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p-summable sequence spaces with 2-inner products

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Abstract. We revisit the space ℓ^p of p-summable sequences of real numbers. In particular, we show that this space is actually contained in a (weighted) 2-inner product space. For $p > 2$, we also obtain a result which describes how the weighted 2-inner product space is associated to the weights.

Key words. 2-inner product spaces, 2-normed spaces, p-summable sequences, weights.

1 Introduction

The inner product spaces have been, up to now, the most useful spaces in practical applications of functional analysis. These spaces were initially introduced by D. Hilbert [6] in 1912. By $\ell^p = \ell^p(\mathbb{R})$ we denote the space of all p-summable sequences of real numbers. We know that for $p \neq 2$, the space ℓ^p is not an inner product space, since the usual norm $\|x\|_p = (\sum_{k=1}^{\infty} |x_k|^p)^{\frac{1}{p}}$ on ℓ^p does not satisfy the parallelogram law. So, it can not be derived from an inner product for $p \neq 2$. There is a semi-inner product on ℓ^p as in [10], but having a semi-inner product is not as nice as having an inner product. Konca et al. [8] defined a weighted inner product on ℓ^p for $p \neq 2$, and obtained a larger space.

We also know that the space ℓ^p which is equipped with usual 2-norm $\|\cdot,\cdot\|_p$ defined by Gunawan [4] is not a 2-inner product space for $p \neq 2$, because the 2-norm $\|\cdot,\cdot\|_p$ does not satisfy the parallelogram law. One question arises: can we define a 2-norm on ℓ^p which satisfies the parallelogram law? The reason why we are interested in the parallelogram law is because we eventually wish to define a 2-inner product, possibly with weights, on ℓ^p, so that we can define orthogonality and many other notions on this space.

In this paper, we shall discuss a weighted 2-norm $\|\cdot,\cdot\|_{2,v}$, which is not equivalent to the usual 2-norm $\|\cdot,\cdot\|_p$ on ℓ^p, but satisfies the parallelogram law. We discuss the properties of the weighted 2-norm $\|\cdot,\cdot\|_{2,v}$ and its relationship with the usual 2-norm $\|\cdot,\cdot\|_p$ on ℓ^p. We also find that the associated 2-inner product is actually defined on a larger space. How this larger space is related to the original one will be discussed in this paper. In addition, for $p > 2$, we describe how the weighted 2-inner product space is associated to the weights.

2 Results for ℓ^p as 2-normed spaces

We first recall the notion of 2-inner product spaces and 2-normed spaces, which have been introduced in [2] and [3]. Let X be a real vector space of dimension $d \geq 2$. The real-valued function $\langle\cdot,\cdot|\cdot\rangle$ which satisfies the following properties on X^3 is called a 2-*inner product* on X, and the pair $(X, \langle\cdot,\cdot|\cdot\rangle)$ is called a 2-*inner product space*:

(I1) $\langle x, x|z\rangle \geq 0$; $\langle x, x|z\rangle = 0$ if and only if x and z are linearly dependent,

(I2) $\langle x, y|z\rangle = \langle y, x|z\rangle$,

(I3) $\langle x, x|z\rangle = \langle z, z|x\rangle$,

(I4) $\langle \alpha x, y|z\rangle = \alpha \langle x, y|z\rangle$, for $\alpha \in \mathbb{R}$,

(I5) $\langle x_1 + x_2, y|z\rangle = \langle x_1, y|z\rangle + \langle x_2, y|z\rangle$.

The function $\|\cdot,\cdot\|$, which satisfies the following four properties, is called a 2-*norm* and the pair $(X, \|\cdot,\cdot\|)$ is called a 2-*normed space*:

(N1) $\|x, z\| \geq 0$, for $x, z \in X$, $\|x, z\| = 0$ if and only if x and z are linearly dependent,

(N2) $\|x, z\| = \|z, x\|$, for $x, z \in X$,

(N3) $\|\alpha x, z\| = |\alpha|\,\|x, z\|$, for $x, y \in X$ and $\alpha \in \mathbb{R}$,

(N4) $\|x + y, z\| \leq \|x, z\| + \|y, z\|$, for $x, y, z \in X$.

A sequence $x = (x_j)$ in a 2-normed space $(X, \|\cdot,\cdot\|)$ is *convergent* if there is an ξ in X such that $\lim\limits_{j\to\infty} \|x_j - \xi, y\| = 0$ and $\lim\limits_{j\to\infty} \|x_j - \xi, z\| = 0$ for some linearly independent vectors $y, z \in X$. Moreover, a sequence $x = (x_j)$ in $(X, \|\cdot,\cdot\|)$ is called a *Cauchy sequence* if $\lim_{i,\, j\to\infty} \|x_i - x_j, y\| = 0$ and $\lim_{i,\, j\to\infty} \|x_i - x_j, z\| = 0$ for some linearly independent vectors $y, z \in X$. A 2-normed space is *complete* if every Cauchy sequence there is convergent. A complete 2-normed space is called a 2-*Banach space*.

