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TWO EQUIVALENT n-NORMS ON THE SPACE OF p-SUMMABLE SEQUENCES

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Abstract

We prove the (strong) equivalence between two known n-norms on the space ℓ^p of p-summable sequences (of real numbers). The first n-norm is derived from Gähler's formula [3], while the second is due to Gunawan [7]. The equivalence is proved by using the properties of the volume of n-dimensional parallelepipeds in ℓ^p.

1. Introduction

In the 1960's, S. Gähler [2], [3], [4], [5] developed the theory of n-normed spaces. An *n-norm* on a real vector space X (of dimension at least n) is a mapping $\|\cdot,\dots,\cdot\|\colon X^n \to \mathbb{R}$ which satisfies the following four conditions:

(1.1) $\|x_1,\dots,x_n\| = 0$ if and only if $x_1,\dots,x_n$ are linearly dependent;
(1.2) $\|x_1,\dots,x_n\|$ is invariant under permutation;
(1.3) $\|\alpha x_1,\dots,x_n\| = |\alpha|\,\|x_1,\dots,x_n\|$ for $\alpha \in \mathbb{R}$;
(1.4) $\|x_1 + x_1', x_2,\dots,x_n\| \le \|x_1,x_2,\dots,x_n\| + \|x_1',x_2,\dots,x_n\|$.

The pair $(X, \|\cdot,\dots,\cdot\|)$ is called an *n-normed space*. Note that in this space, we have $\|x_1 + y, x_2,\dots,x_n\| = \|x_1,x_2,\dots,x_n\|$ for any $y = c_2x_2 + \cdots + c_nx_n$. See [1], [8], [9], [10], [13], [17] for various results on n-normed spaces. See also [14] for a related topic.

If X is a normed space, then, according to Gähler, the following formula defines an n-norm on X:

$$\|x_1,\dots,x_n\|' := \sup_{\substack{f_i \in X',\ \|f_i\| \le 1 \\ i=1,\dots,n}} \begin{vmatrix} f_1(x_1) & \cdots & f_1(x_n) \\ \vdots & \ddots & \vdots \\ f_n(x_1) & \cdots & f_n(x_n) \end{vmatrix}.$$

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Here X' denotes the dual of X, which consists of bounded linear functionals on X.

For $X = \ell^p$ $(1 \le p < \infty)$, the space of p-summable sequences (of real numbers), the above formula reduces to

$$\|x_1, \dots, x_n\|_p' := \sup_{\substack{z_i \in \ell^{p'},\ \|z_i\|_{p'} \le 1 \\ i=1,\dots,n}} \begin{vmatrix} x_{1j}z_{1j} & \cdots & \sum x_{1j}z_{nj} \\ \vdots & \ddots & \vdots \\ \sum x_{nj}z_{1j} & \cdots & \sum x_{nj}z_{nj} \end{vmatrix},$$

where $\|\cdot\|_{p'}$ denotes the usual norm on $X' = \ell^{p'}$ and each of the sums is taken over $j \in \mathbb{N}$. Here p' denotes the dual exponent of p, so that $\frac{1}{p} + \frac{1}{p'} = 1$.

In 2001, Gunawan [7] defined a different n-norm on ℓ^p $(1 \le p < \infty)$ by

$$\|x_1, \dots, x_n\|_p := \left[\frac{1}{n!} \sum_{j_1} \cdots \sum_{j_n} \text{abs} \begin{vmatrix} {}_{1j_1} & \cdots & x_{nj_1} \\ \vdots & \ddots & \vdots \\ x_{1j_n} & \cdots & x_{nj_n} \end{vmatrix}^p \right]^{1/p},$$

where $x_i = (x_{ij}),\ i = 1, \dots, n$. For $p = 2$, this formula reduces to

$$\|x_1, \dots, x_n\|_2 = \begin{vmatrix} x_1, x_1\rangle & \cdots & \langle x_1, x_n\rangle \\ \vdots & \ddots & \vdots \\ \langle x_n, x_1\rangle & \cdots & \langle x_n, x_n\rangle \end{vmatrix}^{1/2},$$

where $\langle x_i, x_j\rangle$ denotes the usual inner product on ℓ^2. Here $\|x_1, \dots, x_n\|_2$ represents the volume of the n-dimensional parallelepiped spanned by $x_1, \dots, x_n$ in ℓ^2.

