CHAPTER I

THE HARDY-LITTLEWOOD MAXIMAL FUNCTION

We give here a brief introduction to the theory of maximal functions, which we summarize from Chapters I and III of Stein [33].

§1.1 Introduction: A theorem for the maximal function

We shall assume throughout this chapter that all functions are real-valued.

Let f be a locally integrable function on $\mathbf{R}^n$. The Hardy-Littlewood maximal function $\mathcal{M}f$ is defined by

$$\mathcal{M}f(x) = \sup_{r \in \mathbf{R}^+} \frac{1}{m(B_x(r))} \int_{B_x(r)} |f(y)|\, dy, \quad x \in \mathbf{R}^n,$$

where $m(B_x(r))$ denotes the measure of the ball $B_x(r)$. (Note that if f_1 and f_2 are equivalent, then $\mathcal{M}f_1(x) = \mathcal{M}f_2(x)$ everywhere.)

The following theorem gives some fundamental facts about $\mathcal{M}f$.

Theorem 1.1 (Hardy-Littlewood)

(a) If $f \in L^1(\mathbf{R}^n)$, then

$$m\{x : \mathcal{M}f(x) > \alpha\} \leq \frac{C}{\alpha} \|f\|_1,$$

for every $\alpha > 0$.

(b) The L^p-inequality

$$\|\mathcal{M}f\|_p \le C_p \, \|f\|_p, \quad f \in L^p(\mathbf{R}^n),$$

holds whenever $1 < p \le \infty$.

Remark. From the definition of $\mathcal{M}f$, it is possible that $\mathcal{M}f(x) = \infty$ for some $x \in \mathbf{R}^n$. The above theorem, however, guarantees that $\mathcal{M}f$ is finite almost everywhere whenever $f \in L^p(\mathbf{R}^n)$, for any $1 \le p \le \infty$.

§1.2 A covering lemma of Vitali-type

We shall use the following covering lemma to prove the theorem.

Lemma 1.2 (Vitali)

Let E be a measurable set in $\mathbf{R}^n$ and $\{B_\alpha\}$ be a family of balls of bounded diameter covering E. Then from this family of balls we can extract a countable disjoint subfamily $\{B_k\}$ satisfying

$$\sum_k m(B_k) \ge C \, m(E),$$

with C depending only on the dimension n.

Proof. We extract the subfamily $\{B_k\}$ from the family of balls $\{B_\alpha\}$ as follows. First choose B_1 so that $\operatorname{diam}(B_1) \ge \frac{1}{2} \sup \{\operatorname{diam}(B_\alpha)\}$. (The choice is not unique but this is not of great importance.) Next choose B_2 disjoint from B_1 so that $\operatorname{diam}(B_2) \ge \frac{1}{2} \sup \{\operatorname{diam}(B_\alpha) : B_\alpha \text{ disjoint from } B_1\}$. We repeat this procedure to get the sequence of balls $B_1,\ B_2,\dots,\ B_k,\ B_{k+1},\dots$, where B_{k+1} is disjoint from $B_1,\ B_2,\dots,\ B_k$ and $\operatorname{diam}(B_{k+1}) \geq \frac{1}{2}\sup\{\operatorname{diam}(B_\alpha) : B_\alpha \text{ disjoint from } B_1,\ B_2,\dots,\ B_k\}$. (This sequence could terminate at B_l, for some l, if there were no balls in $\{B_\alpha\}$ disjoint from $B_1,\ B_2,\dots,\ B_l$.)

Let us assume that $\sum\limits_k m(B_k) < \infty$ (for otherwise the result is trivial). Now denote by B_k^* the ball having the same centre as B_k but with diameter five times as large. Then we claim that $\bigcup B_k^* \supseteq E$.

