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WEAK TYPE INEQUALITIES FOR FRACTIONAL INTEGRAL OPERATORS ON GENERALIZED NON-HOMOGENEOUS MORREY SPACES

Idha Sihwaningrum Sri Maryani
(*Jenderal Soedirman University, Indonesia*)
and
H. Gunawan
(*Bandung Institute of Technology, Indonesia)*

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Abstract. We obtain weak type $(1,q)$ inequalities for fractional integral operators on generalized non-homogeneous Morrey spaces. The proofs use some properties of maximal operators. Our results are closely related to the strong type inequalities in [13, 14, 15].

Key words: *weak type inequalitiy fractional integral operator, (generalized) non-homogeneous MOrrey psace*

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1 Introduction

The work of Nazarov et al. [10], Tolsa[17, and Verdera [18] reveal some important ideas of the spaces of non-homogeneous type. By a non-homogeneous space we mean a (metric) measure space-here we consider only $\mathbf{R}^d$-equipped with a Borel measure μ satisfying the growth condition of order n with $0 < n \le d$, that is there exists a constant $C > 0$ such that

$$\mu(B(a,r)) \le C r^n \tag{1}$$

for every ball $B(a,r)$ centered at $a \in \mathbf{R}^d$ with radius $r > 0$. The growth condition replaces the *doubling condition*:

$$\mu(B(a,2r)) \le C\mu(B(a,r))$$

which plays an important role in the space of homogeneous type.

In the setting of non-homogeneous spaces described above, we define the fractional integral operator I_α $(0 < \alpha < n \leq d)$ by the formula

$$I_\alpha f(x) := \int_{\mathbf{R}^d} \frac{f(y)}{|x-y|^{n-\alpha}}\, \mathrm{d}\mu(y)$$

for suitable functions f on $\mathbf{R}^d$. Note that if $n = d$ and μ is the usual Lebesgue measure on $\mathbf{R}^d$, then I_α is the classical fractional integral operator introduced by Hardy and Littlewood[5,6] and Sobolev[16]. The classical fractional integral operator I_α is known to be bounded from the Lebesgue space $L^p(\mathbf{R}^d)$ to $L^q(\mathbf{R}^d)$ where $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{d}$ for $1 < p < \frac{d}{\alpha}$. This result has been extended in many ways-see for examples [4, 8, 11] and the references therein.

For $p = 1$, we have a weak type inequality for I_α and on non-homogeneous Lebesgue spaces such an inequality has been studied, among others, by García-Cuerva, Gatto, and Martell in [2, 3]. One of their results is the following theorem. (Here and after, we denote by C a positive constant which may be different from line to line.)

Theorem 1.1[2,3]**.** $\frac{1}{q} = 1 - \frac{\alpha}{n}$*, then for any function* $f \in L^1(\mu)$ *we have*

$$\mu\{x \in \mathbf{R}^d : |I_\alpha f(x)| > \gamma\} \leq C \left(\frac{\|f\|_{L^1(\mu)}}{\gamma} \right)^q, \qquad \gamma > 0.$$

The proof of Theorem 1.1 uses the weak type inequality for the maximal operator

$$Mf(x) := \sup_{r>0} \frac{1}{r^n} \int_{B(x,r)} |f(y)|\, \mathrm{d}\mu(y).$$

In this paper, we shall prove the weak type inequality for I_α on generalized non-homogeneous Morrey spaces (which we shall define later). The proof will employ the following inequality for the maximal operator M.

Theorem 1.2[2,12]**.** *For any positive weight* w *on* $\mathbf{R}^d$ *and any function* $f \in L^1_{\mathrm{loc}}(\mu)$*, we have*

$$\int_{\{x\in\mathbf{R}^d : Mf(x)>\gamma\}} w(x)\, \mathrm{d}\mu(x) \leq \frac{C}{\gamma} \int_{\mathbf{R}^d} |f(x)| Mw(x)\, \mathrm{d}\mu(x), \qquad \gamma > 0.$$

Our main results are presented as Theorems 2.2 and 2.3in the next section. Related results can be found in [13, 14, 15].

2 Main Results

For $1 \leq p < \infty$ and a suitable function $\phi : (0,\infty) \to (0,\infty)$, we define the generalized non-homogeneous Morrey space $\mathcal{M}^{p,\phi}(\mu) = \mathcal{M}^{p,\phi}(\mathbf{R}^d, \mu)$ to be the space of all functions $f \in L^p_{\mathrm{loc}}(\mu)$ for which

$$\|f\|_{\mathcal{M}^{p,\phi}(\mu)} := \sup_{B=B(a,r)} \frac{1}{\phi(r)} \left(\frac{1}{r^n} \int_B |f(x)|^p \mathrm{d}\mu(x) \right)^{1/p} < \infty.$$

(We refer the reader to [1] for the definition of analogous spaces in the homogeneous case.) Throughout this paper, we will always assume that ϕ is an almost decreasing function, that is there exists a constant $C>0$ such that $\phi(t)\le C\phi(s)$ whenever $s<t$.