For simplicity, we use the sum notation $\sum\limits_k$ instead of $\sum\limits_{k=1}^{\infty}$. From [4], we know that ℓ^p $(1 \le p \le \infty)$ can be equipped with the following 2-norm $\|\cdot,\cdot\|_p$:

$$\|x_1, x_2\|_p := \left(\frac{1}{2} \sum_{k_1} \sum_{k_2} \left\| \begin{matrix} x_{1k_1} & x_{1k_2} \\ x_{2k_1} & x_{2k_2} \end{matrix} \right\|^p \right)^{\frac{1}{p}}, \tag{2.1}$$

where $x_1 = (x_{1k})$ and $x_2 = (x_{2k})$. For $p = \infty$, the formula reduces to

$$\|x_1, x_2\|_\infty := \sup_{k_1} \sup_{k_2} \left\| \begin{matrix} x_{1k_1} & x_{1k_2} \\ x_{2k_1} & x_{2k_2} \end{matrix} \right\|.$$

The 2-norm given by (2.1) does not satisfy the parallelogram law for $p \neq 2$. To show this, we can take for example $x_1 = (1, 0, 0, \dots)$, $y_1 = (0, 1, 0, \dots)$, $x_2 = (0, \dots, 0, 1, 0, \dots)$ (1 is in the k_0^{th} term) in ℓ^p, then we have $\|x_1, x_2\|_p = 1$, $\|y_1, x_2\|_p = 1$, $\|x_1 + y_1, x_2\|_p = \|x_1 - y_1, x_2\|_p = 2^{\frac{1}{p}}$. Hence,

$$2\|x_1, x_2\|_p^2 + 2\|y_1, x_2\|_p^2 \neq \|x_1 - y_1, x_2\|_p^2 + \|x_1 + y_1, x_2\|_p^2.$$

We can define a different 2-norm on ℓ^p, for example

$$\|x_1, x_2\|_* := \left(\frac{1}{2} \sum_{k_1} \sum_{k_2} \frac{1}{k_1^p k_2^p} \left\| \begin{matrix} x_{1k_1} & x_{1k_2} \\ x_{2k_1} & x_{2k_2} \end{matrix} \right\|^p \right)^{\frac{1}{p}}, \ 1 \le p < \infty, \ \frac{1}{p} + \frac{1}{q} = 1$$

which is not equivalent to $\|\cdot,\cdot\|_p$. This 2-norm, however, does not satisfy the parallelogram law. Take $x = (1, 0, 0, \dots)$, $y = (0, 1, 0, \dots)$, $z = (0, 0, 1, 0, \dots)$ and compute $\|x, z\|_* = \frac{1}{3}$, $\|y, z\|_* = \frac{1}{6}$ and $\|x + y, z\|_* = \|x - y, z\|_* = (\frac{1}{3^p} + \frac{1}{6^p})^{\frac{1}{p}}$. Then

$$2\|x, z\|_*^2 + 2\|y, z\|_*^2 \neq \|x + y, z\|_*^2 + \|x - y, z\|_*^2.$$

Our question is if we can define a 2-norm on ℓ^p which satisfies the parallelogram law, or not. We will give the answer of this question in the following subsections.

2.1 Results for $1 \le p \le 2$

In this subsection, we let $1 \le p \le 2$, unless otherwise stated. First, we recall that $\ell^p \subseteq \ell^2$ (as sets). With respect to the 2-norms on these spaces, we have the following proposition.

Proposition 2.1 *If $x, z \in \ell^p$ with $\|x, z\|_p < \infty$, then $x, z \in \ell^2$ with $\|x, z\|_2 < \infty$.*

Proof. Let $x, z \in \ell^p$ with $\|x, z\|_p < \infty$. Since $\ell^p \subseteq \ell^2$, we have $x, z \in \ell^2$. Moreover,

$$\begin{aligned}
\|x, z\|_2^2 &= \frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^2 = \frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^{2-p}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^p \\
&= \sum_{k_1>k_2}\sum_{k_2}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^{2-p}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^p \\
&\le \sup_{k_1>k_2}\sup_{k_2}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^{2-p}\sum_{k_1>k_2}\sum_{k_2}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^p \\
&\le \left(\sum_{k_1>k_2}\sum_{k_2}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^p\right)^{\frac{2-p}{p}}\sum_{k_1>k_2}\sum_{k_2}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^p \\
&= \left(\sum_{k_1>k_2}\sum_{k_2}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^p\right)^{\frac{2}{p}} = \left(\frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^p\right)^{\frac{2}{p}}.
\end{aligned}$$

Taking the square roots of both sides, we get

$$\left(\frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^2\right)^{\frac{1}{2}} \le \left(\frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^p\right)^{\frac{1}{p}}.$$

Therefore $x, z \in \ell^2$ with $\|x, z\|_2 < \infty$. ■

Following Proposition 2.1, we realize that ℓ^p-sequences are in $\left(\ell^2, \|\cdot,\cdot\|_2\right)$. Hence ℓ^p can be equipped with the 2-inner product

$$\langle x, y \,|z\rangle := \frac{1}{2}\sum_{k_1}\sum_{k_2}\left|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right|\left|\begin{matrix} y_{k_1} & y_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right|,$$

and the 2-norm

$$\|x, z\|_2 := \left(\frac{1}{2}\sum_{k_1}\sum_{k_2}\left|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right|^2\right)^{\frac{1}{2}}.$$

Being an induced 2-norm from the 2-inner product, the 2-norm $\|\cdot,\cdot\|_2$ of course satisfies the parallelogram law

$$\|x+y, z\|_2^2 + \|x-y, z\|_2^2 = 2\,\|x, z\|_2^2 + 2\,\|y, z\|_2^2$$

for every $x, y, z \in \ell^p$. (We can check later that as a subspace of $\left(\ell^2, \|\cdot,\cdot\|_2\right)$, ℓ^p is not closed.)

