

# On the $n$ -Normed Space of $p$ -Integrable Functions

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## Abstract

The space  $L^p(X)$ , where  $X$  is a measure space with at least  $n$  disjoint subsets of positive measure, can be equipped with an  $n$ -norm, which makes  $L^p(X)$  an  $n$ -normed space. The purpose of this paper is to study some properties of this  $n$ -normed space. In particular, we examine the completeness of the  $n$ -normed space and prove a contractive mapping theorem on this space.

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## 1 Introduction

Let  $X$  be a (real) vector space (of dimension at least  $n$ , where  $n$  is a fixed number in  $\mathbb{N}$ ). A mapping  $\|\cdot, \dots, \cdot\| : X^n \rightarrow \mathbb{R}$  satisfying the following properties:

- (1.1)  $\|x_1, \dots, x_n\| = 0$  if and only if  $x_1, \dots, x_n$  are linearly dependent,  
(1.2)  $\|x_1, \dots, x_n\|$  is invariant under permutation,  
(1.3)  $\|\alpha x_1, \dots, x_n\| = |\alpha| \|x_1, \dots, x_n\|$  for every  $x_1, \dots, x_n \in X$  and  $\alpha \in \mathbb{R}$ ,  
(1.4)  $\|x + y, x_2, \dots, x_n\| \leq \|x, x_2, \dots, x_n\| + \|y, x_2, \dots, x_n\|$  for every  $x, y, x_2, \dots, x_n \in X$ ,

is called an  $n$ -norm on  $X$ , and the pair  $(X, \|\cdot, \dots, \cdot\|)$  is called an  $n$ -normed space.

Geometrically, the value of  $\|x_1, \dots, x_n\|$  may be interpreted as the volume of the  $n$ -dimensional parallelepiped spanned by  $x_1, \dots, x_n$  in  $X$ . The concept of  $n$ -normed spaces was developed by Gähler in the period of 1964-1970 [3, 4, 5, 6, 7]. More recent works may be found in [1, 8, 10, 11, 12, 14, 15, 16, 17].

Let  $(X, \|\cdot, \dots, \cdot\|)$  be an  $n$ -normed space. A sequence  $(x_k)$  in  $X$  is said to converge to an  $x \in X$  (in the  $n$ -norm) if

$$\lim_{k \rightarrow \infty} \|x_k - x, y_2, \dots, y_n\| = 0,$$

for every  $y_2, \dots, y_n \in X$ . Also, a sequence  $(x_k)$  in  $X$  is called a *Cauchy* sequence if

$$\lim_{k, l \rightarrow \infty} \|x_k - x_l, y_2, \dots, y_n\| = 0,$$

for every  $y_2, \dots, y_n \in X$ .

If every Cauchy sequence  $(x_k)$  in  $X$  converges to some  $x \in X$ , then  $X$  is said to be *complete*. A complete  $n$ -normed space is called an  $n$ -Banach space.

On the space  $L^p(X)$  ( $1 \leq p < \infty$ ), the following an  $n$ -norm was defined by Gunawan in [9],

$$\|f_1, \dots, f_n\|_p := \left( \frac{1}{n!} \int_X \cdots \int_X \left\| \begin{array}{ccc} f_1(x_1) & \cdots & f_n(x_1) \\ f_1(x_2) & \cdots & f_n(x_2) \\ \vdots & \ddots & \vdots \\ f_1(x_n) & \cdots & f_n(x_n) \end{array} \right\|^p dx_1 \cdots dx_n \right)^{\frac{1}{p}}.$$

The aim of this note is to study  $L^p(X)$ ,  $1 \leq p < \infty$ , as an  $n$ -normed space with the above  $n$ -norm. We investigate the completeness of this  $n$ -normed space, and use the result to prove a contractive mapping theorem on this space.

## 2 Main Results

Let  $X$  be a measure space with at least  $n$  disjoint subsets of positive measure. Recall that  $L^p(X)$ ,  $1 \leq p < \infty$ , is the space of equivalence classes (modulo

equivalence almost everywhere) of functions such that  $\int_X |f(x)|^p dx < \infty$  and the function  $\|f\|_p := \left( \int_X |f(x)|^p dx \right)^{\frac{1}{p}}$  defines a norm on  $L^p(X)$ .

Before we reveal our main results, we present some lemmas and proposition.