Our first theorem is closely related to Theorem 3.3 in [14].

Theorem 2.1. *If the function* $\phi:(0,\infty)\to(0,\infty)$ *satisfies*

$$\int_r^\infty \frac{\phi(t)}{t}\,\mathrm{d}t \le C\phi(r), \qquad r>0,$$

then for any function $f\in\mathcal{M}^{1,\phi}(\mu)$ *and any ball* $B(a,r)\subseteq\mathbf{R}^d$ *we have*

$$\mu\{x\in B(a,r): Mf(x)>\gamma\}\le\frac{C}{\gamma}r^n\phi(r)\|f\|_{\mathcal{M}^{1,\phi}(\mu)}, \qquad \gamma>0.$$

Proof. For any function $f\in\mathcal{M}^{1,\phi}(\mu)$ and the characteristic function $\chi_{B(a,r)}$, we observe that

$$\begin{aligned}\int_{\mathbf{R}^d}|f(x)|M\chi_{B(a,r)}(x)\,\mathrm{d}\mu \le& \int_{B(a,2r)}|f(x)|M\chi_{B(a,r)}(x)\,\mathrm{d}\mu\\ &+\sum_{k=1}^\infty\int_{B(a,2^{k+1}r)\setminus B(a,2^kr)}|f(x)|M\chi_{B(a,r)}(x)\,\mathrm{d}\mu.\end{aligned}$$

Since μ satisfies the growth condition (1), we have $M\chi_{B(a,r)}(x)\le C$ and $M\chi_{B(a,r)}(x)\le C2^{-kn}$ whenever $x\in B(a,2^{k+1}r)\setminus B(a,2^kr)$ (where $k=1,2,3,\cdots$). Now, as ϕ is almost increasing, we have

$$\phi(2^{k+1}r)\le C\int_{2^kr}^{2^{k+1}r}\frac{\phi(t)}{t}\,\mathrm{d}t$$

for $k=1,2,3,\cdots$. Consequently,

$$\begin{aligned}&\int_{\mathbf{R}^d}|f(x)|M\chi_{B(a,r)}(x)\,\mathrm{d}\mu\\
&\le C\left(\int_{B(a,2r)}|f(x)|\,\mathrm{d}\mu+\sum_{k=1}^\infty\int_{B(a,2^{k+1}r)\setminus B(a,2^kr)}|f(x)|2^{-kn}\,\mathrm{d}\mu\right)\\
&\le C\left((2r)^n\phi(2r)\|f\|_{\mathcal{M}^{1,\phi}(\mu)}+\sum_{k=1}^\infty 2^{-kn}(2^{k+1}r)^n\phi(2^{k+1}r)\|f\|_{\mathcal{M}^{1,\phi}(\mu)}\right)\\
&=Cr^n\|f\|_{\mathcal{M}^{1,\phi}(\mu)}\sum_{k=0}^\infty\phi(2^{k+1}r)\\
&\le Cr^n\|f\|_{\mathcal{M}^{1,\phi}(\mu)}\sum_{k=0}^\infty\int_{2^kr}^{2^{k+1}r}\frac{\phi(t)}{t}\,\mathrm{d}t\\
&\le Cr^n\|f\|_{\mathcal{M}^{1,\phi}(\mu)}\int_r^\infty\frac{\phi(t)}{t}\,\mathrm{d}t\\
&\le Cr^n\phi(r)\|f\|_{\mathcal{M}^{1,\phi}(\mu)}.\end{aligned}$$

Next, by applying Theorem 1.2, we find that for $\gamma > 0$,

$$\begin{aligned}\mu\{x \in B(a,r) : Mf(x) > \gamma\} &= \int_{\{x\in B(a,r):Mf(x)>\gamma\}} \chi_{B(a,r)}(x)\, \mathrm{d}\mu \\ &\le \frac{C}{\gamma}\int_{\mathbf{R}^d} |f(x)| M\chi_{B(a,r)}(x)\, \mathrm{d}\mu \\ &\le \frac{C}{\gamma} r^n \phi(r) \|f\|_{\mathcal{M}^{1,\phi}(\mu)},\end{aligned}$$

as desired.

