solve some interpolation problems on $X_1 \times \cdots \times X_N$ and present some concrete examples.

## 1. Introduction

Let $X$ be a compact Hausdorff topological space that contains at least $m$ points [4]. A set of continuous, complex or real functions $\Phi := \{\phi_1, \ldots, \phi_m\}$ on $X$ is called a *Chebyshev system* on $X$ if it satisfies the following condition: for any $m$ distinct points $x_1, \ldots, x_m$ on $X$, we have

$$D_\Phi(x_1, \ldots, x_m) := \det[\phi_j(x_i)] \neq 0$$

where $[\phi_j(x_i)]$ is an $m \times m$ matrix whose element on $i^{th}$-row and $j^{th}$-column is $\phi_j(x_i)$. For example, $\{\sin x, \ldots, \sin mx\}$ is a Chebyshev system on $X := [0, \pi]$, which forms an $m$-dimensional space of (real-valued) functions of the form $\alpha_1 \sin x + \cdots + \alpha_m \sin mx$, where $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$.

Chebyshev systems are used to solve interpolation problems [3]. Given $m$ points $x_1, \ldots, x_m$ on $X$ and $m$ (real or complex) numbers $c_1, \ldots, c_m$, there exists a unique function $\phi$ in the linear span of $\Phi$ that interpolates $(x_1, c_1), \ldots, (x_m, c_m)$.

In this paper, we shall consider the product of $N$ Chebyshev systems $\Phi_1, \ldots, \Phi_N$ on (compact Hausdorff) topological spaces $X_1, \ldots, X_N$. To be precise, if the set $\Phi_n := \{\phi_{n1}, \ldots, \phi_{nm_n}\}$ is a Chebyshev system on $X_n$ ($n = 1, \ldots, N$), then we talk about the set of functions of $N$ variables which has the form

$$\phi_{j_1,\ldots,j_N}(x_1, \ldots, x_N) := \phi_{1j_1}(x_1) \cdots \phi_{Nj_N}(x_N) = \prod_{n=1}^{N} \phi_{nj_n}(x_n).$$

Then, given a grid of points on the Cartesian product $X_1 \times \cdots \times X_N$, we shall see that there is an interesting relationship between the determinant corresponding to the product of the Chebyshev systems computed at the grid and the determinants corresponding to each system computed at the associated points.

We shall use the formula which states the above relationship to construct a procedure of solving certain interpolation problems on $X_1 \times \cdots \times X_N$. In this regard, some

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concrete examples of 2-dimensional interpolation using the product of two Chebyshev systems will be provided. For related works, see [1, 2, 6, 7, 8].

## 2. Preliminary Results: The Kronecker Products

Suppose that we have two Chebyshev systems $\Phi_1$ and $\Phi_2$ on topological spaces $X_1$ and $X_2$, which form $m_1$- and $m_2$-dimensional spaces of functions, respectively. Given a grid of points $(x_{1i}, x_{2k})$ on $X_1 \times X_2$, we are interested in finding a function of the form $\phi(x_1, x_2) = \sum_{j=1}^{m_1} \sum_{l=1}^{m_2} \alpha_{jl}\phi_{1j}(x_1)\phi_{2l}(x_2)$ such that

$$\phi(x_{1i}, x_{2k}) = c_{ik}, \quad i = 1, \dots, m_1,\ k = 1, \dots, m_2,$$

where $c_{ik},\ i = 1, \dots, m_1,\ k = 1, \dots, m_2$, are (real or complex) numbers.

