GENERAL INTRODUCTION

Let f be a locally integrable function on $\mathbf{R}^n$. Then Lebesgue's theorem on the differentiation of integrals asserts that

$$\lim_{r\to 0} \frac{1}{m(B^n)} \int_{B^n} f(x-ry)\,dy = f(x), \quad \text{a.e.},$$

where $m(B^n)$ denotes the mass of the unit ball B^n. In order to prove this, one defines the so-called maximal function $\mathcal{M}f$ by

$$\mathcal{M}f(x) = \sup_{r\in\mathbf{R}^+} \frac{1}{m(B^n)} \int_{B^n} |f(x-ry)|\,dy, \quad x \in \mathbf{R}^n.$$

One fundamental fact about $\mathcal{M}f$ that attracts our interest is the L^p-inequality

$$\|\mathcal{M}f\|_p \le C_p\,\|f\|_p, \quad f \in L^p(\mathbf{R}^n),$$

for all $1 < p \le \infty$. If one has this, the theorem of Lebesgue follows immediately. The case $n = 1$ was first studied by G. H. Hardy and J. E. Littlewood [19], while the general case was due to J. Marcinkiewicz and A. Zygmund [23] and N. Wiener [40]. Certain covering lemmas were used to prove the result. Its modern proof, however, usually involves an interpolation theorem due to Marcinkiewicz [22]. In Chapter I, we give a proof of Lebesgue's differentiation theorem, following E. M. Stein [33].

One may inquire what happens if the unit ball B^n in the definition of $\mathcal{M}f$ is replaced by the unit sphere S^{n-1}. Indeed, if $f \in \mathcal{S}(\mathbf{R}^n)$, the spherical mean

$$M_r f(x) = \int_{S^{n-1}} f(x - r\sigma)\, d\sigma, \quad x \in \mathbf{R}^n,$$

where $d\sigma$ denotes the surface measure on S^{n-1} of total mass 1, is well-defined for every $r \in \mathbf{R}^+$. One may then define the spherical maximal function $\mathcal{M}f$ by

$$\mathcal{M}f(x) = \sup_{r \in \mathbf{R}^+} |M_r f(x)|, \quad x \in \mathbf{R}^n.$$

Stein [34] discussed this subject and proved that for $n \geq 3$, the *a priori* inequality

$$\|\mathcal{M}f\|_p \leq C_p \, \|f\|_p, \quad f \in \mathcal{S}(\mathbf{R}^n),$$

holds exactly when $n/(n-1) < p \leq \infty$. His proof is based on the Fourier transform and a g-function argument. The result was extended to $L^p(\mathbf{R}^n)$ by Stein and S. Wainger [35]. A consequence of this is that if $f \in L^p(\mathbf{R}^n)$ where $n/(n-1) < p \leq \infty$, then the spherical means $M_r f(x)$ tends to $f(x)$ almost everywhere, as r tends to 0. We explain Stein's approach to the spherical maximal function in Chapter II.

Later M. Cowling and G. Mauceri [11] and [12] developed a new approach to the study of maximal functions using the Mellin transform. For each $f \in \mathcal{S}(\mathbf{R}^n)$, they defined the maximal function $\mathcal{M}_\phi f$ associated to a distribution ϕ on $\mathbf{R}^n$ by the formula

$$\mathcal{M}_\phi f(x) = \sup_{r \in \mathbf{R}^+} |(\phi_r * f)(x)|, \quad x \in \mathbf{R}^n,$$

where ϕ_r denotes the dilate of ϕ. When ϕ is surface measure on S^{n-1}, $\mathcal{M}_\phi f$ defines the spherical maximal function and Stein's result can be reproved. We outline the Mellin transform approach of Cowling and Mauceri [11] in Chapter III.

A generalization of Stein's treatment of spherical maximal functions was also given by Cowling and Mauceri [13]. One of their results, namely the L^2-estimate, asserts that the inequality

$$\|\mathcal{M}_\phi f\|_2 \leq C\,\|f\|_2, \quad f \in \mathcal{S}(\mathbf{R}^n),$$

holds whenever ϕ is compactly supported and $|\hat{\phi}_r(\sigma)| \leq C\,(1+r)^{-\alpha},\ r \in \mathbf{R}^+,\ \sigma \in S^{n-1}$, for some $\alpha > \frac{1}{2}$. Its proof involves a study of Riesz operators, the theory of fractional differentiation, some properties of Bessel functions, and an argument using g-functions.

In Chapter IV, we present a different proof of a similar L^2-estimate, which is one of our own contributions to the area. We use the Mellin transform technique to prove that the L^2-inequality holds provided that ϕ is compactly supported, $|\hat{\phi}(r\sigma)| \leq C\,(1+r)^{-\epsilon}$ and $\left|\frac{\partial}{\partial r}\hat{\phi}(r\sigma)\right| \leq C\,(1+r)^{-1-\epsilon}$, $r \in \mathbf{R}^+,\ \sigma \in S^{n-1}$, for some $\epsilon > 0$. In Chapter V, we also give a similar result for maximal functions associated to some surface measures on $\mathbf{R}^n$. These results have been submitted for publication [17].

Some related work on the theory of maximal functions in $\mathbf{R}^n$ may be found in N. E. Aguilera [1], Aguilera and E. O. Harboure [2], J. Bourgain [3] and [4], C. Calderón [5], R. R. Coifman and G. Weiss [8], A. Córdoba [9], J. L.

Rubio de Francia [24] and [25], and C. D. Sogge and Stein [30], [31] and [32].

In the late 1980's, Cowling and C. Meaney [14] worked on a maximal function on compact Lie groups. For every continuous function f on G, they defined the maximal function $\mathcal{M}f$, using a particular one-parameter family of measures μ_r, by the formula

$$\mathcal{M}f(x) = \sup_{r\in(0,R)} |(\mu_r * f)(x)|, \quad x \in G,$$

for some positive number R. Using the g-function technique and the decay estimates on the Fourier transform of μ_r and its derivatives, they verified the *a priori* inequality

$$\|\mathcal{M}f\|_p \leq C_p \, \|f\|_p, \quad f \in C(G),$$

for all p greater than some index p_0 in (1,2).

Our other major contribution to the subject is presented in Chapter VI. We generalize the result of Cowling and Meaney [14] to a class of maximal functions on compact semisimple Lie groups. More Lie group and representation theoretic arguments are involved in our treatment, including some formulae for characters and dimensions, a study of root systems, the theory of weights and some properties of the Weyl group. In addition, we also provide an example concerning the sharpness of the estimate. This result will appear in [18].

Some related results on the theory of maximal functions on manifolds may be found in M. Christ [6] and Sogge and Stein [32].