Thus, on ℓ^p, we have two definitions of n-norms, one is derived from Gähler's formula and the other is due to Gunawan. For $p = 2$, one may verify that the two n-norms are identical (see [6]). The aim of this paper is to prove the (strong) equivalence between the two n-norms for $1 \le p < \infty$. We do so by invoking the volume formula of n-dimensional parallelepipeds in ℓ^p, which involves the notion of semi-inner products [11]. This result solves the problem posed in [15]. See also [12] for related results.

2. Main results

On a normed space $(X, \|\cdot\|)$, we may define the functional $g\colon X^2 \to R$ by

$$g(x,y) := \frac{\|x\|}{2}(\lambda_+(x,y) + \lambda_-(x,y)), \qquad \text{where} \qquad \lambda_\pm(x,y) := \lim_{t \to \pm 0} \frac{\|x+ty\| - \|x\|}{t}.$$

The functional g satisfies the following properties:

(2.1) $g(x,x) = \|x\|^2$ for every $x \in X$;

(2.2) $g(\alpha x, \beta y) = \alpha\beta g(x, y)$ for every $x, y \in X$, $\alpha, \beta \in R$;
(2.3) $g(x, x + y) = \|x\|^2 + g(x, y)$ for every $x, y \in X$;
(2.4) $|g(x, y)| \leq \|x\| \|y\|$ for every $x, y \in X$.

If, in addition, the functional $g(x, y)$ is linear with respect to $y \in X$, then g is called a *semi-inner product* on X [14].

For example, for $1 \leq p < \infty$, the functional

$$g(x, y) := \|x\|_p^{2-p} \sum_j |x_j|^{p-1} \operatorname{sgn}(x_j)\, y_j, \qquad x = (x_j), y = (y_j) \in \ell^p, \tag{1}$$

defines a semi-inner product on ℓ^p, where $\|\cdot\|_p$ is the usual norm on ℓ^p.

By using the semi-inner product g, we can define an orthogonality relation on X by

$$x \perp_g y \Leftrightarrow g(x, y) = 0.$$

In general, $x \perp_g y$ does not imply $y \perp_g x$, since g is not always commutative.

Next, we can define the *g-orthogonal projection* of y on x by

$$y_x := \frac{g(x, y)}{\|x\|^2} x,$$

and obtain the *g-orthogonal complement* $y - y_x$. Note here that $x \perp_g y - y_x$. Moreover, the g-orthogonal projection of y on $S = \operatorname{span}\{x_1, \ldots, x_k\}$ with $\Gamma(x_1, \ldots, x_k) := \det[g(x_i, x_j)] \neq 0$ is given by

$$y_S := -\frac{1}{\Gamma(x_1, \ldots, x_k)} \begin{vmatrix} 0 & x_1 & \cdots & x_k \\ g(x_1, y) & g(x_1, x_1) & \cdots & g(x_1, x_k) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_k, y) & g(x_k, x_1) & \cdots & g(x_k, x_k) \end{vmatrix},$$

and the g-orthogonal complement $y - y_S$ is given by

$$y - y_S := \frac{1}{\Gamma(x_1, \ldots, x_k)} \begin{vmatrix} y & x_1 & \cdots & x_k \\ g(x_1, y) & g(x_1, x_1) & \cdots & g(x_1, x_k) \\ \vdots & \vdots & \ddots & \vdots \\ g(x_k, y) & g(x_k, x_1) & \cdots & g(x_k, x_k) \end{vmatrix}.$$

Observe here that $x_i \perp_g y - y_S$ for $i = 1, \ldots, k$.