To make sure that such a function exists, we compute the determinant that corresponds to the product $\Phi$ (of $\Phi_1$ and $\Phi_2$) at the given grid — namely $D_\Phi([x_{1i}, x_{2k}]) =$

$$\begin{vmatrix} \phi_{11}(x_{11})\,[\phi_{2l}(x_{2k})] & \phi_{12}(x_{11})\,[\phi_{2l}(x_{2k})] & \cdots & \phi_{1m_1}(x_{11})\,[\phi_{2l}(x_{2k})] \\ \phi_{11}(x_{12})\,[\phi_{2l}(x_{2k})] & \phi_{12}(x_{12})\,[\phi_{2l}(x_{2k})] & \cdots & \phi_{1m_1}(x_{12})\,[\phi_{2l}(x_{2k})] \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{11}(x_{1m_1})\,[\phi_{2l}(x_{2k})] & \phi_{12}(x_{1m_1})\,[\phi_{2l}(x_{2k})] & \cdots & \phi_{1m_1}(x_{1m_1})\,[\phi_{2l}(x_{2k})] \end{vmatrix}.$$

The matrix involved here is nothing but the Kronecker product $[\phi_{1j}(x_{1i})] \otimes [\phi_{2l}(x_{2k})]$, whose size is $m_1m_2 \times m_1m_2$ (see [5]). As is known, we have

$$\begin{aligned} D_\Phi([x_{1i}, x_{2k}]) &= \det\left([\phi_{1j}(x_{1i})] \otimes [\phi_{2l}(x_{2k})]\right) \\ &= \{\det[\phi_{1j}(x_{1i})]\}^{m_2} \cdot \{\det[\phi_{2l}(x_{2k})]\}^{m_1} \neq 0 \end{aligned}$$

since both $\det[\phi_{1j}(x_{1i})]$ and $\det[\phi_{2l}(x_{2k})]$ are nonzero. This guarantees the existence of the function we are looking for.

This observation suggests that dealing with the product of two Chebyshev systems amounts to dealing with the Kronecker product of the two corresponding matrices. Thus, in order to study the product of $N$ Chebyshev systems, we shall now discuss the Kronecker product of $N$ matrices in general.

Let $A_1$ and $A_2$ be two matrices of size $m_1 \times n_1$ and $m_2 \times n_2$ respectively. The Kronecker product of $A_1$ and $A_2$, denoted by $A_1 \otimes A_2$, is given by the formula

$$A_1 \otimes A_2 := \begin{bmatrix} a_{11}A_2 & a_{12}A_2 & \cdots & a_{1n_1}A_2 \\ a_{21}A_2 & a_{22}A_2 & \cdots & a_{2n_1}A_2 \\ \vdots & \vdots & \ddots & \vdots \\ a_{m_11}A_2 & a_{m_12}A_2 & \cdots & a_{m_1n_1}A_2 \end{bmatrix}.$$

Note that the size of $A_1 \otimes A_2$ is $m_1m_2 \times n_1n_2$.

The Kronecker product is associative: if $A_1, A_2$ and $A_3$ are three arbitrary matrices, then

$$(A_1 \otimes A_2) \otimes A_3 = A_1 \otimes (A_2 \otimes A_3).$$

Hence we can write the Kronecker product of $A_1, A_2$ and $A_3$ as $A_1 \otimes A_2 \otimes A_3$ without brackets. We should note, however, that the Kronecker product is not commutative.

Nonetheless, if $A_1$ and $A_2$ are square matrices of size $m_1 \times m_1$ and $m_2 \times m_2$ respectively, then we have

$$\det(A_1 \otimes A_2) = \{\det A_1\}^{m_2} \cdot \{\det A_2\}^{m_1} = \det(A_2 \otimes A_1).$$

Furthermore, the following theorem holds.