To verify our claim, take any ball B_α from the family that covers E. We have to show that $B_\alpha \subseteq \bigcup B_k^*$. We may assume here that B_α is not in $\{B_k\}$, for otherwise there is nothing to show. Now, if the sequence is finite, and terminates at B_l say, then B_α must intersect B_k for some $1 \leq k \leq l$ and $\frac{1}{2}\operatorname{diam}(B_\alpha) \leq \operatorname{diam}(B_k)$ for the smallest k for which B_α intersects B_k. By geometric observation, it is clear that $B_\alpha \subset B_k^*$. If the sequence is infinite, we take the first k for which $\operatorname{diam}(B_{k+1}) < \frac{1}{2}\operatorname{diam}(B_\alpha)$ (which we may since $\operatorname{diam}(B_k) \to 0$ as $k \to \infty$). The ball B_α must then intersect one of $B_1,\ B_2,\dots,\ B_k$ or it would have been chosen as the $(k+1)^{\text{th}}$ ball instead of B_{k+1} since its diameter is more than twice that of B_{k+1}. The same situation now happens as in the previous case.

The claim is therefore clear and eventually we have

$$m(E) \leq \sum_k m(B_k^*) = 5^n \sum_k m(B_k),$$

proving the lemma. □

§1.3 The proof of the theorem

For every $\alpha > 0$, write $E_\alpha = \{x : \mathcal{M}f(x) > \alpha\}$. Then, from the definition of $\mathcal{M}f$, for each $x \in E_\alpha$, there exists a ball B_x centered at x such that

$$\int_{B_x} |f(y)|\, dy > \alpha\, m(B_x).$$

Now consider the family of balls $\{B_x : x \in E_\alpha\}$. These balls clearly have bounded diameter and cover E_α. Hence, by Lemma 1.2, we can select from them a countable disjoint subfamily $\{B_k\}$ satisfying

$$\sum_k m(B_k) \geq C\, m(E_\alpha).$$

Thus we have

$$\begin{aligned} \|f\|_1 &= \int_{\mathbf{R}^n} |f(y)|\, dy \\ &\geq \int_{\bigcup B_k} |f(y)|\, dy \\ &= \sum_k \int_{B_k} |f(y)|\, dy \\ &\geq \sum_k \alpha\, m(B_k) \\ &\geq C\, \alpha\, m(E_\alpha), \end{aligned}$$

proving part (a) of the theorem.

To prove part (b), we note that the case $p = \infty$ is trivial, and so assume that $1 < p < \infty$. Having $\alpha > 0$ fixed for the moment, we define

$$f_1(x) = \begin{cases} f(x), & \text{if } |f(x)| > \alpha/2, \\ 0, & \text{otherwise,} \end{cases}$$

being the large part of f. Clearly $|f(x)| \leq |f_1(x)| + \alpha/2,\ x \in \mathbf{R}^n$, and hence

$$\mathcal{M}f(x) \leq \mathcal{M}f_1(x) + \alpha/2, \quad x \in \mathbf{R}^n.$$

Also, $f_1 \in L^1(\mathbf{R}^n)$, given that $f \in L^p(\mathbf{R}^n)$. Therefore, by part (a), we have

$$\begin{aligned} m\{x : \mathcal{M}f(x) > \alpha\} &\le m\{x : \mathcal{M}f_1(x) > \alpha/2\} \\ &\le \frac{2C}{\alpha}\, \|f_1\|_1 \\ &= \frac{2C}{\alpha} \int_{|f(x)|>\alpha/2} |f(x)|\, dx. \end{aligned}$$

Now let λ be the distribution function of $\mathcal{M}f$, given by the formula

$$\lambda(\alpha) = m\{x : \mathcal{M}f(x) > \alpha\}, \quad \alpha > 0.$$

We remark that λ is a monotonic nonincreasing function of α. In particular, we have (see [27], p. 172)

$$\int_{\mathbf{R}^n} |\mathcal{M}f(x)|^p\, dx = p \int_0^\infty \alpha^{p-1}\, \lambda(\alpha)\, d\alpha.$$