A more general result is formulated in the following proposition which describes the monotonicity property of the 2-norms on ℓ^p spaces.

Proposition 2.2 *Let $1 \leq p \leq q \leq \infty$. If $x, z \in \ell^p$, then $x, z \in \ell^q$ with $\|x, z\|_q \leq \|x, z\|_p$. The converse is not true, because there exist $x, z \in \ell^q$ with $\|x, z\|_q < \infty$ but $\|x, z\|_p = \infty$ for $1 \leq p < q \leq \infty$.*

Proof. Let $1 \leq p \leq q \leq \infty$ and $x, z \in \ell^p$. Since $\ell^p \subseteq \ell^q$, we have $x, z \in \ell^q$. Moreover we have

$$\begin{aligned}
\|x, z\|_q^q &= \frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^q = \frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^{q-p}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^p \\
&= \sum_{k_1>k_2}\sum_{k_2}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^{q-p}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^p \\
&\leq \sup_{k_1>k_2}\sup_{k_2}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^{q-p}\sum_{k_1>k_2}\sum_{k_2}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^p \\
&\leq \left(\sum_{k_1>k_2}\sum_{k_2}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^p\right)^{\frac{q-p}{p}}\sum_{k_1>k_2}\sum_{k_2}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^p \\
&= \left(\frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^p\right)^{\frac{q}{p}}.
\end{aligned}$$

Taking the q-th roots of both sides, we get $\|x, z\|_q \leq \|x, z\|_p$.

To show that the converse is not true, one may take $x = (x_k) = \left(\frac{1}{k^{1/p}}\right)_{k\in\mathbb{N}}$, where $1 \leq p < \infty$ and $z = (1, 0, \dots)$. Then

$$\begin{aligned}
\|x, z\|_p &= \left(\frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^p\right)^{\frac{1}{p}} = \left(\frac{1}{2}\sum_{k_1}\sum_{k_2}\left|\frac{z_{k_2}}{k_1^{\frac{1}{p}}} - \frac{z_{k_1}}{k_2^{\frac{1}{p}}}\right|^p\right)^{\frac{1}{p}} \\
&= \left(\frac{1}{2}\sum_{k\neq 1}\frac{1}{k} + \frac{1}{2}\sum_{k\neq 1}\frac{1}{k}\right)^{\frac{1}{p}} = \left(\sum_{k\neq 1}\frac{1}{k}\right)^{\frac{1}{p}} = \infty.
\end{aligned}$$

But

$$\begin{aligned}
\|x, z\|_q &= \left(\frac{1}{2}\sum_{k_1}\sum_{k_2}\left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2}\end{matrix}\right\|^q\right)^{\frac{1}{q}} = \left(\frac{1}{2}\sum_{k_1}\sum_{k_2}\left|\frac{z_{k_2}}{k_1^{\frac{1}{p}}} - \frac{z_{k_1}}{k_2^{\frac{1}{p}}}\right|^q\right)^{\frac{1}{q}} \\
&= \left(\frac{1}{2}\sum_{k\neq 1}\frac{1}{k^{\frac{q}{p}}} + \frac{1}{2}\sum_{k\neq 1}\frac{1}{k^{\frac{q}{p}}}\right)^{\frac{1}{q}} = \left(\sum_{k\neq 1}\frac{1}{k^{\frac{q}{p}}}\right)^{\frac{1}{q}} < \infty,
\end{aligned}$$

since $\frac{q}{p} > 1$. ■

Now we ask whether $\|\cdot,\cdot\|_2$ is equivalent with the usual 2-norm $\|\cdot,\cdot\|_p$ on ℓ^p. We already have $\|x,z\|_2 \leq \|x,z\|_p$ for every $x, z \in \ell^p$, but the following proposition prevents the two 2-norms to be equivalent.

Proposition 2.3 *Let $1 \leq p < 2$. There is no constant $C > 0$ such that $\|x,z\|_2 \geq C\,\|x,z\|_p$ for every $x, z \in \ell^p$.*

Proof. Let $z := (1, 0, \dots)$. For each $n \in \mathbb{N}$, take $x^{(n)} = (x_k^{(n)}) = \left(\frac{1}{k^{\frac{1}{p}+\frac{1}{n}}}\right)$. Then we have

$$\begin{aligned}\left\|x^{(n)}, z\right\|_2^2 &= \frac{1}{2}\sum_{k_1}\sum_{k_2}\begin{vmatrix} x_{k_1}^{(n)} & x_{k_2} \\ z_{k_1}^{(n)} & z_{k_2}\end{vmatrix}^2 = \sum_{k\neq 1}\left(\frac{1}{k^{\frac{1}{p}+\frac{1}{n}}}\right)^2 = \sum_{k\neq 1}\frac{1}{k^{\frac{2}{p}+\frac{2}{n}}} \\ &\leq \sum_{k\neq 1}\frac{1}{k^{\frac{2}{p}}} < \infty \text{ (independent of } n)\end{aligned}$$

while

$$\left\|x^{(n)}, z\right\|_p^p = \sum_{k\neq 1}\frac{1}{k^{1+\frac{p}{n}}} < \infty \text{ (dependent of } n)$$

which tends to ∞ as $n \to \infty$. Hence

$$\frac{\left\|x^{(n)}, z\right\|_2}{\left\|x^{(n)}, z\right\|_p} \to 0$$

as $n \to \infty$. So, there is no constant $C > 0$ such that $\|x,z\|_2 \geq C\,\|x,z\|_p$ for every $x,\ z \in \ell^p$. ■