**Theorem 1.** *For each $i = 1, \ldots, N$ ($N \geq 2$), let $A_i$ be a matrix of size $m_i \times m_i$. Then*

$$\det(A_1 \otimes A_2 \otimes \cdots \otimes A_N) = \prod_{i=1}^{N} \{\det A_i\}^{\frac{P}{m_i}}$$

*where $P = \prod_{i=1}^{N} m_i$.*

*Proof.* We will prove the formula by mathematical induction. We have stated that the formula holds for $N = 2$. Now suppose the formula holds for $N = k \geq 2$, that is,

$$\det(A_1 \otimes \cdots \otimes A_k) = \prod_{i=1}^{k} \{\det A_i\}^{\frac{P_k}{m_i}}$$

where $P_k = \prod_{i=1}^{k} m_i$. Then, we have

$$\begin{aligned}
\det(A_1 \otimes \cdots \otimes A_k \otimes A_{k+1}) &= \det((A_1 \otimes \cdots \otimes A_k) \otimes A_{k+1}) \\
&= \{\det(A_1 \otimes \cdots \otimes A_k)\}^{m_{k+1}} \cdot \{\det A_{k+1}\}^{P_k} \\
&= \prod_{i=1}^{k} \{\det A_i\}^{\frac{P_k m_{k+1}}{m_i}} \cdot \{\det A_{k+1}\}^{\frac{P_k m_{k+1}}{m_{k+1}}} \\
&= \prod_{i=1}^{k+1} \{\det A_i\}^{\frac{P}{m_i}},
\end{aligned}$$

where $P = \prod_{i=1}^{k+1} m_i$. This means that the formula holds for $N = k + 1$. We therefore conclude that the formula holds for every $N \geq 2$. □

## 3. The Product of $N$ Chebyshev Systems

For $n = 1, \ldots, N$, let $\Phi_n := \{\phi_{n1}, \ldots, \phi_{nm_n}\}$ be a Chebyshev system on a topological space $X_n$, which forms an $m_n$-dimensional space of functions on $X_n$. Given a grid of points $(x_{1i_1}, \ldots, x_{Ni_N})$ on $X_1 \times \cdots \times X_N$, where $i_n = 1, \ldots, m_n,\ n = 1, \ldots, N$, we are interested in finding a function of the form

$$\phi(x_1, \ldots, x_N) = \sum_{j_1}^{m_1} \cdots \sum_{j_N=1}^{m_N} \alpha_{j_1,\ldots,j_N} \phi_{1j_1}(x_1) \cdots \phi_{Nj_N}(x_N)$$

such that

$$\phi(x_{1i_1}, \ldots, x_{Ni_N}) = c_{i_1,\ldots,i_N}, \quad i_n = 1, \ldots, m_n,\ n = 1, \ldots, N, \tag{3.1}$$

where $c_{i_1,\ldots,i_N}$ are (real or complex) numbers. To know that such a function exists, we need to show that the determinant corresponding to the product $\Phi$ (of $\Phi_1, \ldots, \Phi_N$) computed at the given grid, namely $D_\Phi([(x_{1i_1}, \ldots, x_{Ni_N})])$, is nonzero.

In order to do so, we have the following fact.

**Fact 2.** $D_\Phi([(x_{1i_1}, \ldots, x_{Ni_N})]) = \det\left([\phi_{1j_1}(x_{1i_1})] \otimes [\phi_{2j_2}(x_{2i_2})] \otimes \ldots \otimes [\phi_{Nj_N}(x_{Ni_N})]\right)$.

*Proof.* We prove the formula by mathematical induction. Having known that the formula holds for $N = 2$, we assume that it holds for $N = k \geq 2$, that is,

$$D_\Phi([(x_{1i_1}, \ldots, x_{ki_k})]) = \det\left([\phi_{1j_1}(x_{1i_1})] \otimes \cdots \otimes [\phi_{kj_k}(x_{ki_k})]\right).$$