Now, let $\{x_1, \ldots, x_n\}$ be a linearly independent set of vectors in X. Then, we can obtain the *left g-orthogonal sequence* $x_1^\circ, \ldots, x_n^\circ$ through the following procedure: $x_1^\circ = x_1$ and

$$x_i^\circ = x_i - (x_i)_{S_{i-1}},$$

where $S_{i-1} = \operatorname{span}\{x_1^\circ, \ldots, x_{i-1}^\circ\}$ for $i = 2, \ldots, n$. Note here that $x_i \perp_g x_j$ whenever $i < j$.

Next, as in [11], we can define the 'volume' of the n-dimensional parallelepipeds spanned by $x_1, \ldots, x_n$ by

$$V\left(x_1, \ldots, x_n\right) := \prod_{i=1}^{n} \left\|x_i^\circ\right\|. \tag{2}$$

Since g may not be commutative, the value of $V\left(x_1, \ldots, x_n\right)$ may not be invariant under permutation of $(x_1, \ldots, x_n)$. If $x_1, \ldots, x_n$ are linearly dependent, then we shall define $V(x_1, \ldots, x_n) = 0$.

The following theorem gives an estimate for the volume of an n-dimensional parallelepiped in ℓ^p in terms of Gunawan's n-norm.

THEOREM 2.1. *Let* $\{x_1, \ldots, x_n\}$ *be any set in* ℓ^p. *Then we have*

$$(n!)^{1/p-1}\|x_1, \ldots, x_n\|_p \le V(x_{i_1}, \ldots, x_{i_n}) \le (n!)^{1/p}\|x_1, \ldots, x_n\|_p$$

for any permutation $(i_1, \ldots, i_n)$ *of* $(1, \ldots, n)$.

PROOF. The upper estimate is already proved in [11]. Now, to prove the lower estimate, it suffices to show

$$(n!)^{1/p-1}\|x_1, \ldots, x_n\|_p \le V(x_1, \ldots, x_n)$$

because $\|x_1, \ldots, x_n\|_p$ is invariant under permutations of $(x_1, \ldots, x_n)$. Assuming that $x_1, \ldots, x_n$ are linearly independent, let $x_1^\circ, \ldots, x_n^\circ$ be the left g-orthogonal sequence obtained from $x_1, \ldots, x_n$. Then, by basic properties of an n-norm, we have

$$\|x_1, \ldots, x_n\|_p = \|x_1^\circ, \ldots, x_n^\circ\|_p.$$

Next (see [7, Fact 3.1]), we have

$$\|x_1^\circ, \ldots, x_n^\circ\|_p \le (n!)^{1/p'} \prod_{i=1}^{n} \|x_i^\circ\|_p$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. It thus follows that

$$(n!)^{1/p-1}\|x_1, \ldots, x_n\|_p \le \prod_{i=1}^{n} \|x_i^\circ\|_p = V(x_1, \ldots, x_n),$$

as desired.

□

The following theorem gives an estimate for the volume in terms of Gähler's n-norm.

THEOREM 2.2. *Let $\{x_1, \ldots, x_n\}$ be any set in ℓ^p. Then, we have*

$$(n!)^{-1}\|x_1, \ldots, x_n\|'_p \le V(x_{i_1}, \ldots, x_{i_n}) \le \|x_1, \ldots, x_n\|'_p$$

for any permutation $(i_1, \ldots, i_n)$ of $(1, \ldots, n)$.

PROOF. As before, it suffices to show

$$(n!)^{-1}\|x_1, \ldots, x_n\|'_p \le V(x_1, \ldots, x_n) \le \|x_1, \ldots, x_n\|'_p$$

for any linearly independent set $\{x_1, \ldots, x_n\}$ in ℓ^p. The lower estimate follows from the inequality

$$\|x_1, \ldots, x_n\|'_p \le (n!)^{1/p}\|x_1, \ldots, x_n\|_p$$

(see [15, Fact 2.5]) and the lower estimate is in Theorem 2.1.