Hence, using the above estimate and interchanging the orders of integration, we obtain

$$\begin{aligned} \|\mathcal{M}f\|_p^p &= \int_{\mathbf{R}^n} |\mathcal{M}f(x)|^p\, dx \\ &= p \int_0^\infty \alpha^{p-1}\, \lambda(\alpha)\, d\alpha \\ &\le 2Cp \int_0^\infty \alpha^{p-2} \int_{|f(x)|>\alpha/2} |f(x)|\, dx\, d\alpha \\ &= 2Cp \int_{\mathbf{R}^n} |f(x)| \int_0^{2|f(x)|} \alpha^{p-2}\, d\alpha\, dx \\ &= \frac{Cp\, 2^p}{p-1} \int_{\mathbf{R}^n} |f(x)|^p dx \\ &= \big(C_p\, \|f\|_p\big)^p, \end{aligned}$$

which concludes the proof of Theorem 1.1. □

Remark. It is worth observing that $C_p = O\big(\frac{1}{p-1}\big)$, as $p \to 1$.

§1.4 The Marcinkiewicz interpolation theorem

The argument used in the proof of Theorem 1.1(b) brings us to the Marcinkiewicz interpolation theorem below, which we will find useful later.

Let T be an operator from $L^p(\mathbf{R}^n)$ to $L^q(\mathbf{R}^n)$, where $1 \le p \le \infty$ and $1 \le q \le \infty$. We then say that T is *of type* (p, q) if

$$\|Tf\|_q \le C\,\|f\|_p, \quad f \in L^p(\mathbf{R}^n).$$

For $1 \le q < \infty$, we say that T is *of weak-type* (p, q) if

$$m\{x : |Tf(x)| > \alpha\} \le \left(\frac{C}{\alpha}\,\|f\|_p\right)^q, \quad f \in L^p(\mathbf{R}^n);$$

while for $q = \infty$, T is said to be *of weak-type* (p, q) if it is of type (p, q).

We remark that if T is of type (p, q), then T is also of weak-type (p, q). Indeed, for $1 \le q < \infty$, we have

$$\begin{aligned} \alpha^q\, m\{x : |Tf(x)| > \alpha\} &\le \int_{|Tf(x)|>\alpha} |Tf(x)|^q\, dx \\ &\le \int_{\mathbf{R}^n} |Tf(x)|^q\, dx \\ &= \|Tf\|_q^q \\ &\le (C\,\|f\|_p)^q, \end{aligned}$$

provided that T is of type (p, q).

Now denote by $L^{p_1}(\mathbf{R}^n) + L^{p_2}(\mathbf{R}^n)$ the space of all functions $f = f_1 + f_2$, where $f_1 \in L^{p_1}(\mathbf{R}^n)$ and $f_2 \in L^{p_2}(\mathbf{R}^n)$. When T is defined on $L^{p_1}(\mathbf{R}^n) + L^{p_2}(\mathbf{R}^n)$, we say that T is *sub-additive* if

$$|T(f_1 + f_2)(x)| \le |Tf_1(x)| + |Tf_2(x)|, \quad \text{a.e.},$$

for all $f_1 \in L^{p_1}(\mathbf{R}^n)$ and $f_2 \in L^{p_2}(\mathbf{R}^n)$.

One may observe that if $p_1 < p_2$, then $L^p(\mathbf{R}^n) \subseteq L^{p_1}(\mathbf{R}^n) + L^{p_2}(\mathbf{R}^n)$, whenever $p_1 \le p \le p_2$.

We now come to the Marcinkiewicz interpolation theorem.

THEOREM 1.4 (Marcinkiewicz)

Let T be an operator from $L^1(\mathbf{R}^n) + L^q(\mathbf{R}^n)$, where $1 < q \le \infty$, to the space of measurable functions in $\mathbf{R}^n$. Suppose that T is sub-additive and is simultaneously of weak-type (1,1) and of weak-type (q, q). Then T is of type (p, p), whenever $1 < p < q$.

PROOF. The main technique of the proof is to split a function into its large and small parts. For the case $q = \infty$, the proof is similar to that of Theorem 1.1(b). We shall only prove the theorem under the assumption that $1 < q < \infty$.