Then, for $N = k + 1$, we have

$$D_\Phi([(x_{1i_1}, \ldots, x_{ki_k}, x_{(k+1)i_{k+1}})]) = \begin{vmatrix} a_{11}B & a_{21}B & \cdots & a_{P_k1}B \\ a_{21}B & a_{22}B & \cdots & a_{P_k1}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{P_k1}B & a_{P_k2}B & \cdots & a_{P_kP_k}B \end{vmatrix}$$

where $[a_{ij}] = [\phi_{1j_1}(x_{1i_1})] \otimes \cdots \otimes [\phi_{kj_k}(x_{ki_k})]$ is the matrix associated to the product of $\Phi_1, \ldots, \Phi_k$ at the grid $[(x_{1i_1}, \ldots, x_{ki_k})]$, $P_k = \prod_{n=1}^k m_n$, and $B = [\phi_{(k+1)j_{k+1}}(x_{(k+1)i_{k+1}})]$. Hence we obtain that $D_\Phi([(x_{1i_1}, \ldots, x_{ki_k}, x_{(k+1)i_{k+1}})])$ is equal to

$$\det\left([\phi_{1j_1}(x_{1i_1})] \otimes \cdots \otimes [\phi_{kj_k}(x_{ki_k})] \otimes \left[\phi_{(k+1)j_{k+1}}(x_{(k+1)i_{k+1}})\right]\right).$$

This completes the proof. □

**Corollary 3.** $D_\Phi([(x_{1i_1}, \ldots, x_{Ni_N})]) = \prod_{n=1}^{N} \{\det[\phi_{nj_n}(x_{ni_n})]\}^{\frac{P}{m_n}}$, *where* $P = \prod_{n=1}^N m_n$.

It thus follows from Corollary 3 that the system of equations (3.1) has a unique solution, since each $\det[\phi_{nj_n}(x_{ni_n})]$ is nonzero.

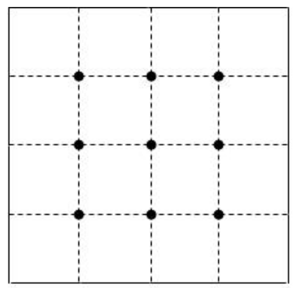
The formula in Corollary 3 is interesting, in the sense that it gives the relationship between the determinant that corresponds to the product of Chebyshev systems computed at the grid of points and the determinants of each system computed at the associated points. It also relates three types of products, namely the tensor product (of the Chebyshev systems), the Cartesian product (of the topological spaces), and the Kronecker product (of the associated matrices).

## 4. An Application: 2-Dimensional Interpolation

Here we shall give an application of the product of two Chebyshev systems in 2-dimensional interpolation.

4.1. **Problem 1: Full Grid.** Here we apply directly our previous results to the problem, as we illustrate in the following examples.

**Example 1.** Suppose we want to interpolate $(\frac{1}{4}, \frac{1}{4}, \frac{1}{2})$, $(\frac{1}{4}, \frac{1}{2}, 1), (\frac{1}{4}, \frac{3}{4}, 2), (\frac{1}{2}, \frac{1}{4}, 1)$, $(\frac{1}{2}, \frac{1}{2}, 2), (\frac{1}{2}, \frac{3}{4}, 1), (\frac{3}{4}, \frac{1}{4}, \frac{1}{2}), (\frac{3}{4}, \frac{1}{2}, 1), (\frac{3}{4}, \frac{3}{4}, 1)$ using the product of two Chebyshev systems on [0,1]. The points are given as the following $3 \times 3$ grid on $[0, 1] \times [0, 1]$:



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Since we want to interpolate points on the $3 \times 3$ grid, we use two Chebyshev systems that consist of 3 functions on $[0, 1]$.

(a) If we use $\{\sin \pi x, \sin 2\pi x, \sin 3\pi x\}$ and $\{1, \cos \pi y, \cos 2\pi y\}$ as Chebyshev systems, then the general interpolant has the form

$$\begin{aligned} U(x, y) = & a_{11} \sin \pi x + a_{12} \sin \pi x \cos \pi y + a_{13} \sin \pi x \cos 2\pi y \\ & + a_{21} \sin 2\pi x + a_{22} \sin 2\pi x \cos \pi y + a_{23} \sin 2\pi x \cos 2\pi y \\ & + a_{31} \sin 3\pi x + a_{32} \sin 3\pi x \cos \pi y + a_{33} \sin 3\pi x \cos 2\pi y. \end{aligned}$$