Proposition 2.4 *As a set, ℓ^p is not closed but dense in $\left(\ell^2, \|\cdot,\cdot\|_2\right)$.*

Proof. To show that ℓ^p is not closed in $\left(\ell^2, \|\cdot,\cdot\|_2\right)$, for each $n \in \mathbb{N}$, we take $x^{(n)} := \left(1, \frac{1}{2^{1/p}}, \dots, \frac{1}{n^{1/p}}, 0, 0, \dots\right)$, and let $z_1 := (1, 0, \dots)$ and $z_2 := (0, 1, 0, \dots)$. Clearly $\left(x^{(n)}\right)$ converges to $x := \left(\frac{1}{k^{1/p}}\right)$ in $\|\cdot,\cdot\|_2$ with respect to the set $\{z_1, z_2\}$, that is, $\left\|x^{(n)} - x, z_1\right\|_2 \to 0$ and $\left\|x^{(n)} - x, z_2\right\|_2 \to 0$ as $n \to \infty$. But $x^{(n)} \in \ell^p$ for each $n \in \mathbb{N}$, while $x \notin \ell^p$. Therefore ℓ^p is not closed in $\left(\ell^2, \|\cdot,\cdot\|_2\right)$.

To show that ℓ^p is dense in $\left(\ell^2, \|\cdot,\cdot\|_2\right)$, we observe that every $x = (x_k) \in \ell^2$ can be approximated arbitrarily close by $x^{(n)} := (x_1, x_2, \dots, x_n, 0, 0, \dots) \in \ell^p$, where $n \in \mathbb{N}$. Indeed, $\left\|x^{(n)} - x, z_1\right\|_2 \to 0$ and $\left\|x^{(n)} - x, z_2\right\|_2 \to 0$ as $n \to \infty$, where $z_1 := (1, 0, 0, \dots)$ and $z_2 := (0, 1, 0, \dots)$. ■

2.2 Results for $2 < p < \infty$

Throughout this subsection, we let $2 < p < \infty$, unless otherwise stated. As we have seen in previous subsection and Proposition 2.1 in [8], the set ℓ^p is larger than ℓ^2 for $p > 2$, so that the usual 2-inner product and 2-norm on ℓ^2 are not defined for all sequences in ℓ^p. As suggested in the case of ℓ^p as normed spaces, to define a 2-inner product or a new 2-norm on ℓ^p which satisfies the parallelogram law, we shall use weights. Choose $v = (v_k) \in \ell^p$ where $v_k > 0,\ k \in \mathbb{N}$, and define the mapping $\langle \cdot, \cdot | \cdot \rangle_v$ which maps every triple of sequences $x = (x_k)$, $y = (y_k)$ and $z = (z_k)$ to

$$\langle x, y|z\rangle_v := \frac{1}{2}\sum_{k_1}\sum_{k_2} |v_{k_1}v_{k_2}|^{p-2} \begin{vmatrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2} \end{vmatrix} \begin{vmatrix} y_{k_1} & y_{k_2} \\ z_{k_1} & z_{k_2} \end{vmatrix}. \tag{2.2}$$

We also define $\|\cdot,\cdot\|_{2,v}$ which maps every sequence $x = (x_k)$ and $z = (z_k)$ to

$$\|x, z\|_{2,v} := \sqrt{\langle x, x|z\rangle_v} := \left[\frac{1}{2}\sum_{k_1}\sum_{k_2} |v_{k_1}v_{k_2}|^{p-2} \begin{vmatrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2} \end{vmatrix}^2\right]^{\frac{1}{2}}. \tag{2.3}$$

Note that both mappings are well-defined on ℓ^p. Indeed, for $x = (x_k)$, $y = (y_k)$ and $z = (z_k)$ in ℓ^p, it follows from Hölder's inequality that

$$\begin{aligned} &\tfrac{1}{2}\sum_{k_1}\sum_{k_2} |v_{k_1}v_{k_2}|^{p-2} \begin{vmatrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2} \end{vmatrix} \begin{vmatrix} y_{k_1} & y_{k_2} \\ z_{k_1} & z_{k_2} \end{vmatrix} \\ &\le \left[\tfrac{1}{2}\sum_{k_1}\sum_{k_2} |v_{k_1}v_{k_2}|^{p}\right]^{\frac{p-2}{p}} \left[\tfrac{1}{2}\sum_{k_1}\sum_{k_2} \begin{vmatrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2} \end{vmatrix}^{\frac{p}{2}} \begin{vmatrix} y_{k_1} & y_{k_2} \\ z_{k_1} & z_{k_2} \end{vmatrix}^{\frac{p}{2}}\right]^{\frac{2}{p}} \\ &\le \left(\tfrac{1}{2}\right)^{\frac{p-2}{p}} \|v\|_p^{2(p-2)} \left[\tfrac{1}{2}\sum_{k_1}\sum_{k_2} \left\|\begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2} \end{matrix}\right\|^{p}\right]^{\frac{1}{p}} \left[\tfrac{1}{2}\sum_{k_1}\sum_{k_2} \left\|\begin{matrix} y_{k_1} & y_{k_2} \\ z_{k_1} & z_{k_2} \end{matrix}\right\|^{p}\right]^{\frac{1}{p}}. \end{aligned} \tag{2.4}$$

Moreover, we have the following proposition, whose proof is left to the reader.