Next, to prove the upper estimate, let $x_1^\circ, \ldots, x_n^\circ$ be the left g-orthogonal sequence obtained from $x_1, \ldots, x_n$. Recall that

$$\|x_1^\circ, \ldots, x_n^\circ\|'_p = \sup_{i=1,\ldots,n} z_i \in \ell^{p'}, \|z_i\|_{p'} \le 1 \begin{vmatrix} \sum x_{1j}^\circ z_{1j} & \cdots & \sum x_{1j}^\circ z_{nj} \\ \vdots & \ddots & \vdots \\ \sum x_{nj}^\circ z_{1j} & \cdots & \sum x_{nj}^\circ z_{nj} \end{vmatrix}.$$

For each $i = 1, \ldots, n$, take $z_i = (z_{ij})$ where $z_{ij} := \|x_i^\circ\|_p^{1-p}|x_{ij}^\circ|^{p-1}\mathrm{sgn}(x_{ij}^\circ)$. We observe that $\|z_i\|_{p'} = 1$, and hence

$$\|x_1^\circ, \ldots, x_n^\circ\|'_p \ge \begin{vmatrix} \|x_1^\circ\|_p^{-1}\, g(x_1^\circ, x_1^\circ) & \cdots & \|x_n^\circ\|_p^{-1}\, g(x_n^\circ, x_1^\circ) \\ \vdots & \ddots & \vdots \\ \|x_1^\circ\|_p^{-1}\, g(x_1^\circ, x_n^\circ) & \cdots & \|x_n^\circ\|_p^{-1}\, g(x_n^\circ, x_n^\circ) \end{vmatrix},$$

where g is the functional defined by the formula (1). Here $g(x_i^\circ, x_j^\circ) = 0$ for $i < j$ and $g(x_i^\circ, x_i^\circ) = \|x_i^\circ\|_p^2$ for $i = 1, \ldots, n$, and so the determinant on the right-hand side is equal to $\prod_{i=1}^n \|x_i^\circ\|_p = V(x_1, \ldots, x_n)$. Meanwhile, the left-hand side is equal to $\|x_1, \ldots, x_n\|'_p$. Therefore, we obtain $\|x_1, \ldots, x_n\|'_p \ge V(x_1, \ldots, x_n)$, which is what we wanted to prove.

□

As a consequence of Theorems 2.1 and 2.2, we get the following result, which tells us that Gunawan's and Gähler's n-norms on ℓ^p are (strongly) equivalent.

THEOREM 2.3. *For any $x_1, \ldots, x_n \in \ell^p$, we have*

$$(n!)^{1/p-1}\|x_1, \ldots, x_n\|_p \le \|x_1, \ldots, x_n\|'_p \le (n!)^{1/p}\|x_1, \ldots, x_n\|_p.$$

NOTE. The upper estimate follows from [15, Fact 2.5].

3. Concluding remarks

We have just seen that Gähler's and Gunawan's n-norms on ℓ^p are (strongly) equivalent. While Gähler's formula uses the functionals on ℓ^p as a normed space, Gunawan's uses the Plücker coordinates of the vectors (thanks to Norman Wildberger who brought these coordinates into our attention). In this case, Gunawan's formula allows us to compute the value of the n-norm $\|x_1, \dots, x_n\|_p$ directly from the coordinates of the vectors $x_1, \dots, x_n$.