Substituting the values from the given points and reducing the associated matrix into a row echelon form, we get

$$\left[\begin{array}{ccccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{\sqrt{2}}{2} + \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2} \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \frac{3}{4} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \frac{-\sqrt{2}}{4} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & \frac{\sqrt{2}}{2} - \frac{1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{-1}{2} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & \frac{1}{2} \end{array}\right].$$

Hence

$$\begin{aligned} U(x, y) = & \left(\frac{\sqrt{2}}{2} + \frac{1}{2}\right) \sin \pi x - \frac{1}{2} \sin \pi x \cos \pi y - \frac{1}{2} \sin \pi x \cos 2\pi y \\ & + \frac{1}{4} \sin 2\pi x - \frac{\sqrt{2}}{4} \sin 2\pi x \cos \pi y + \frac{1}{4} \sin 2\pi x \cos 2\pi y \\ & + \left(\frac{\sqrt{2}}{2} - \frac{1}{2}\right) \sin 3\pi x - \frac{1}{2} \sin 3\pi x \cos \pi y + \frac{1}{2} \sin 3\pi x \cos 2\pi y \end{aligned}$$

interpolates the given points.

(b) If we use $\{1, \cos \pi x, \cos 2\pi x\}$ and $\{1, y, y^2\}$ as Chebyshev systems, then the general interpolant has the form

$$\begin{aligned} U(x, y) =& a_{11} + a_{12} y + a_{13} y^2 + a_{21} \cos \pi x + a_{22} y \cos \pi x + a_{23} y^2 \cos \pi x \\ &+ a_{31} \cos 2\pi x + a_{32} y \cos 2\pi x + a_{33} y^2 \cos 2\pi x. \end{aligned}$$

Substituting the values from the given points and solving the system of linear equations, we obtain that

$$\begin{aligned} U(x, y) =& 2y + \frac{\sqrt{2}}{2} \cos \pi x - 3\sqrt{2} y \cos \pi x + 4\sqrt{2} y^2 \cos \pi x \\ &+ 2 \cos 2\pi x - 14 y \cos 2\pi x + 16 y^2 \cos 2\pi x \end{aligned}$$

interpolate the given points.

(c) Let us now use $\{1, x, x^2\}$ and $\{\sin \pi y, \sin 2\pi y, \sin 2\pi y\}$ as Chebyshev systems. Similar to the above process, we get

$$\begin{aligned} U(x, y) =& \left(\frac{3\sqrt{2}}{4} - 1\right) \sin \pi y + \frac{-5}{2} \sin 2\pi y + \left(\frac{3\sqrt{2}}{4} + 1\right) \sin 3\pi y \\ &+ \left(\frac{-\sqrt{2}}{2} + 8\right) x \sin \pi y + 9x \sin 2\pi y + \left(\frac{-\sqrt{2}}{2} - 8\right) x \sin 3\pi y \\ &- 8x^2 \sin \pi y - 8x^2 \sin 2\pi y + 8x^2 \sin 3\pi y \end{aligned}$$

as an interpolant.

The graph of functions that interpolate the given points are shown in Figure 1.