Theorem 2.1 enables us to obtain an inequality in which the fractional integral operator is controlled by the maximal operator. The classical setting of this inequality is available in [7].

Theorem 2.2. *Suppose that for some $0 \le \lambda < n-\alpha$, we have*

$$\int_r^\infty t^{\alpha-1}\phi(t)dt \le Cr^{\lambda+\alpha-n}, \qquad r > 0.$$

Then, for any function $f \in \mathcal{M}^{1,\phi}(\mu)$, we have

$$|I_\alpha f(x)| \le C\,[Mf(x)]^{1-\frac{\alpha}{n-\lambda}} \|f\|_{\mathcal{M}^{1,\phi}(\mu)}^{\alpha/(n-\lambda)}, \qquad x \in \mathbf{R}^d.$$

Proof. Let $f \in \mathcal{M}^{1,\phi}(\mu)$ and $x \in \mathbf{R}^d$. For every $r > 0$, we have

$$\begin{aligned}|I_\alpha f(x)| &\le \int_{|x-y|<r} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, \mathrm{d}\mu(y) + \int_{|x-y|\ge r} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, \mathrm{d}\mu(y) \\ &=: A + B.\end{aligned}$$

Observe that for the first term we obtain

$$\begin{aligned}A &= \int_{|x-y|<r} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, \mathrm{d}\mu(y) \\ &= \sum_{j=-\infty}^{-1} \int_{2^j r<|x-y|\le 2^{j+1}r} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, \mathrm{d}\mu(y) \\ &\le \sum_{j=-\infty}^{-1} \frac{1}{(2^j r)^{n-\alpha}} \int_{|x-y|\le 2^{j+1}r} |f(y)|\, \mathrm{d}\mu(y) \\ &= \sum_{j=-\infty}^{-1} 2^n (2^j r)^\alpha \frac{1}{(2^{j+1}r)^n} \int_{B(x,2^{j+1}r)} |f(y)|\, \mathrm{d}\mu(y) \\ &\le 2^n r^\alpha Mf(x) \sum_{j=-\infty}^{-1} 2^{j\alpha} \\ &\le Cr^\alpha Mf(x).\end{aligned}$$

Meanwhile, for the second term, we have the following estimate:

$$\begin{aligned} B &= \int_{|x-y|\geq r} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, \mathrm{d}\mu(y) \\ &= \sum_{j=0}^{\infty} \int_{2^j r<|x-y|\leq 2^{j+1}r} \frac{|f(y)|}{|x-y|^{n-\alpha}}\, \mathrm{d}\mu(y) \\ &\leq \sum_{j=0}^{\infty} \frac{1}{(2^j r)^{n-\alpha}} \int_{|x-y|\leq 2^{j+1}r} |f(y)|\, \mathrm{d}\mu(y) \\ &= \sum_{j=0}^{\infty} 2^n (2^j r)^{\alpha} \frac{1}{(2^{j+1}r)^n} \int_{B(x,2^{j+1}r)} |f(y)|\, d\mu(y) \\ &\leq C\|f\|_{\mathcal{M}^{1,\phi}(\mu)} \sum_{j=0}^{\infty} (2^j r)^{\alpha} \phi(2^{j+1}r). \end{aligned}$$

As ϕ is almost decreasing, we observe that for $j=0,1,2,\cdots$,

$$(2^j r)^{\alpha}\phi(2^{j+1}r) \leq C \int_{2^j r}^{2^{j+1}r} t^{\alpha-1}\phi(t)\mathrm{d}t.$$

This last inequality and our assumption then lead us to

$$\begin{aligned} B &\leq C\|f\|_{\mathcal{M}^{1,\phi}(\mu)} \sum_{j=0}^{\infty} \int_{2^j r}^{2^{j+1}r} t^{\alpha-1}\phi(t)\mathrm{d}t \\ &\leq C\|f\|_{\mathcal{M}^{1,\phi}(\mu)} \int_r^{\infty} t^{\alpha-1}\phi(t)\mathrm{d}t \\ &\leq Cr^{\lambda+\alpha-n}\|f\|_{\mathcal{M}^{1,\phi}(\mu)}. \end{aligned}$$

Now, by choosing

$$r = \left(\frac{Mf(x)}{\|f\|_{\mathcal{M}^{1,\phi}(\mu)}}\right)^{\frac{1}{\lambda-n}},$$

we obtain

$$\begin{aligned} |I_\alpha f(x)| &\leq Cr^{\alpha}\left(Mf(x) + r^{\lambda-n}\|f\|_{\mathcal{M}^{1,\phi}(\mu)}\right) \\ &\leq C\,[Mf(x)]^{1-\frac{\alpha}{n-\lambda}}\|f\|_{\mathcal{M}^{1,\phi}(\mu)}^{\alpha/(n-\lambda)}. \end{aligned}$$

This completes the proof.

