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ORTHOGONALITY IN n-NORMED SPACES

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Abstract. We introduce several notions of orthogonality in n-normed spaces. We show that in the standard case where the space is an inner product space and the n-norm is the square root of the Gramian, our notions of orthogonality coincide with the usual one. Our definitions also improve those proposed by Khan and Siddiqui [15], Cho and Kim [3], and Godini [9].

1. INTRODUCTION

Several notions of orthogonality in a normed space have been developed by many authors. For example, the following definitions of Pythagorean, isosceles, and the Birkhoff-James orthogonality in a (real) normed space $(X, \|\cdot\|)$ are known:

P-orthogonality: x is P-orthogonal to y (denoted by $x \perp_{\rm P} y$) if only if

$$\|x+y\|^2 = \|x\|^2 + \|y\|^2.$$

I-orthogonality: x is I-orthogonal to y (denoted by $x \perp_{\rm I} y$) if only if

$$\|x+y\| = \|x-y\|.$$

BJ-orthogonality: x is BJ-orthogonal to y (denoted by $x \perp_{\rm BJ} y$) if only if

$$\|x+\alpha y\| \geq \|x\| \text{ for every } \alpha \in \mathbb{R}.$$

If X is actually equipped with an inner product $\langle \cdot, \cdot \rangle$, then one may observe that $x \perp_{\rm P} y$, $x \perp_{\rm I} y$, and $x \perp_{\rm BJ} y$ are all equivalent to the condition that $\langle x, y \rangle = 0$, for

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which we have the usual orthogonality $x \perp y$. However, in a normed space which is not an inner product space, one does not imply another. For further properties of these notions of orthogonality, see, for example, [17]. Related results may be found in [1, 4, 5, 10, 16, 20].

These notions of orthogonality have been extended to 2-normed spaces by several researchers (see, for example, [3, 15]). A (real) *2-normed space* is a (real) vector space X equipped with a *2-norm* $\|\cdot,\cdot\| : X \times X \to \mathbb{R}$ satisfying the following properties:

(1.1) $\|x, y\| \geq 0$ for every $x, y \in X$; $\|x, y\| = 0$ if and only if x and y are linearly dependent;
(1.2) $\|x, y\| = \|y, x\|$ for every $x, y \in X$;
(1.3) $\|x, \alpha y\| = |\alpha|\|x, y\|$ for every $x, y \in X$ and $\alpha \in \mathbb{R}$;
(1.4) $\|x, y + z\| \leq \|x, y\| + \|x, z\|$ for every $x, y, z \in X$.

As the notions of orthogonality in normed spaces are inspired by that in inner product spaces, the notions of orthogonality in 2-normed spaces are also connected to that in 2-inner product spaces. A (real) *2-inner product space* is a (real) vector space X equipped with an *2-inner product* $\langle\cdot,\cdot|\cdot\rangle : X \times X \times X \to \mathbb{R}$ satisfying

(1.5) $\langle x, x|z\rangle \geq 0$ for every $x, z \in X$; $\langle x, x|z\rangle = 0$ if and only if x and z are linearly dependent;
(1.6) $\langle x, y|z\rangle = \langle y, x|z\rangle$ for every $x, y, z \in X$;
(1.7) $\langle \alpha x, y|z\rangle = \alpha\langle x, y|z\rangle$ for every $x, y, z \in X$ and $\alpha \in \mathbb{R}$;
(1.8) $\langle x_1 + x_2, y|z\rangle = \langle x_1, y|z\rangle + \langle x_2, y|z\rangle$ for every $x_1, x_2, y, z \in X$.

For historical background about 2-inner product spaces and 2-normed spaces, we refer the reader to [6, 7] and [8], respectively.

In [14], it is shown that the 'standard' definition of orthogonality in a 2-inner product space $(X, \langle\cdot,\cdot|\cdot\rangle)$, where $\dim(X) \geq 3$, is the following:

Definition 1.1 (G-orthogonality in 2-inner product spaces)
x is *G-orthogonal to* y, denoted by $x \perp_G y$, if and only if there exists a subspace V of X with $\mathrm{codim}(V) = 1$ such that $\langle x, y|z\rangle = 0$ for all $z \in V$.