If one insists on involving functionals on ℓ^p, one may actually use the fact that the dual space $\ell^{p'}$ is also an n-normed space, as well as ℓ^p $(1 \le p < \infty)$. Thus, one may define the following n-norm on ℓ^p:

$$\|x_1, \dots, x_n\|_p^* := \sup_{z_i \in \ell^{p'},\ \|z_1,\dots,z_n\|_{p'} \le 1} \begin{vmatrix} x_{1j} z_{1j} & \cdots & \sum x_{1j} z_{nj} \\ \vdots & \ddots & \vdots \\ \sum x_{nj} z_{1j} & \cdots & \sum x_{nj} z_{nj} \end{vmatrix},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. Here $\|\cdot, \dots, \cdot\|_{p'}$ is Gunawan's n-norm on $\ell^{p'}$. For $p = 2$, we observe that

$$\|x_1, \dots, x_n\|_2^* = \|x_1, \dots, x_n\|_2,$$

so that we have the duality

$$\|x_1, \dots, x_n\|_2 := \sup_{z_i \in \ell^2,\ \|z_1,\dots,z_n\|_2 \le 1} \begin{vmatrix} x_{1j} z_{1j} & \cdots & \sum x_{1j} z_{nj} \\ \vdots & \ddots & \vdots \\ \sum x_{nj} z_{1j} & \cdots & \sum x_{nj} z_{nj} \end{vmatrix}.$$

For other values of p, one may show that the above n-norm is at least equivalent to Gunawan's (and hence it is also equivalent to Gähler's). See [6] for related results.

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References

[1] X. Y. Chen and M. M. Song, Characterizations on isometries in linear n-normed spaces, *Nonlinear Anal.*, **72** (2010), 1895–1901.
[2] S. Gähler, Lineare 2-normierte räume, *Math. Nachr.*, **28** (1964), 1–43.
[3] S. Gähler, Untersuchungen über verallgemeinerte m-metrische Räume. I, *Math. Nachr.*, **40** (1969), 165–189.
[4] S. Gähler, Untersuchungen über verallgemeinerte m-metrische Räume. II, *Math. Nachr.*, **40** (1969), 229–264.
[5] S. Gähler, Untersuchungen über verallgemeinerte m-metrische Räume. III, *Math. Nachr.*, **41** (1970), 23–26.

[6] S. M. Gozali, H. Gunawan and O. Neswan, On n-norms and bounded n-linear functionals in a Hilbert space, *Ann. Funct. Anal.*, **1** (2010), 72–79.

[7] H. Gunawan, The space of p-summable sequences and its natural n-norms, *Bull. Austral. Math. Soc.*, **64** (2001), 137–147.

[8] H. Gunawan, On n-inner products, n-norms, and the Cauchy–Schwarz inequality, *Sci. Math. Japan*, **55** (2002), 53–60.

[9] H. Gunawan, A generalization of Bessel's inequality and Parseval's identity, *Period. Math. Hungar.*, **44** (2002), 177–181.

[10] H. Gunawan and Mashadi, On n-normed spaces, *Int. J. Math. Math. Sci.*, **27** (2001), 631–639.

[11] H. Gunawan, W. Setya-Budhi, Mashadi and S. Gemawati, On volume of n-dimensional parallelepipeds in ℓ^p spaces, *Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat.*, **16** (2005), 48–54.

[12] T. R. Kristianto, R. A. Wibawa-Kusumah and H. Gunawan, Equivalence relations of n-norms on a vector spaces, *Mat. Vesnik*, to appear.

[13] R. Malčeski, The Ascoli–Mazur theorem for n-normed spaces, *Mat. Bilten*, **28** (2004), 35–44 (in Macedonian).

[14] P. M. Miličić, On the Gram–Schmidt projection in normed spaces, *Univ. Beograd Publ. Elektrotehn. Fak. Ser. Mat.*, **4** (1993), 89–96.

[15] A. Mutaqin and H. Gunawan, Equivalence of n-norms on the space of p-summable sequences, *J. Indones. Math. Soc.*, **16** (2010), 39–49.

[16] A. Misiak, n-inner product spaces, *Math. Nachr.*, **140** (1989), 299–319.

[17] C.-G. Park and T. M. Rassias, Isometries on linear n-normed spaces, *JIPAM. J. Inequal. Pure Appl. Math.*, **7** (2006), Article 168, 7 pp.