**Lemma 2.1.** *For every  $f_1, \dots, f_n \in L^p(X)$ , we have*

$$\|f_1, \dots, f_n\|_p \leq (n!)^{1-\frac{1}{p}} \|f_1\|_p \cdots \|f_n\|_p.$$

*Proof.* Let  $\Phi$  be a set of all permutations of  $\{1, \dots, n\}$ . By the triangle inequality for real numbers and Minkowski's inequality, we have

$$\begin{aligned} & \|f_1, \dots, f_n\|_p \\ &= \left( \frac{1}{n!} \int_X \cdots \int_X |\det(f_i(x_j))|^p dx_1 \cdots dx_n \right)^{\frac{1}{p}} \\ &= \left( \frac{1}{n!} \int_X \cdots \int_X \left| \sum_{\phi=(j_1, \dots, j_n) \in \Phi} \operatorname{sgn}(\phi) f_1(x_{j_1}) f_2(x_{j_2}) \cdots f_n(x_{j_n}) \right|^p dx_{j_1} \cdots dx_{j_n} \right)^{\frac{1}{p}} \\ &\leq \left( \frac{1}{n!} \int_X \cdots \int_X \left( \sum_{(j_1, \dots, j_n) \in \Phi} |f_1(x_{j_1}) f_2(x_{j_2}) \cdots f_n(x_{j_n})| \right)^p dx_{j_1} \cdots dx_{j_n} \right)^{\frac{1}{p}} \\ &\leq (n!)^{-\frac{1}{p}} \sum_{(j_1, \dots, j_n) \in \Phi} \left( \left[ \int_X |f_1(x_{j_1})|^p dx_{j_1} \right]^{\frac{1}{p}} \cdots \left[ \int_X |f_n(x_{j_n})|^p dx_{j_n} \right]^{\frac{1}{p}} \right) \\ &= (n!)^{-\frac{1}{p}} \sum_{(j_1, \dots, j_n) \in \Phi} \|f_1\|_p \cdots \|f_n\|_p \\ &= (n!)^{1-\frac{1}{p}} \|f_1\|_p \cdots \|f_n\|_p, \end{aligned}$$

for every  $f_1, \dots, f_n \in L^p(X)$ , as claimed.  $\square$

Now, as shown in [9], we can derive a norm from the  $n$ -norm in a certain way. Indeed, if  $\{a_1, \dots, a_n\}$  is a linearly independent set in  $L^p(X)$ , then one may observe that

$$\|f\|_p^* := \left( \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|f, a_{i_2}, \dots, a_{i_n}\|_p^p \right)^{\frac{1}{p}} \quad (2.1)$$

defines a norm on  $L^p(X)$ . The mapping  $\|\cdot\|_p^*$  in (2.1) can be easily seen to satisfy the properties of a norm. In particular, we may check that if  $\|f\|_p^* = 0$ , then  $f = 0$  almost everywhere. Indeed, if  $\|f\|_p^* = 0$ , then we obtain  $\|f, a_{i_2}, \dots, a_{i_n}\|_p = 0$  for every  $\{i_2, \dots, i_n\} \subset \{1, \dots, n\}$ . This means that  $f$  is in the linear span of  $\{a_{i_2}, \dots, a_{i_n}\}$  almost everywhere, for every

$\{i_2, \dots, i_n\} \subset \{1, \dots, n\}$ . This forces us to conclude that  $f = 0$  almost everywhere.

We know that  $L^p(X)$  equipped with  $\|\cdot\|_p$  is complete. Now, we will show that  $L^p(X)$  as an  $n$ -normed space is complete with respect to the  $n$ -norm, through the following proposition.

**Proposition 2.2.** *Let  $\{a_1, \dots, a_n\}$  be a linearly independent set in  $L^p(X)$ , and the norm  $\|\cdot\|_p^*$  be defined by (2.1). Then  $\|\cdot\|_p^*$  is equivalent to the usual norm  $\|\cdot\|_p$ . Precisely, we have*

$$\frac{n\|a_1, \dots, a_n\|_p}{(2n-1)(\|a_1\|_p + \dots + \|a_n\|_p)} \|f\|_p \leq \|f\|_p^*$$

and

$$\|f\|_p^* \leq (n!)^{1-\frac{1}{p}} \left( \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|a_{i_2}\|_p^p \cdots \|a_{i_n}\|_p^p \right)^{\frac{1}{p}} \|f\|_p,$$

for every  $f \in L^p(X)$ .