Proposition 2.5 *The mappings in (2.2) and (2.3) define a weighted 2-inner product and a weighted 2-norm, respectively, on ℓ^p.*

We observe that the equation (2.2) can be rewritten as

$$\begin{aligned} \langle x, y\,|z\,\rangle_v &= \frac{1}{2}\sum_{k_1}\sum_{k_2} |v_{k_1}v_{k_2}|^{p-2} \begin{vmatrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2} \end{vmatrix} \begin{vmatrix} y_{k_1} & y_{k_2} \\ z_{k_1} & z_{k_2} \end{vmatrix} \\ &= \begin{vmatrix} \sum_k |v_k|^{p-2} x_k y_k & \sum_k |v_k|^{p-2} x_k z_k \\ \sum_k |v_k|^{p-2} z_k y_k & \sum_k |v_k|^{p-2} {z_k}^2 \end{vmatrix} = \begin{vmatrix} \langle x, y\rangle_v & \langle x, z\rangle_v \\ \langle z, y\rangle_v & \langle z, z\rangle_v \end{vmatrix}, \end{aligned}$$

where $\langle x, y\rangle_v := \sum_k |v_k|^{p-2} x_k y_k$ (see in [8]). Thus $\|\cdot,\cdot\|_{2,v}$ is a standard 2-norm on ℓ^p, with respect to the weighted inner product $\langle\cdot,\cdot\rangle_v$.

From (2.4), we see that the inequality

$$\|x, z\|_{2,v} \le 2^{\frac{1}{p}-\frac{1}{2}} \|v\|_p^{p-2} \|x, z\|_p \tag{2.5}$$

holds for every $x, z \in \ell^p$. The two 2-norms, however, are not equivalent on ℓ^p, due to the following result.

Proposition 2.6 *There is no constant $C = C_v > 0$ such that*

$$\|x, z\|_p \le C\|x, z\|_{2,v}$$

for every $x, z \in \ell^p$.

Proof. Take $z := (1, 0, \dots)$. Suppose that such a constant exists. Then, for $x := e_n = (0, \dots, 0, 1, 0, \dots)$, where the 1 is the n^{th} term ($n \ge 2$), we have

$$1 \le C|v_1 v_n|^{\frac{p-2}{2}}$$

for each $n \ge 2$ (because $\|x, z\|_p = 1$ and $\|x, z\|_{2,v} = |v_1 v_n|^{\frac{p-2}{2}}$ for $n \ge 2$). But this cannot be true, since $v_n \to 0$ as $n \to \infty$. ∎

According to Proposition 2.6, it is possible for us to find a sequence in ℓ^p which is divergent with respect to the 2-norm $\|\cdot,\cdot\|_p$, but convergent with respect to the 2-norm $\|\cdot,\cdot\|_{2,v}$.

Example 2.1 *Let $x^{(n)} := e_n \in \ell^p$, where $e_n = (0, \dots, 0, 1, 0, \dots)$ (the 1 is the n^{th} term) and let $\{z_1, z_2\}$ be a linearly independent set where $z_1 := (1, 0, 0, \dots)$ and $z_2 := (0, 1, 0, \dots)$. Then $\left\|x^{(m)} - x^{(n)}, z_i\right\|_p = 2^{\frac{1}{p}} 0$ as $m, n \to \infty$ for $i = 1, 2$. Since $(x^{(n)})$ is not a Cauchy sequence with respect to $\|\cdot,\cdot\|_p$, it is not convergent with respect to $\|\cdot,\cdot\|_p$. However, $\left\|x^{(n)}, z_i\right\|_{2,v} = |v_i v_n|^{\frac{p-2}{2}} \to 0$ as $n \to \infty$, since $v_n \to 0$ as $n \to \infty$ for $i = 1, 2$. Hence, $(x^{(n)})$ is convergent with respect to the 2-norm $\|\cdot,\cdot\|_{2,v}$.*

If we wish, we can also define another weighted 2-norm $\|\cdot,\cdot\|_{\beta,v}$ on ℓ^p, where $1 \le \beta \le p < \infty$, by

$$\|x, z\|_{\beta,v} := \left[\frac{1}{2}\sum_{k_1}\sum_{k_2} |v_{k_1} v_{k_2}|^{p-\beta} \left\| \begin{array}{cc} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2} \end{array} \right\|^{\beta}\right]^{\frac{1}{\beta}}.$$

Here p may be less than 2. Note that if $\beta = p$, then $\|\cdot,\cdot\|_{\beta,v} = \|\cdot,\cdot\|_p$.

The following proposition gives a relationship between two such weighted 2-norms on ℓ^p.

Proposition 2.7 *Let $1 \le \beta < \gamma \le p$. Then we have*

$$\|x,z\|_{\beta,v} \le 2^{\frac{1}{\gamma}-\frac{1}{\beta}} \|v\|_p^{\frac{2p(\gamma-\beta)}{\gamma\beta}} \|x,z\|_{\gamma,v}$$

for every $x, z \in \ell^p$.