We say that this definition is 'standard' because when X is a standard 2-inner product space, that is, when X is actually equipped with an inner product $\langle\cdot,\cdot\rangle$ and the 2-inner product

$$\langle x, y|z\rangle := \begin{vmatrix} \langle x, y\rangle & \langle x, z\rangle \\ \langle z, y\rangle & \langle z, z\rangle \end{vmatrix},$$

then we have $x \perp_G y$ if and only if $x \perp y$ (see [14]). The above definition of G-orthogonality also improves the one proposed by Cho and Kim [3]. (As shown in [14], Cho and Kim's definition of orthogonality in 2-inner product spaces are void.)

Accordingly, we define P-, I-, and BJ-orthogonality in a 2-normed space $(X, \|\cdot,\cdot\|)$ of dimension 3 or higher as follows:

Definition 1.2 (P-, I-, and BJ-orthogonality in 2-normed spaces)
(a) $x \perp_{\rm P} y$ if only if there exists a subspace V of X with $\text{codim}(V) = 1$ such that

$$\|x + y, z\|^2 = \|x, z\|^2 + \|y, z\|^2 \quad \text{for every } z \in V;$$

(b) $x \perp_{\rm I} y$ if only if there exists a subspace V of X with $\text{codim}(V) = 1$ such that

$$\|x + y, z\| = \|x - y, z\| \quad \text{for every } z \in V;$$

(c) $x \perp_{\rm BJ} y$ if only if there exists a subspace V of X with $\text{codim}(V) = 1$ such that

$$\|x + \alpha y, z\| \geq \|x, z\| \quad \text{for every } z \in V \text{and } \alpha \in \mathbb{R}.$$

These definitions improve those formulated by Khan and Siddiqui [15].

In this paper, we shall extend these notions of orthogonality to n-normed spaces and show that in the standard case they are equivalent to the usual one.

Throughout this paper, X will always denote a real vector space, unless otherwise stated.

2. MAIN RESULTS

Let $n \geq 2$ be a nonnegative integer and X be a vector space of dimension n or higher. A real-valued function $\|\cdot, \dots, \cdot\|$ on X^n satisfying the following four properties

(2.1) $\|x_1, \dots, x_n\| = 0$ if and only if $x_1, \dots, x_n$ are linearly dependent;
(2.2) $\|x_1, \dots, x_n\|$ is invariant under permutation;
(2.3) $\|\alpha x_1, x_2, \dots, x_n\| = |\alpha| \, \|x_1, x_2, \dots, x_n\|$ for any $\alpha \in \mathbb{R}$;
(2.4) $\|x + x', x_2, \dots, x_n\| \leq \|x, x_2, \dots, x_n\| + \|x', x_2, \dots, x_n\|$,

is called an *n-norm* on X, and the pair $(X, \|\cdot, \dots, \cdot\|)$ is called an *n-normed space.*

Next, a real-valued function $\langle \cdot, \cdot | \cdot, \dots, \cdot \rangle$ on X^{n+1} satisfying the following five properties

(2.5) $\langle x_1, x_1 | x_2, \dots, x_n \rangle \geq 0$; $\langle x_1, x_1 | x_2, \dots, x_n \rangle = 0$ if and only if $x_1, x_2, \dots, x_n$ are linearly dependent;
(2.6) $\langle x_1, x_1 | x_2, \dots, x_n \rangle = \langle x_{i_1}, x_{i_1} | x_{i_2}, \dots, x_{i_n} \rangle$ for every permutation $(i_1, \dots, i_n)$ of $(1, \dots, n)$;
(2.7) $\langle x, y | x_2, \dots, x_n \rangle = \langle y, x | x_2, \dots, x_n \rangle$;
(2.8) $\langle \alpha x, y | x_2, \dots, x_n \rangle = \alpha \langle x, y | x_2, \dots, x_n \rangle, \ \alpha \in \mathbb{R}$;
(2.9) $\langle x + x', y | x_2, \dots, x_n \rangle = \langle x, y | x_2, \dots, x_n \rangle + \langle x', y | x_2, \dots, x_n \rangle$,

is called an *n-inner product* on X, and the pair $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$ is called an *n-inner product space.*

If $(X, \langle\cdot,\cdot|\cdot,\dots,\cdot\rangle)$ is an n-inner product space, then one may define an n-norm on X by the formula

$$\|x_1, x_2, \dots, x_n\| := \sqrt{\langle x_1, x_1|x_2, \dots, x_n\rangle}.$$

Here we have the parallelogram law

$$\|x+y, x_2, \dots, x_n\|^2 + \|x-y, x_2, \dots, x_n\|^2 = 2\|x, x_2, \dots, x_n\|^2 + 2\|y, x_2, \dots, x_n\|^2,$$

and the polarization identity

$$4\langle x, y|x_2, \dots, x_n\rangle = \|x+y, x_2, \dots, x_n\|^2 - \|x-y, x_2, \dots, x_n\|^2.$$

For historical background of n-normed spaces and n-inner product spaces, see [18].