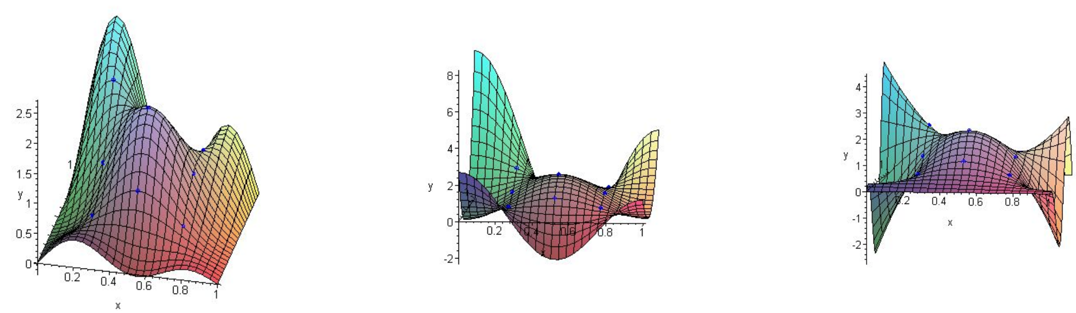
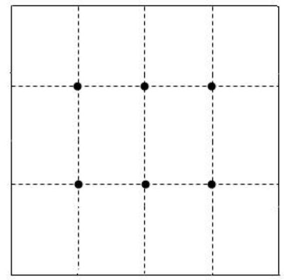


FIGURE 1. The graph of the interpolants using the product of Chebyshev systems $\{\sin \pi x, \sin 2\pi x, \sin 3\pi x\}$ and $\{1, \cos \pi y, \cos 2\pi y\}$ (left); $\{1, \cos \pi x, \cos 2\pi x\}$ and $\{1, y, y^2\}$ (center); $\{1, x, x^2\}$ and $\{\sin \pi y, \sin 2\pi y, \sin 2\pi y\}$ (right).

**Example 2.** Suppose we want to interpolate $(\frac{1}{4}, \frac{1}{3}, 4), (\frac{1}{4}, \frac{2}{3}, 1), (\frac{1}{2}, \frac{1}{3}, 1), (\frac{1}{2}, \frac{2}{3}, 4), (\frac{3}{4}, \frac{1}{3}, 4), (\frac{3}{4}, \frac{2}{3}, 1)$ using the product of two Chebyshev system on [0,1]. The points are given as the following $3 \times 2$ grid on $[0, 1] \times [0, 1]$:



Since we want to interpolate points on the $3\times 2$ grid, we use two Chebyshev systems that consist of 3 functions and 2 functions.

(a) If we use $\{\sin \pi x, \sin 2\pi x, \sin 3\pi x\}$ and $\{1, \cos \pi y\}$ as Chebyshev systems, then we get

$$U(x,y) = \left(\frac{5}{4} + \frac{5\sqrt{2}}{4}\right)\sin \pi x + \left(\frac{3\sqrt{2}}{2} - \frac{3}{2}\right)\sin \pi x \cos \pi y + \left(\frac{5\sqrt{2}}{4} - \frac{5}{4}\right)\sin 3\pi x + \left(\frac{3\sqrt{2}}{2} + \frac{3}{2}\right)\sin 3\pi x \cos \pi y$$

as an interpolant.

(b) If we use $\{1, \cos \pi x, \cos 2\pi x\}$ and $\{1, y\}$ as Chebyshev systems, then we get

$$U(x,y) = 7 - 9y + 9\cos 2\pi x - 18y\cos 2\pi x$$

as an interpolant.

(c) If we use $\{1, x, x^2\}$ and $\{\sin \pi y, \sin 2\pi y\}$ as Chebyshev systems, then we get

$$U(x,y) = \frac{5\sqrt{3}}{3}\sin \pi y + 7\sqrt{3}\sin 2\pi y - 32\sqrt{3}x\sin 2\pi y + 32\sqrt{3}x^2\sin 2\pi y$$

as an interpolant.

The graph of the functions that interpolate the given points are shown in Figure 2.