Proof. Suppose that $x, z \in \ell^p$. We compute

$$\begin{aligned}
\|x,z\|_{\beta,v} &= \left[\frac{1}{2}\sum_{k_1}\sum_{k_2} |v_{k_1}v_{k_2}|^{p-\beta} \left\| \begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2} \end{matrix} \right\|^{\beta}\right]^{\frac{1}{\beta}} \\
&= \left[\frac{1}{2}\sum_{k_1}\sum_{k_2} |v_{k_1}v_{k_2}|^{\frac{p\,(\gamma-\beta)}{\gamma}} |v_{k_1}v_{k_2}|^{\frac{(p-\gamma)\,\beta}{\gamma}} \left\| \begin{matrix} x_{k_1} & z_{k_1} \\ x_{k_2} & z_{k_2} \end{matrix} \right\|^{\beta}\right]^{\frac{1}{\beta}} \\
&\le \left(\frac{1}{2}\right)^{\frac{1}{\beta}-\frac{1}{\gamma}} \left\{ \left(\sum_{k_1}\sum_{k_2} |v_{k_1}v_{k_2}|^{p}\right)^{\frac{\gamma-\beta}{\gamma}} \left[\sum_{k_1}\sum_{k_2} |v_{k_1}v_{k_2}|^{p-\gamma} \left\| \begin{matrix} x_{k_1} & z_{k_1} \\ x_{k_2} & z_{k_2} \end{matrix} \right\|^{\gamma}\right]^{\frac{\beta}{\gamma}} \right\}^{\frac{1}{\beta}} \\
&= 2^{\frac{1}{\gamma}-\frac{1}{\beta}} \|v\|_p^{\frac{2p(\gamma-\beta)}{\gamma\beta}} \|x,z\|_{\gamma,v}
\end{aligned}$$

as desired. ∎

Corollary 2.8 *If $1 \le \beta < 2 < \gamma \le p$, then there are constants $C_{1,v},\ C_{2,v} > 0$ such that*

$$C_{1,v}\|x,z\|_{\beta,v} \le \|x,z\|_{2,v} \le C_{2,v}\|x,z\|_{\gamma,v}$$

for every $x,\ z \in \ell^p$.

The following proposition is an analog of Proposition 4.1 in [8].

Proposition 2.9 *If $x, z \in \ell^p$ with $\|x,z\|_p < \infty$, then $x, z \in \ell^2_v$ with $\|x,z\|_{2,v} < \infty$. The converse is not true, because there exist x, z such that $\|x,z\|_{2,v} < \infty$ but $\|x,z\|_p = \infty$.*

Proof. Let $x, z \in \ell^p$ with $\|x,z\|_p < \infty$. It follows from (2.5) that $\|x,z\|_{2,v} \le 2^{\frac{1}{p}-\frac{1}{2}} \|v\|_p^{p-2} \|x,z\|_p$, which means that $x, z \in \ell^2_v$ with $\|x,z\|_{2,v} < \infty$.

It remains to show that the converse is not true. Using the same idea as in the proof of Proposition 4.1 in [8], we choose $m_1 \in \mathbb{N}$ such that $v_{m_1}^{p-2} < \frac{1}{2}$, $m_2 > m_1$ such that

$v_{m_2}^{p-2} < \frac{1}{2^2}$, $m_3 > m_2$ such that $v_{m_3}^{p-2} < \frac{1}{2^3}$, and so on. This process never stops, and so we obtain an increasing sequence of nonnegative integers (m_j) such that $v_{m_j}^{p-2} < 2^{-j}$ for every $j \in \mathbb{N}$. As before, we put $x_k = 1$ for $k = m_1,\ m_2,\ m_3, \dots$ and $x_k = 0$ otherwise, and take $z := (1, 0, \dots)$. Hence we have

$$\begin{aligned} \|x,z\|_{2,v}^2 &= \frac{1}{2}\sum_{k_1}\sum_{k_2} |v_{k_1}v_{k_2}|^{p-2} \begin{vmatrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2} \end{vmatrix}^2 \leq \sum_k |v_1 v_k|^{p-2} x_k^2 = |v_1|^{p-2} \sum_j v_{m_j}^{p-2} \\ &< |v_1|^{p-2} \sum_j \frac{1}{2^j} = |v_1|^{p-2} < \infty. \end{aligned}$$

Meanwhile,

$$\|x,z\|_p^p = \frac{1}{2}\sum_{k_1}\sum_{k_2} \left\| \begin{matrix} x_{k_1} & x_{k_2} \\ z_{k_1} & z_{k_2} \end{matrix} \right\|^p = \sum_k |x_k|^p = \sum_j 1 = \infty.$$

This is what we want to show. ■

2.3 Further Results for $2 < p < \infty$

We shall now discuss the completeness of the space $(\ell_v^2, \|\cdot,\cdot\|_{2,v})$. To do so, we need some lemmas.