Note that any inner product space $(X, \langle\cdot,\cdot\rangle)$ can be equipped with the *standard* n-inner product (also known as the *simple* n-inner product)

$$\langle x, y|x_2, \dots, x_n\rangle := \begin{vmatrix} \langle x, y\rangle & \langle x, x_2\rangle & \dots & \langle x, x_n\rangle \\ \langle x_2, y\rangle & \langle x_2, x_2\rangle & \dots & \langle x_2, x_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_n, y\rangle & \langle x_n, x_2\rangle & \dots & \langle x_n, x_n\rangle \end{vmatrix}.$$

and its induced n-norm

$$\|x_1, \dots, x_n\| := \sqrt{\det\big(\langle x_i, x_j\rangle\big)},$$

which is known as the *standard* n-norm. Thus a standard n-inner product space is a standard n-normed space, and vice versa. The reader might recognize the determinant $\det\big(\langle x_i, x_j\rangle\big)$ as the *Gramian* of $x_1, \dots, x_n$. In general, $\|x_1, \dots, x_n\|$ has a geometrical interpretation as the volume of the n-dimensional parallelepiped spanned by $x_1, \dots, x_n$ in X. For further discussion about n-normed spaces and n-inner product spaces, see [11, 13, 18, 19].)

Our first result below shows that, in an n-inner product space $(X, \langle\cdot,\cdot|\cdot,\dots,\cdot\rangle)$, we cannot define G-orthogonality between x and y by the condition that

$$\langle x, y|x_2, \dots, x_n\rangle = 0 \quad \text{for all } x_2, \dots, x_n \notin \operatorname{span}\{x, y\}$$

as suggested by [3] and [9].

Theorem 2.1 *Let $(X, \langle\cdot,\cdot|\cdot,\dots,\cdot\rangle)$ be a standard n-inner product space of dimension $n+1$ or higher. Then, the condition that $\langle x, y|x_2, \dots, x_n\rangle = 0$ for all $x_2, \dots, x_n \notin \operatorname{span}\{x, y\}$ is satisfied only by $x = 0$ or $y = 0$.*

Proof. We shall prove the theorem through its contraposition. Suppose that $x \neq 0$ and $y \neq 0$. Our task is to show that there exist $x_2, \dots, x_n \notin \operatorname{span}\{x, y\}$ such that $\langle x, y|x_2, \dots, x_n\rangle \neq 0$. To do so, we consider several cases.

Case 1. If x and y are linearly dependent, that is, $y = kx$ for $k \neq 0$, then

$$\langle x, y|x_2, \ldots, x_n\rangle = k\langle x, x|x_2, \ldots, x_n\rangle = k\|x, x_2, \ldots, x_n\|^2.$$

Now choose $x_2, \ldots, x_n \notin \text{span}\{x\}$ such that $\{x, x_2, \ldots, x_n\}$ is linearly independent. Then, we have

$$\langle x, y|x_2, \ldots, x_n\rangle = k\|x, x_2, \ldots, x_n\|^2 \neq 0.$$

Case 2. If x and y are linearly independent, then we consider the following two subcases.

Case 2a. If $x \not\perp y$ or $\langle x, y\rangle \neq 0$, then we may choose an orthonormal sequence $x_2, \ldots, x_n \in \{x, y\}^{\perp}$, so that

$$\langle x, y|x_2, \ldots, x_n\rangle = \langle x, y\rangle\|x_2\|^2 \cdots \|x_n\|^2 = \langle x, y\rangle \neq 0.$$