*Proof.* For every  $f \in L^p(X)$  and any subset  $\{i_2, \dots, i_n\}$  of  $\{1, 2, \dots, n\}$ , we have

$$\|f, a_{i_2}, \dots, a_{i_n}\|_p \leq (n!)^{1-\frac{1}{p}} \|f\|_p \|a_{i_2}\|_p \cdots \|a_{i_n}\|_p,$$

by Lemma 2.1. Hence we obtain

$$\begin{aligned} \|f\|_p^* &= \left( \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|f, a_{i_2}, \dots, a_{i_n}\|_p^p \right)^{\frac{1}{p}} \\ &\leq (n!)^{1-\frac{1}{p}} \left( \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|a_{i_2}\|_p^p \cdots \|a_{i_n}\|_p^p \right)^{\frac{1}{p}} \|f\|_p. \end{aligned}$$

To prove the reverse inequality, we observe that

$$\begin{aligned} &\|f\|_p^p \|a_1, \dots, a_n\|_p^p \\ &= \frac{1}{n!} \int_X \cdots \int_X \left| f(x) \begin{vmatrix} a_1(x_1) & \cdots & a_n(x_1) \\ \vdots & \ddots & \vdots \\ a_1(x_n) & \cdots & a_n(x_n) \end{vmatrix} \right|^p dx dx_1 \cdots dx_n. \end{aligned}$$

By Minkowski's inequality, we have

$$\begin{aligned}
 & \left( \frac{1}{n!} \int_X \cdots \int_X \left| f(x) \begin{vmatrix} a_1(x_1) & \cdots & a_n(x_1) \\ \vdots & \ddots & \vdots \\ a_1(x_n) & \cdots & a_n(x_n) \end{vmatrix} \right|^p dx dx_1 \cdots dx_n \right)^{\frac{1}{p}} \\
 & \leq \left( \frac{1}{n!} \int_X \cdots \int_X \left| a_1(x_1) \begin{vmatrix} f(x) & \cdots & a_n(x) \\ \vdots & \ddots & \vdots \\ f(x_n) & \cdots & a_n(x_n) \end{vmatrix} \right|^p dx dx_1 \cdots dx_n \right)^{\frac{1}{p}} + \\
 & \cdots + \left( \frac{1}{n!} \int_X \cdots \int_X \left| a_1(x_n) \begin{vmatrix} f(x_1) & \cdots & a_n(x_1) \\ \vdots & \ddots & \vdots \\ f(x) & \cdots & a_n(x) \end{vmatrix} \right|^p dx dx_1 \cdots dx_n \right)^{\frac{1}{p}} + \\
 & + \left( \frac{1}{n!} \int_X \cdots \int_X \left| a_2(x) \begin{vmatrix} a_1(x_1) & f(x_1) & \cdots & a_n(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ a_1(x_n) & f(x_n) & \cdots & a_n(x_n) \end{vmatrix} \right|^p dx dx_1 \cdots dx_n \right)^{\frac{1}{p}} + \\
 & \cdots + \left( \frac{1}{n!} \int_X \cdots \int_X \left| a_n(x) \begin{vmatrix} a_1(x_1) & \cdots & f(x_1) \\ \vdots & \ddots & \vdots \\ a_1(x_n) & \cdots & f(x_n) \end{vmatrix} \right|^p dx dx_1 \cdots dx_n \right)^{\frac{1}{p}} \\
 & = n \|a_1\|_p \|f, a_2, \dots, a_n\|_p + \|a_2\|_p \|f, a_1, a_3, \dots, a_n\|_p + \\
 & \cdots + \|a_n\|_p \|f, a_1, \dots, a_{n-1}\|_p.
 \end{aligned}$$