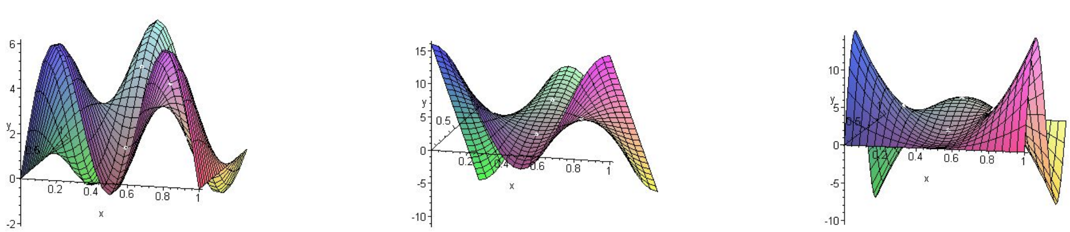


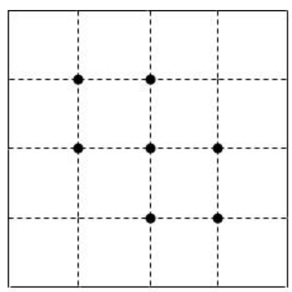
FIGURE 2. The graph of the interpolants using the product of Chebyshev systems $\{\sin \pi x, \sin 2\pi x, \sin 3\pi x\}$ and $\{1, \cos \pi y\}$ (left); $\{1, \cos \pi x, \cos 2\pi x\}$ and $\{1, y\}$ (center); $\{1, x, x^2\}$ and $\{\sin \pi y, \sin 2\pi y\}$ (right).

4.2. **Problem 2: Part of Grid.** Let $G = \{(x_n, y_n, c_n) : n = 1, 2, \ldots, m\}$ be any set of points on $X_1 \times X_2 \times \mathbb{F}$, where $X_1$ and $X_2$ are topological spaces. Then we can find $H = \{(x_i, y_k, c_{ik}) : i = 1, 2, \ldots, m_1; k = 1, 2, \ldots, m_2\}$ that has a grid form such that $H$ is the 'minimal' grid that contains $G$. Here we assume that $G$ itself is not a grid, so that $m < m_1 m_2$. Now let $\{\phi_1, \phi_2, \ldots, \phi_{m_1}\}$ and $\{\psi_1, \psi_2, \ldots, \psi_{m_2}\}$ be Chebyshev systems on $X_1$ and $X_2$ respectively. Then we can use

$$U(x,y) = \sum_{j=1}^{m_1} \sum_{l=1}^{m_2} a_{jl} \phi_j(x) \psi_l(y) \tag{4.1}$$

as an interpolant of $G$. Substituting the points on $G$ to (4.1), we obtain the system of linear equations with $m$ equations and $m_1 m_2$ variables. The system is guaranteed to have a solution because it is part of the $m_1 m_2 \times m_1 m_2$ system which has a solution. Moreover, since $m < m_1 \cdot m_2$, the system has many solutions. This implies that there are many possible values for $a_{jl}$'s such that (4.1) interpolates the given points.

**Example 3.** Suppose we want to interpolate $(\frac{1}{4}, \frac{1}{2}, 2), (\frac{1}{4}, \frac{3}{4}, 1), (\frac{1}{2}, \frac{1}{4}, 2), (\frac{1}{2}, \frac{1}{2}, 3), (\frac{1}{2}, \frac{3}{4}, 2), (\frac{3}{4}, \frac{1}{4}, 1), (\frac{3}{4}, \frac{1}{2}, 2)$ using the product of two Chebyshev system on [0,1]. The points are given on the following subset of a $3 \times 3$ grid on $[0, 1] \times [0, 1]$:



Accordingly, we use two Chebyshev systems that consist of 3 functions.

(a) If we use $\{\sin \pi x, \sin 2\pi x, \sin 3\pi x\}$ and $\{1, \cos \pi y, \cos 2\pi y\}$ as the Chebyshev systems, then the general interpolant has the form

$$\begin{aligned} U(x,y) = & a_{11} \sin \pi x + a_{12} \sin \pi x \cos \pi y + a_{13} \sin \pi x \cos 2\pi y \\ & + a_{21} \sin 2\pi x + a_{22} \sin 2\pi x \cos \pi y + a_{23} \sin 2\pi x \cos 2\pi y \\ & + a_{31} \sin 3\pi x + a_{32} \sin 3\pi x \cos \pi y + a_{33} \sin 3\pi x \cos 2\pi y. \end{aligned}$$