Lemma 2.10 *For every $x, y, z \in \ell_v^2$, we have*

$$\|x,y\|_{2,v}^2 + \|x,z\|_{2,v}^2 \leq \left(\|y\|_{2,v}^2 + \|z\|_{2,v}^2\right) \|x\|_{2,v}^2 .$$

Proof. The proof can be done similarly as in [7]. ■

Lemma 2.11 *For any linearly independent set $\{a, b\}$ in ℓ_v^2, we have*

$$\frac{4\,\|a,b\|_{2,v}^2\,\|x\|_{2,v}^2}{9\Big(\|b\|_{2,v} + \|a\|_{2,v}\Big)^2} \leq \|x,a\|_{2,v}^2 + \|x,b\|_{2,v}^2 \leq \left(\|a\|_{2,v}^2 + \|b\|_{2,v}^2\right) \|x\|_{2,v}^2 .$$

Proof. The proof can be done similarly as in [7]. ■

It follows from Lemma 2.11 that

$$\frac{2\|a,b\|_{2,v}}{3\left(\|b\|_{2,v} + \|a\|_{2,v}\right)} \|x\|_{2,v} \leq \|x\|^* \leq \left(\|a\|_{2,v}^2 + \|b\|_{2,v}^2\right)^{\frac{1}{2}} \|x\|_{2,v} ,$$

where

$$\|x\|^* = \left(\|x,a\|_{2,v}^2 + \|x,b\|_{2,v}^2\right)^{\frac{1}{2}}.$$

Note that $\|\cdot\|^*$ defines a norm on ℓ_v^2. Lemma 2.11 tells us that $\|\cdot\|^*$ and $\|\cdot\|_{2,v}$ are equivalent. This result can be used to understand the topology of ℓ_v^2 as a 2-normed space and it will facilitate the process to show that $(\ell_v^2, \|\cdot,\cdot\|_{2,v})$ is a 2-Banach space.

Corollary 2.12 *If $\left(x^{(n)}\right)$ is a Cauchy sequence in ℓ_v^2 with respect to the $\|\cdot,\cdot\|_{2,v}$, then it is also a Cauchy sequence with respect to the $\|\cdot\|_{2,v}$. If it is convergent with respect to the $\|\cdot\|_{2,v}$, then it is also convergent with respect to the $\|\cdot,\cdot\|_{2,v}$.*

Corollary 2.13 *The space $\left(\ell_v^2, \|\cdot,\cdot\|_{2,v}\right)$ is complete. In other words, ℓ_v^2 is a 2-Banach space with respect to the $\|\cdot,\cdot\|_{2,v}$. Accordingly, $(\ell_v^2, \langle\cdot,\cdot|\cdot\rangle_v)$ is a 2-Hilbert space.*

Proof. We know from Theorem 4.2 in [8] that $\left(\ell_v^2, \|\cdot\|_{2,v}\right)$ is complete. Following Corollary 2.12, the desired result is obtained. ■

We know that ℓ^p is complete when it is equipped with the usual 2-norm $\|\cdot,\cdot\|_p$. The following proposition tells us that it is no longer so when it is equipped with $\|\cdot,\cdot\|_{2,v}$.

Proposition 2.14 *As a set, ℓ^p is not closed but dense in $(\ell_v^2, \|\cdot,\cdot\|_{2,v})$.*

Proof. As in the proof of Proposition 2.9, we construct an increasing sequence of nonnegative integers (k_j) such that $v_{k_j}^{p-2} < 2^{-j}$ for every $j \in \mathbb{N}$. Next, for each $j \in \mathbb{N}$, we define $x^{(n)} = \left(x_k^{(n)}\right)$ by $x_k^{(n)} = 1$ for $k = k_1, k_2, \ldots, k_n$ and $x_k^{(n)} = 0$ otherwise. Let $\{z_1, z_2\}$ be a linearly independent set where $z_1 := (1, 0, \ldots)$ and $z_2 := (0, 1, 0, \ldots)$. Then we see that $\left(x^{(n)}\right)$ forms a Cauchy sequence in ℓ_v^2 since for $m > n$ we have

$$\begin{aligned}
\left\|x^{(n)} - x^{(m)}, z_i\right\|_{2,v}^2 &= \frac{1}{2}\sum_{k_1}\sum_{k_2} |v_{k_1}v_{k_2}|^{p-2} \begin{vmatrix} x_{k_1}^{(n)} - x_{k_1}^{(m)} & x_{k_2}^{(n)} - x_{k_2}^{(m)} \\ z_{ik_1} & z_{ik_2} \end{vmatrix}^2 \\
&\le \sum_k |v_i v_k|^{p-2} \left|x_k^{(n)} - x_k^{(m)}\right|^2 = |v_i|^{p-2} \sum_{j=n+1}^{m} v_{k_j}^{p-2} \\
&= |v_i|^{p-2} \sum_{j=n+1}^{m} \frac{1}{2^j} \to 0\,,
\end{aligned}$$

as $m, n \to \infty$ for each $i = 1, 2$. Since ℓ_v^2 is complete, $\left(x^{(n)}\right)$ is convergent and we know that the limit is the sequence $x = (x_k)$ where $x_k = 1$ for $k = k_1, k_2, k_3, \ldots$ and $x_k = 0$ otherwise. While $x^{(n)} \in \ell^p$ for every $n \in \mathbb{N}$, the limit $x \notin \ell^p$. This shows that ℓ^p is not closed in $(\ell_v^2, \|\cdot,\cdot\|_{2,v})$.

To see that ℓ^p is dense in $\left(\ell_v^2, \|\cdot,\cdot\|_{2,v}\right)$, we observe that every $x = (x_k) \in \ell_v^2$ can be approximated by $x^{(n)} := (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$ for sufficiently large values of $n \in \mathbb{N}$. ■

Following Proposition 2.14, let us now study ℓ^2_v further as the ambient space, instead of ℓ^p. We would like to know how the space ℓ^2_v depends on the choice of the weight v.