Case 2b. If $x \perp y$ or $\langle x, y\rangle = 0$, then we may choose a nonzero vector $x_2 = x+y+u \notin \text{span}\{x, y\}$ where u is a fixed nonzero vector perpendicular to $\text{span}\{x, y\}$, and nonzero vectors $x_3, \ldots, x_n \notin \text{span}\{x, y\}$ where $x_3 \perp \text{span}\{x, y\}$ and $x_i \perp \text{span}\{x, y, x_3, \ldots, x_{i-1}\}$ for $i = 4, \ldots, n$. Hence, we have

$$\begin{aligned}
\langle x, y|x_2, \ldots, x_n\rangle &= \begin{vmatrix} 0 & \langle x, x_2\rangle & 0 & \cdots & 0 \\ \langle x_2, y\rangle & \langle x_2, x_2\rangle & \langle x_2, x_3\rangle & \cdots & \langle x_2, x_n\rangle \\ 0 & \langle x_3, x_2\rangle & \langle x_3, x_3\rangle & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \langle x_n, x_2\rangle & 0 & \cdots & \langle x_n, x_n\rangle \end{vmatrix} \\
&= -\langle x, x_2\rangle\langle x_2, y\rangle\|x_3\|^2 \cdots \|x_n\|^2 \\
&= -\|x\|^2\|y\|^2\|x_3\|^2 \cdots \|x_n\|^2 \neq 0.
\end{aligned}$$

Thus, in any case, we can always find $x_2, \ldots, x_n \notin \text{span}\{\text{x}, \text{y}\}$ such that $\langle x, y|x_2, \ldots, x_n\rangle \neq 0$. This proof is therefore complete. □

As in 2-inner product spaces and 2-normed spaces, we define the notions of G-orthogonality in n-inner product spaces and P-, I-, and BJ-orthogonality in n-normed spaces as follows. For the rest of this section, we assume that X is a vector space of dimension $n + 1$ or higher.

Definition 2.2 (G-orthogonality) Let $(X, \langle\cdot, \cdot|\cdot, \ldots, \cdot\rangle)$ be an n-inner product space. For $x, y \in X$, we say that x is *G-orthogonal to* y and write $x \perp_G y$ if and only if there exists a subspace V of X with $\text{codim}(V) = 1$ such that

$$\langle x, y|x_2, \ldots, x_n\rangle = 0 \quad \text{for every } x_2, \ldots, x_n \in V.$$

Definition 2.3 (P-, I-, and BJ-orthogonality) Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space. For $x, y \in X$, we define

(a) $x \perp_{\mathrm{P}} y$ if only if there exists a subspace V of X with $\mathrm{codim}(V) = 1$ such that

$$\|x+y, x_2, \dots, x_n\|^2 = \|x, x_2, \dots, x_n\|^2 + \|y, x_2, \dots, x_n\|^2 \quad \text{for every } x_2, \dots, x_n \in V;$$

(b) $x \perp_{\mathrm{I}} y$ if only if there exists a subspace V of X with $\mathrm{codim}(V) = 1$ such that

$$\|x+y, x_2, \dots, x_n\| = \|x-y, x_2, \dots, x_n\| \quad \text{for every } x_2, \dots, x_n \in V;$$

(c) $x \perp_{\mathrm{BJ}} y$ if only if there exists a subspace V of X with $\mathrm{codim}(V) = 1$ such that

$$\|x+\alpha y, x_2, \dots, x_n\| \geq \|x, x_2, \dots, x_n\| \quad \text{for every } x_2, \dots, x_n \in V \text{and } \alpha \in \mathbb{R}.$$

Theorem 2.4 *In an n-inner product space $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$, P-, I-, and BJ-orthogonality are equivalent to G-orthogonality.*

Proof. The proof follows from the properties of the n-inner product. For instance, to prove that $x \perp_{\mathrm{BJ}} y$ implies $x \perp_G y$, we take a subspace V of X with $\mathrm{codim}(V) = 1$ such that

$$\|x+\alpha y, x_2, \dots, x_n\| \geq \|x, x_2, \dots, x_n\|$$

for every $x_2, \dots, x_n \in V$ and $\alpha \in \mathbb{R}$. Now fix $x_2, \dots, x_n \in V$ for the moment. Then, squaring both sides, we get

$$2\alpha \langle x, y | x_2, \dots, x_n \rangle + \alpha^2 \|y, x_2, \dots, x_n\|^2 \geq 0$$

for every $\alpha \in \mathbb{R}$. This forces us to have $\langle x, y | x_2, \dots, x_n \rangle^2 \leq 0$ or $\langle x, y | x_2, \dots, x_n \rangle = 0$. Since this is true for every $x_2, \dots, x_n$, we conclude that $x \perp_G y$. □

Theorem 2.5 *Let $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$ be an n-inner product space. Then,*
(a) *$x \perp_G x$ if and only if $x = 0$;*
(b) *$x \perp_G y$ if and only if $y \perp_G x$;*
(c) *if $x \perp_G y$, then $\alpha x \perp_G \beta y$ for every $\alpha, \beta \in \mathbb{R}$.*