Substituting the values from the given points and reducing the associated matrix into a row echelon form, we get

$$\left[\begin{array}{ccccccccc|c} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{2} & \sqrt{2} + \frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & \sqrt{2} - 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & \frac{-\sqrt{2}}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & \frac{-1}{2} & \sqrt{2} - \frac{3}{2} \end{array}\right].$$

There are many functions that interpolate the given points, one of them is:

$$U(x, y) = \left(\frac{-1}{2}\sqrt{2} + \frac{1}{2}\right)\sin\pi x - \sin\pi x\cos 2\pi y + \left(\sqrt{2} - 1\right)\sin 2\pi x$$
$$+ \left(\sqrt{2} - \frac{3}{2}\right)0.25\sin 2\pi x\cos 2\pi y.$$

(b) If we use $\{1, \cos\pi x, \cos 2\pi x\}$ and $\{1, y, y^2\}$ as the Chebyshev systems, then the general interpolant has the form

$$U(x, y) = a_{11} + a_{12}y + a_{13}y^2 + a_{21}\cos\pi x + a_{22}y\cos\pi x + a_{23}y^2\cos\pi x$$
$$+ a_{31}\cos 2\pi x + a_{32}y\cos 2\pi x + a_{33}y^2\cos 2\pi x.$$

The same process to the above, so

$$U(x, y) = -2 + 16y - 16y^2 - \cos 2\pi x.$$

is one of the functions that interpolate the given points.

(c) If we use $\{1, x, x^2\}$ and $\{\sin\pi y, \sin 2\pi y, \sin 2\pi y\}$ as the Chebyshev systems, then one of the functions that interpolates the given points is

$$U(x, y) = \left(\frac{-5}{2} + \sqrt{2}\right)\sin\pi y + \left(2 - \sqrt{2}\right)\sin 2\pi y + \left(\sqrt{2} - \frac{3}{2}\right)\sin 3\pi y$$
$$+ 16x\sin\pi y + \left(-4 + 2\sqrt{2}\right)x\sin 2\pi y - 16x^2\sin\pi y.$$

The graph of the functions that interpolate the given points are shown in Figure 3.

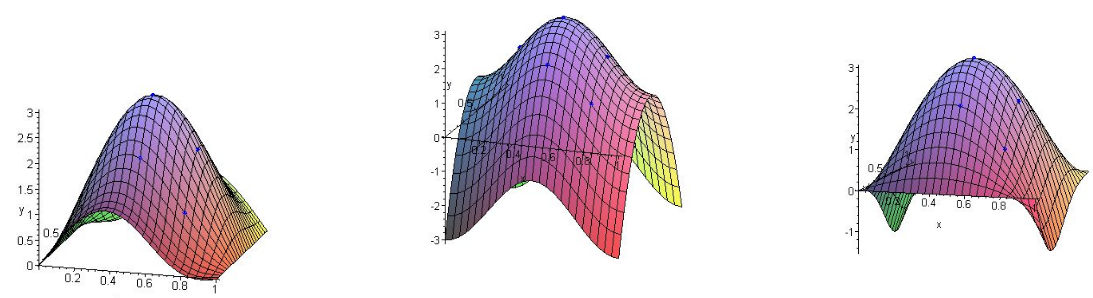


FIGURE 3. The graph of the interpolants using the product of Chebyshev systems $\{\sin\pi x, \sin 2\pi x, \sin 3\pi x\}$ and $\{1, \cos\pi y, \cos 2\pi y\}$ (left); $\{1, \cos\pi x, \cos 2\pi x\}$ and $\{1, y, y^2\}$ (center); $\{1, x, x^2\}$ and $\{\sin\pi y, \sin 2\pi y, \sin 2\pi y\}$ (right).

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