Suppose $f \in L^p(\mathbf{R}^n)$, with $1 < p < q$. For each fixed $\alpha > 0$, we split f into $f_1 \in L^1(\mathbf{R}^n)$ and $f_2 \in L^q(\mathbf{R}^n)$ where

$$f_1(x) = \begin{cases} f(x), & \text{if } |f(x)| > \alpha, \\ 0, & \text{otherwise,} \end{cases}$$

and

$$f_2(x) = \begin{cases} f(x), & \text{if } |f(x)| \le \alpha, \\ 0, & \text{otherwise.} \end{cases}$$

Then, by sub-additivity of T, we have

$$\{x : |Tf(x)| > \alpha\} \subseteq \{x : |Tf_1(x)| > \alpha/2\} \cup \{x : |Tf_2(x)| > \alpha/2\},$$

and consequently

$$\lambda(\alpha) = m\{x : |Tf(x)| > \alpha\}$$

$$\leq m\{x : |Tf_1(x)| > \alpha/2\} + m\{x : |Tf_2(x)| > \alpha/2\}.$$

Now, following the hypothesis that T is simultaneously of weak-type (1,1) and of weak-type (q,q), we have

$$\begin{aligned}\lambda(\alpha) &\leq \frac{2C_1}{\alpha}\int_{\mathbf{R}^n}|f_1(x)|\,dx + \left(\frac{2C_q}{\alpha}\right)^q\int_{\mathbf{R}^n}|f_2(x)|^q\,dx\\ &= \frac{2C_1}{\alpha}\int_{|f(x)|>\alpha}|f(x)|\,dx + \left(\frac{2C_q}{\alpha}\right)^q\int_{|f(x)|\leq\alpha}|f(x)|^q\,dx.\end{aligned}$$

We finally observe that

$$\begin{aligned}\|Tf\|_p^p &= \int_{\mathbf{R}^n}|Tf(x)|^p\,dx\\ &= p\int_0^\infty \alpha^{p-1}\,\lambda(\alpha)\,d\alpha\\ &\leq 2C_1p\int_0^\infty \alpha^{p-2}\int_{|f(x)|>\alpha}|f(x)|dx\,d\alpha\\ &\qquad + (2C_q)^qp\int_0^\infty \alpha^{p-q-1}\int_{|f(x)|\leq\alpha}|f(x)|dx\,d\alpha\\ &= 2C_1p\int_{\mathbf{R}^n}|f(x)|\int_0^{|f(x)|}\alpha^{p-2}\,d\alpha\,dx\\ &\qquad + (2C_q)^qp\int_{\mathbf{R}^n}|f(x)|^q\int_{|f(x)|}^\infty \alpha^{p-q-1}d\alpha\,dx\\ &= \frac{2C_1p}{p-1}\int_{\mathbf{R}^n}|f(x)|^p\,dx + \frac{(2C_q)^qp}{q-p}\int_{\mathbf{R}^n}|f(x)|^p\,dx\\ &= \left(\frac{2C_1p}{p-1} + \frac{(2C_q)^qp}{q-p}\right)\|f\|_p^p\\ &= \big(C_p\,\|f\|_p\big)^p,\end{aligned}$$

completing the proof of the theorem. □

Remark. As before, one might like to observe that $C_p = O\big(\frac{1}{p-1}\big)$, as $p \to 1$.

§1.5 Approximations of the identity

Suppose ϕ is an integrable function on $\mathbf{R}^n$. For each $r \in \mathbf{R}^+$, denote by ϕ_r the dilate of ϕ given by $\phi_r(x) = \phi(x/r)\, r^{-n},\ x \in \mathbf{R}^n$.

We have the following theorems.

Theorem 1.5A

Let ψ be the least decreasing radial majorant of ϕ, given by the formula $\psi(x) = \sup\limits_{|y| \geq |x|} |\phi(y)|$, $x \in \mathbf{R}^n$. If ψ is integrable, then

$$\sup_{r \in \mathbf{R}^+} |(\phi_r * f)(x)| \leq C\, \mathcal{M}f(x), \quad x \in \mathbf{R}^n,$$

for all $f \in L^p(\mathbf{R}^n)$, with $1 \leq p \leq \infty$. Here $C = \int_{\mathbf{R}^n} \psi(x)\, dx$.