Now, with the use of Theorems 2.1 and 2.2, we obtain the following weak type $(1,q)$ inequality for I_α. Our result is analogous to that of [9] in homogeneous setting.

Theorem 2.3. *If* $\frac{1}{q} = 1 - \frac{\alpha}{n-\lambda}$ *and* ϕ *satisfies the conditions in Theorems* 2.1 *and* 2.2*, then for any function* $f \in \mathcal{M}^{1,\phi}(\mu)$ *and any ball* $B(a,r) \subseteq \mathbf{R}^d$ *we have*

$$\mu\{x \in B(a,r) : |I_\alpha f(x)| > \gamma\} \leq Cr^n\phi(r)\left(\frac{\|f\|_{\mathcal{M}^{1,\phi}(\mu)}}{\gamma}\right)^q, \qquad \gamma > 0.$$

Proof. If $|I_\alpha f(x)| > \gamma$, then Theorem 2.2 gives us

$$Mf(x) > \left(\frac{\gamma}{C\|f\|_{\mathcal{M}^{1,\phi}(\mu)}^{\alpha/(n-\lambda)}}\right)^{\frac{n-\lambda}{n-\lambda-\alpha}} = \left(\frac{\gamma}{C\|f\|_{\mathcal{M}^{1,\phi}(\mu)}^{\alpha/(n-\lambda)}}\right)^{q}.$$

Furthermore, by using Theorem 2.1, we get

$$\begin{aligned}
&\mu\{x \in B(a,r) : |I_\alpha f(x)| > \gamma\} \\
&\quad \le \mu\left\{x \in B(a,r) : Mf(x) > \left(\frac{\gamma}{C\|f\|_{\mathcal{M}^{1,\phi}(\mu)}^{\alpha/(n-\lambda)}}\right)^{q}\right\} \\
&\quad \le Cr^n\phi(r)\|f\|_{\mathcal{M}^{1,\phi}(\mu)}\left(\frac{\|f\|_{\mathcal{M}^{1,\phi}(\mu)}^{\alpha/(n-\lambda)}}{\gamma}\right)^{q} \\
&\quad = Cr^n\phi(r)\left(\frac{\|f\|_{\mathcal{M}^{1,\phi}(\mu)}^{1/q+\alpha/(n-\lambda)}}{\gamma}\right)^{q} \\
&\quad = Cr^n\phi(r)\left(\frac{\|f\|_{\mathcal{M}^{1,\phi}(\mu)}}{\gamma}\right)^{q},
\end{aligned}$$

which is the desired inequality.

Remark. Note that when $\phi(t) = t^{\lambda-n}$ with $0 \le \lambda < n-\alpha$, we get $\mathcal{M}^{1,\phi}(\mu) = L^{1,\lambda}(\mu)$, the usual Morrey spaces of non-homogeneous type. In this case, the above inequality reduces to

$$\mu\{x \in B(a,r) : |I_\alpha f(x)| > \gamma\} \le Cr^\lambda\left(\frac{\|f\|_{L^{1,\lambda}(\mu)}}{\gamma}\right)^{q}, \qquad \gamma > 0.$$

Furthermore, if $\lambda = 0$, then $L^{1,0}(\mu) = L^1(\mu)$ and for $\frac{1}{q} = 1 - \dfrac{\alpha}{n}$ we obtain

$$\mu\{x \in B(a,r) : |I_\alpha f(x)| > \gamma\} \le C\left(\frac{\|f\|_{L^1(\mu)}}{\gamma}\right)^{q}, \qquad \gamma > 0.$$

Since the inequality holds for any ball $B(a,r)$, we deduce that

$$\mu\{x \in \mathbf{R}^d : |I_\alpha f(x)| > \gamma\} \le C\left(\frac{\|f\|_{L^1(\mu)}}{\gamma}\right)^{q}, \qquad \gamma > 0.$$

as in Theorem 1.1.

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Faculty of Sciences and Engineering
Jenderal Soedirman University
Purwokerto, 53122

Indonesia

I. Sihwaningrum

E-mail: idha.sihwaningrum@unsoed.ac.id

S. Maryani

E-mail: sri.maryani@unsoed.ac.id

**Corresponding author:*
H. Gunawan
Faculty of Mathematics and Natural Sciences
Bandung Institute of Technology
Bandung 40132
Indonesia

E-mail: hgunawan@math.itb.ac.id