Hence we obtain

$$\begin{aligned}
 \|f\|_p \|a_1, a_2, \dots, a_n\|_p & \leq n \|a_1\|_p \|f, a_2, a_3, \dots, a_n\|_p + \\
 & + \|a_2\|_p \|f, a_1, a_3, \dots, a_n\|_p + \\
 & \cdots + \|a_n\|_p \|f, a_1, a_2, \dots, a_{n-1}\|_p
 \end{aligned}$$

$$\begin{aligned}
 \|f\|_p \|a_2, a_1, \dots, a_n\|_p & \leq n \|a_2\|_p \|f, a_1, a_3, \dots, a_n\|_p + \\
 & + \|a_1\|_p \|f, a_2, a_3, \dots, a_n\|_p + \\
 & \cdots + \|a_n\|_p \|f, a_1, a_2, \dots, a_{n-1}\|_p \\
 & \vdots
 \end{aligned}$$

$$\begin{aligned}
 \|f\|_p \|a_n, a_1, \dots, a_{n-1}\|_p & \leq n \|a_n\|_p \|f, a_1, a_2, \dots, a_{n-1}\|_p + \\
 & + \|a_1\|_p \|f, a_n, a_2, \dots, a_{n-1}\|_p + \\
 & \cdots + \|a_{n-1}\|_p \|f, a_n, a_1, \dots, a_{n-2}\|_p,
 \end{aligned}$$

whence

$$\begin{aligned}
 n \|f\|_p \|a_1, a_2, \dots, a_n\|_p & \leq (2n - 1) \|a_1\|_p \|f, a_2, a_3, \dots, a_n\|_p + \\
 & \cdots + (2n - 1) \|a_n\|_p \|f, a_1, a_2, \dots, a_{n-1}\|_p.
 \end{aligned}$$

Next, we observe that

$$\begin{aligned} \|f, a_2, a_3, \dots, a_n\|_p &\leq \left( \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|f, a_{i_2}, \dots, a_{i_n}\|_p^p \right)^{\frac{1}{p}} = \|f\|_p^* \\ &\vdots \\ \|f, a_1, a_2, \dots, a_{n-1}\|_p &\leq \left( \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|f, a_{i_2}, \dots, a_{i_n}\|_p^p \right)^{\frac{1}{p}} = \|f\|_p^*. \end{aligned}$$

Hence, we obtain

$$n\|f\|_p\|a_1, a_2, \dots, a_n\|_p \leq (2n - 1) (\|a_1\|_p + \dots + \|a_n\|_p) \|f\|_p^*.$$

This completes the proof.  $\square$

**Corollary 2.3.** *If  $A := \{a_1, \dots, a_n\}$  and  $B := \{b_1, \dots, b_n\}$  are two linearly independent sets in  $L^p(X)$ , then the norm defined by (2.1) using  $A$  is equivalent to that using  $B$ .*

**Corollary 2.4.** *The space  $(L^p(X), \|\cdot\|_p^*)$  is complete. In other words, it is a Banach space.*

By Lemma 2.1, if a sequence  $(f_n)$  converges to  $f \in L^p(X)$  with respect to the usual norm  $\|\cdot\|_p$ , then it also converges to  $f$  with respect to the  $n$ -norm  $\|\cdot, \dots, \cdot\|_p$ . Similarly, if  $(f_n)$  is a Cauchy sequence in  $L^p(X)$  with respect to  $\|\cdot, \dots, \cdot\|_p$ , then it is also a Cauchy sequence with respect to  $\|\cdot, \dots, \cdot\|_p$ . Another consequence of Proposition 2.2 is the following theorem.

**Theorem 2.5.** *If a sequence  $(f_n) \in L^p(X)$  converges to some  $f \in L^p(X)$  with respect to  $\|\cdot, \dots, \cdot\|_p$ , then it also converges to  $f$  with respect to  $\|\cdot\|_p^*$ . Also, if  $(f_n)$  is a Cauchy sequence with respect to  $\|\cdot, \dots, \cdot\|_p$ , then it is a Cauchy sequence with respect to  $\|\cdot\|_p$ .*

*Proof.* Let  $\{a_1, \dots, a_n\}$  be a linearly independent set in  $L^p(X)$ , and  $\|\cdot\|_p^*$  be defined by (2.1). Now, if  $(f_n)$  converges to some  $f \in L^p(X)$  with respect to  $\|\cdot, \dots, \cdot\|_p$ , then for every  $\{i_2, \dots, i_n\} \subset \{1, \dots, n\}$  we have

$$\|f_n - f, a_{i_2}, \dots, a_{i_n}\| \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