Proof. First, since ψ is radial, we may put $\psi(r) = \psi(x)$ whenever $r = |x|$. We then observe that

$$\int_{r/2 \leq |x| \leq r} \psi(x)\, dx \geq \psi(r) \int_{r/2 \leq |x| \leq r} dx = K\, \psi(r)\, r^n.$$

As ψ is integrable and is decreasing, we see here that $\psi(r)\, r^n \to 0$, as $r \to 0$ or as $r \to \infty$.

Let us now assume that $f \geq 0$. It then suffices to show that

$$(\psi_r * f)(x) \leq C\, \mathcal{M}f(x), \quad x \in \mathbf{R}^n,$$

for each $r \in \mathbf{R}^+$. But this assertion is translation invariant (with respect to f) and dilation invariant (with respect to ψ), and so it is enough to show that

$$(\psi * f)(0) \leq C\, \mathcal{M}f(0).$$

If $\mathcal{M}f(0) = \infty$, then there is nothing to show. Otherwise, define

$$L(r) = \int_{B_0(r)} f(x)\,dx,\ \ r \in \mathbf{R}^+.$$

By the definition of $\mathcal{M}f(0)$, we have

$$L(r) \le m(B_0(r))\,\mathcal{M}f(0) = m(B^n)\,r^n\,\mathcal{M}f(0), \quad r \in \mathbf{R}^+.$$

Further, we may express

$$L(r) = \int_0^r l(s)\,s^{n-1}\,ds, \quad r \in \mathbf{R}^+,$$

where $l(r) = \int_{S^{n-1}} f(r\sigma)\,d\sigma,\ r \in \mathbf{R}^+$. Hence, using integration by parts (and taking into account our observation on ψ and the fact that $\omega_{n-1} = m(B^n)\,n$), we obtain

$$\begin{aligned}
(\psi * f)(0) &= \int_{\mathbf{R}^n} f(x)\,\psi(x)\,dx \\
&= \int_0^\infty \int_{S^{n-1}} f(r\sigma)\,\psi(r)\,r^{n-1}\,d\sigma\,dr \\
&= \int_0^\infty l(r)\,r^{n-1}\,\psi(r)\,dr \\
&= \int_0^\infty L(r)\,d(-\psi(r)) \\
&\le \mathcal{M}f(0) \int_0^\infty m(B^n)\,r^n\,d(-\psi(r)) \\
&= \mathcal{M}f(0) \int_0^\infty \omega_{n-1}\,\psi(r)\,r^{n-1}\,dr \\
&= \mathcal{M}f(0) \int_0^\infty \int_{S^{n-1}} \psi(r)\,r^{n-1}\,d\sigma\,dr \\
&= \mathcal{M}f(0) \int_{\mathbf{R}^n} \psi(x)\,dx \\
&= C\,\mathcal{M}f(0),
\end{aligned}$$

as required. $\square$

Theorem 1.5B

Let ϕ satisfy the hypothesis of Theorem 1.5A. In addition, suppose that $\int_{\mathbf{R}^n} \phi(x)\,dx = 1$. Then we have

(a) $\|\phi_r * f - f\|_p \to 0$, as $r \to 0$, for all $f \in L^p(\mathbf{R}^n)$, with $1 \le p < \infty$.

(b) $\lim_{r\to 0} (\phi_r * f)(x) = f(x)$, a. e., for all $f \in L^p(\mathbf{R}^n)$, with $1 \le p \le \infty$.

Proof. Suppose $f \in L^p(\mathbf{R}^n)$, with $1 \le p < \infty$. Then define

$$\Delta f(y) = \|f(\,\cdot - y) - f(\,\cdot\,)\|_p, \quad y \in \mathbf{R}^n.$$

We claim that $\Delta f(y) \to 0$, as $y \to 0$. This is obvious if f is continuous with compact support. In general, we write $f = g + h$, where g is continuous with compact support and the L^p-norm of h is at our disposal. Hence $\Delta f \le \Delta g + \Delta h$, with $\Delta g(y) \to 0$, as $y \to 0$, and $\Delta h(y) \le 2\,\|h\|_p,\ y \in \mathbf{R}^n$. The claim is now clear since the norm of h can be chosen to be arbitrarily small.