Let $v = (v_k)$, $w = (w_k) \in V_p$, where V_p is the collection of all sequences $v = (v_k) \in \ell^p$ with $v_k > 0$ for every $k \in \mathbb{N}$. Recall that $v \sim w$ means that v and w are equivalent, that is, there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 v_k \le w_k \le C_2 v_k$$

for every $k \in \mathbb{N}$. Then we have the following theorem, which says that the choice of the weights does not affect the membership nor the topology of the space.

Theorem 2.15 *Let $v, w \in V_p$. Then, the following statements are equivalent:*

(1) $v \sim w$.

(2) There exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 \|x, z\|_{2,v} \le \|x, z\|_{2,w} \le C_2 \|x, z\|_{2,v} \quad x, z \in \ell^p.$$

Proof. It is easy to see that *(1)⇒(2)*. It remains only to show that *(2)⇒(1)*. Assume that there exist constants $C_1 > 0$ and $C_2 > 0$ such that

$$C_1 \|x, z\|_{2,v} \le \|x, z\|_{2,w} \le C_2 \|x, z\|_{2,v} \quad x, z \in \ell^p.$$

Let $x := e_n$, where $n \ge 2$, be fixed but arbitrary, and take $z := (1, 0, \dots)$. Then $x, z \in \ell^p$, so that x and z are in ℓ^2_v as well as in ℓ^2_w. Moreover, we have

$$\|x, z\|_{2,v} = |v_n v_1|^{\frac{p-2}{2}} \quad \text{and} \quad \|x, z\|_{2,w} = |w_n w_1|^{\frac{p-2}{2}}.$$

Hence, from our assumption, we obtain

$$C_1 |v_n v_1|^{\frac{p-2}{2}} \le |w_n w_1|^{\frac{p-2}{2}} \le C_2 |v_n v_1|^{\frac{p-2}{2}},$$

and this holds for every $n \ge 2$. Taking the $\left(\frac{p-2}{2}\right)$-th roots, we conclude that $C_1' v_n \le w_n \le C_2' v_n$ where $C_1' = \frac{v_1}{w_1} C_1^{\frac{2}{p-2}} > 0$ and $C_2' = \frac{v_1}{w_1} C_2^{\frac{2}{p-2}} > 0$. This completes the proof. ∎

3 Closing Remarks

We have shown that the space ℓ^p can be equipped with a (weighted) 2-inner product and its induced 2-norm. Using the 2-inner product, one may define orthogonality on ℓ^p as in [5].

There we might also be interested in bounded bilinear 2-functionals. For example, for $2 < p < \infty$, the 2-functional

$$\begin{aligned} F_z(x_1, x_2) &= \begin{vmatrix} \langle x_1, z_1 \rangle_v & \langle x_1, z_2 \rangle_v \\ \langle x_2, z_1 \rangle_v & \langle x_2, z_2 \rangle_v \end{vmatrix} \\ &= \frac{1}{2} \sum_{k_1} \sum_{k_2} |v_{k_1} v_{k_2}|^{p-2} \begin{vmatrix} x_{1k_1} & x_{1k_2} \\ x_{2k_1} & x_{2k_2} \end{vmatrix} \begin{vmatrix} z_{1k_1} & z_{1k_2} \\ z_{2k_1} & z_{2k_2} \end{vmatrix} \end{aligned}$$

is bilinear and bounded on ℓ_v^2, and its norm can be given by

$$\|F_z\| = \sup_{\|x_1, x_2\|_{2,v} \neq 0} \frac{|F_z(x_1, x_2)|}{\|x_1, x_2\|_{2,v}}.$$

Clearly $\|F_z\| \leq \|z_1, z_2\|_{2,v}$, and by taking $x_i := \frac{z_i}{\sqrt{\|z_1, z_2\|_{2,v}}}$ we obtain $\|F_z\| = \|z_1, z_2\|_{2,v}$.

Moreover, we can prove an analog of the Riesz-Fréchet Theorem (see [1]), which states that for any bounded bilinear 2-functional G on ℓ_v^2, there exists a linearly independent set $\{z_1, z_2\} \in \ell_v^2$ such that

$$G(x_1, x_2) = \begin{vmatrix} \langle x_1, z_1 \rangle_v & \langle x_1, z_2 \rangle_v \\ \langle x_2, z_1 \rangle_v & \langle x_2, z_2 \rangle_v \end{vmatrix}$$

for every $x_1, x_2 \in \ell_v^2$ and $\|G\| = \|z_1, z_2\|_{2,v}$. However, we cannot show the uniqueness of such a set $\{z_1, z_2\}$.

We have first discussed ℓ^p and its natural 2-inner product and then we can generalize the results for all $n > 2$. In this regard,

$$\begin{aligned} \|x, z_2, \ldots, z_n\|_{2,v} \quad : &= \sqrt{\langle x, x | z_2, \ldots, z_n \rangle_v} \\ &= \left[\frac{1}{n!} \sum_{k_1} \cdots \sum_{k_n} |v_{k_1} \cdots v_{k_n}|^{p-2} \begin{vmatrix} x_{k_1} & x_{k_2} & \cdots & x_{k_n} \\ z_{2k_1} & z_{2k_2} & \cdots & z_{2k_n} \\ \vdots & \vdots & \ddots & \vdots \\ z_{nk_1} & z_{nk_2} & \cdots & z_{nk_n} \end{vmatrix}^2 \right]^{\frac{1}{2}} \end{aligned}$$

is an n-norm derived from n-inner product on ℓ^p.

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