It follows that

$$\|f(n) - f\|_p^* \rightarrow 0, \quad \text{as } n \rightarrow \infty,$$

that is,  $(f_n)$  converges to  $f$  with respect to  $\|\cdot\|_p^*$ . By Proposition 2.2, we conclude that  $(f_n)$  also converges to  $f$  with respect to  $\|\cdot\|_p$ . The second statement of the theorem is proved in a similar way.  $\square$

**Corollary 2.6.**  $(L^p(X), \|\cdot, \dots, \cdot\|_p)$  is an  $n$ -Banach space.

*Proof.* Let  $(f_n)$  be a Cauchy sequence in  $L^p(X)$  with respect to  $\|\cdot, \dots, \cdot\|_p$ . Then, by Theorem 2.5,  $(f_n)$  is a Cauchy sequence with respect to  $\|\cdot\|_p$ . We know that  $(L^p(X), \|\cdot\|_p)$  is a Banach space, and so  $(f_n)$  must converge to an element  $f \in L^p(X)$  with respect to  $\|\cdot\|_p$ . By Lemma 2.1,  $(f_n)$  must also converge to  $f$  with respect to  $\|\cdot, \dots, \cdot\|_p$ . Therefore,  $(L^p(X), \|\cdot, \dots, \cdot\|_p)$  is an  $n$ -Banach space.  $\square$

*Remark 2.7.* Up to this point, one may ask what then is the purpose of having an  $n$ -norm on  $L^p(X)$ ? There are two answers to this question. First, we can use the  $n$ -norm to define “volumes” of  $n$ -dimensional parallelepiped spanned by  $n$  elements in  $L^p(X)$ . Second, we did not know the relation between the topology generated by the  $n$ -norm and that by the usual norm on  $L^p(X)$ , until we proved Proposition 2.2. The result enriches our knowledge on particular  $n$ -normed spaces such as  $L^p(X)$  and  $\ell^p(\mathbb{N})$  spaces, as part of an effort in understanding the notion of  $n$ -normed spaces in general.

### 3 An Application

A contractive mapping theorem on standard and finite dimensional  $n$ -normed spaces was formulated by Gunawan and Mashadi [11, 12] in 2001. What distinguishes their work from others’ many years earlier is that they proved the theorem by involving a derived norm from the  $n$ -norm, rather than doing the same steps in  $n$ -normed spaces as in the proof of the analogous theorem in normed spaces. In 2013, Idris, Ekariani and Gunawan [13] formulated a contractive mapping theorem on the infinite dimensional vector space  $\ell^p$  as a 2-normed space. Its generalization for  $\ell^p$  as an  $n$ -normed space, where  $n > 2$ , can be found in [2].

With our previous result, we can now prove the following contractive mapping theorem on  $(L^p(X), \|\cdot, \dots, \cdot\|_p)$ .

**Theorem 3.1.** (Contractive Mapping Theorem) *Let  $T$  be a self-mapping of  $L^p(X)$  which is contractive with respect to a linearly independent set  $\{a_1, \dots, a_n\}$  in  $L^p(X)$ , that is, there exists a constant  $C \in (0, 1)$  such that the inequality*

$$\|Tf - Tg, a_{i_2}, \dots, a_{i_n}\|_p \leq C \|f - g, a_{i_2}, \dots, a_{i_n}\|_p$$

*holds for all  $f, g \in L^p(X)$  and  $\{i_2, \dots, i_n\} \subset \{1, \dots, n\}$ . Then  $T$  has a unique fixed point in  $L^p(X)$ .*

*Proof.* For every  $f, g \in L^p(X)$ , we observe that

$$\begin{aligned} \|Tf - Tg\|_p^* &= \left( \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|Tf - Tg, a_{i_2}, \dots, a_{i_n}\|_p^p \right)^{\frac{1}{p}} \\ &\leq C \left( \sum_{\{i_2, \dots, i_n\} \subset \{1, \dots, n\}} \|f - g, a_{i_2}, \dots, a_{i_n}\|_p^p \right)^{\frac{1}{p}} \\ &= C \|f - g\|_p^*. \end{aligned}$$

This result tells us that  $T$  is a contractive mapping on  $(L^p(X), \|\cdot\|_p^*)$ , which is a Banach space (by Corollary 2.4). Thus  $T$  must have a unique fixed point in  $L^p(X)$ .  $\square$

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