Next, since $\int_{\mathbf{R}^n} \phi_r(x)\,dx = \int_{\mathbf{R}^n} \phi(x)\,dx = 1$, we observe that

$$(\phi_r * f - f)(x) = \int_{\mathbf{R}^n} \{f(x-y) - f(x)\}\,\phi_r(y)\,dy, \quad x \in \mathbf{R}^n,$$

whence (by Minkowski's inequality and the dominated convergence theorem)

$$\begin{aligned}\|\phi_r * f - f\|_p &\le \int_{\mathbf{R}^n} \Delta f(y)\,|\phi_r(y)|\,dy \\ &= \int_{\mathbf{R}^n} \Delta f(ry)\,|\phi(y)|\,dy \\ &\to 0, \quad \text{as } r \to 0,\end{aligned}$$

proving part (a) of the theorem.

We shall now prove part (b). First, we deal with the case when $f \in L^p(\mathbf{R}^n)$, with $1 \le p < \infty$. By part (a), we have

$$\|\phi_r * f - f\|_p \to 0, \quad \text{as } r \to 0.$$

It then remains to show that $\lim_{r\to 0}(\phi_r * f)(x)$ exists almost everywhere. Define

$$\Omega f(x) = |\limsup_{r\to 0}(\phi_r * f)(x) - \liminf_{r\to 0}(\phi_r * f)(x)|, \quad x \in \mathbf{R}^n,$$

representing the oscillation of the family $\{\phi_r * f\}$ as r tends to 0. We then require $\Omega f(x) = 0$ almost everywhere. Again, this is clear when f is continuous with compact support. Otherwise, we write $f = g + h$, where g is continuous with compact support and where the L^p-norm of h is at our disposal. It follows from Theorem 1.5A and Theorem 1.1(b) that for any $\epsilon > 0$, we have

$$m\{x : \Omega h(x) > \epsilon\} \le m\{x : 2\,\mathcal{M}h(x) > \epsilon\} \le \left(\frac{2C}{\epsilon}\,\|h\|_p\right)^p.$$

Since $\Omega f \le \Omega g + \Omega h$ and $\Omega g \equiv 0$, we obtain

$$m\{x : \Omega f(x) > \epsilon\} \le \left(\frac{2C}{\epsilon}\,\|h\|_p\right)^p.$$

But since the L^p-norm of h can be chosen to be arbitrarily small, we conclude $\Omega f = 0$ almost everywhere.

It now remains to handle the case of bounded f. By covering $\mathbf{R}^n$ by a countable collection of balls, it suffices to prove that $\lim_{r\to 0}(\phi_r * f)(x) = f(x)$ for almost every x in a ball B say. To do so, let B_1 be a concentric ball which strictly contains B, and δ be the distance from B to the complement of B_1. Then define

$$f_1(x) = \begin{cases} f(x), & \text{if } x \in B_1, \\ 0, & \text{otherwise,} \end{cases}$$

and put $f_2 = f - f_1$. Obviously $f_1 \in L^1(\mathbf{R}^n)$, and by the previous result the conclusion holds for it. Now, for f_2, we observe that

$$\begin{aligned} |(\phi_r * f_2)(x)| &= \left| \int_{\mathbf{R}^n} f_2(x-y)\, \phi_r(y)\, dy \right| \\ &\le \int_{|y|>\delta} |f_2(x-y)|\, |\phi_r(y)|\, dy \\ &\le \|f_2\|_\infty \int_{|y|>\delta/\epsilon} |\phi(y)|\, dy \\ &\to 0, \quad \text{as } r \to 0, \end{aligned}$$

for all $x \in B$.

The proof of Theorem 1.5B is therefore complete. □

Remark. A special case of Theorem 1.5B(b) is of course Lebesgue's differentiation theorem, in which $\phi = \frac{1}{m(B^n)}\, \chi_{B^n}$.