Proof. The proof follows directly from the definition of G-orthogonality and the properties of the n-inner product. For instance, to prove the 'only if' part of (a), we suppose that $x \perp_G x$ and $x \neq 0$. Then, there exists a subspace V of X with $\mathrm{codim}(V) = 1$ such that $\langle x, x | x_2, \dots, x_n \rangle = \|x, x_2, \dots, x_n\|^2 = 0$ for every $x_2, \dots, x_n \in V$. But, since $\dim(V) \geq n$, we can choose $x_2, \dots, x_n \in V$ such that $\{x, x_2, \dots, x_n\}$ is linearly independent. Hence $\|x, x_2, \dots, x_n\| > 0$, which is a contradiction. □

The next theorem states that in a standard n-inner product space, G-orthogonality is equivalent to the usual orthogonality (with respect to the inner product).

Theorem 2.6 *Let $(X, \langle \cdot, \cdot | \cdot, \dots, \cdot \rangle)$ be a standard n-inner product space. Then, $x \perp_G y$ if and only if $x \perp y$.*

Proof. First we prove the necessary condition. Suppose that $x \perp y$, that is, $\langle x, y\rangle = 0$, and that $x, y \neq 0$. Then we can choose $V = \{x\}^{\perp}$. Clearly V is a subspace of X with $\mathrm{codim}(V) = 1$. Now, for every $x_2, x_3, \ldots, x_n \in V$, we have

$$\langle x, y|x_2, \ldots, x_n\rangle = 0.$$

(Alternatively, one may choose $V = \{y\}^{\perp}$ and get the same result.) This shows that $x \perp_G y$.

For the sufficient condition, suppose that $x \not\perp y$, that is, $\langle x, y\rangle \neq 0$. Clearly x and y are nonzero vectors. To show that $x \not\perp_G y$, let V be an arbitrary subspace of X with $\mathrm{codim}(V) = 1$. Since $\dim(V \cap \{x\}^{\perp}) \geq n-1$, there must exist an orthonormal sequence $x_2, \ldots, x_n \in V$ such that $\langle x, x_i\rangle = 0$ for every $i = 2, \ldots, n$. Accordingly, we have

$$\langle x, y|x_2, \ldots, x_n\rangle = \langle x, y\rangle \|x_2\|^2 \cdots \|x_n\|^2 = \langle x, y\rangle \neq 0.$$

This completes the proof of the theorem. □

3. REMARKS ON THE n-DIMENSIONAL CASE

Let $(X, \|\cdot, \ldots, \cdot\|)$ be a standard n-normed space of dimension n. With respect to the G-orthogonality defined in §2, we find that two arbitrary vectors x and y are G-orthogonal to each other. Given $x, y \neq 0$, just take V to be an $n-1$-dimensional subspace of X such that $x \in V$. Now, for every $x_2, \ldots, x_n \in V$, the set $\{x, x_2, \ldots, x_n\}$ is linearly dependent. Then, supposing that x_n is a linear combination of $x, x_2, \ldots, x_{n-1}$, we get

$$\langle x, y|x_2, \ldots, x_n\rangle = \begin{vmatrix} \langle x, y\rangle & \langle x, x_2\rangle & \ldots & \langle x, x_n\rangle \\ \langle x_2, y\rangle & \langle x_2, x_2\rangle & \ldots & \langle x_2, x_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle x_n, y\rangle & \langle x_n, x_2\rangle & \ldots & \langle x_n, x_n\rangle \end{vmatrix}$$

$$= \begin{vmatrix} \langle x, y\rangle & \langle x, x_2\rangle & \ldots & \langle x, x_n\rangle \\ \langle x_2, y\rangle & \langle x_2, x_2\rangle & \ldots & \langle x_2, x_n\rangle \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 0 \end{vmatrix} = 0.$$

(The same also happens if we take V to be an $n-1$-dimensional subspace of X such that $y \in V$.) This fact is of course undesirable.

To define orthogonality in n-normed spaces of dimension n in general, it seems that we have to use a different way. In light of [13], we can actually derive a norm from the n-norm and then define P-, I-, and BJ-orthogonality using this norm. In the standard case, the derived norm can be obtained from the n-norm in such a way that it coincides with the existing one (see [12]). Therefore, P-, I-, and

BJ-orthogonality defined by using the derived norm will coincide with the usual orthogonality (with respect to the existing inner product).

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