CHAPTER II

# SPHERICAL MAXIMAL FUNCTIONS – I

## The Fourier transform technique

In this chapter, we discuss the Fourier transform approach to the study of maximal functions, which is based on the work of Stein [34] and Stein and Wainger [35].

### §2.1 Introduction: Stein's theorem

For every $f \in \mathcal{S}(\mathbf{R}^n)$, we define the spherical maximal function $\mathcal{M}f$ by

$$\mathcal{M}f(x) = \sup_{r \in \mathbf{R}^+} |M_r f(x)|, \quad x \in \mathbf{R}^n,$$

where $M_r f$ $(r \in \mathbf{R}^+)$ is given by

$$M_r f(x) = \int_{S^{n-1}} f(x - r\sigma)\, d\sigma, \quad x \in \mathbf{R}^n,$$

with $d\sigma$ denoting the rotationally invariant measure on $S^{n-1}$ of total mass 1. We then ask whether we have the *a priori* inequality

$$\|\mathcal{M}f\|_p \leq C_p \, \|f\|_p, \quad f \in \mathcal{S}(\mathbf{R}^n),$$

for some $1 < p \leq \infty$.

For $n = 1$, $S^0$ consists of two points, $\sigma = \pm 1$, and each point has mass $\frac{1}{2}$, say. Here, for each $r \in \mathbf{R}^+$, we have

$$M_r f(x) = \tfrac{1}{2}\{f(x-r) + f(x+r)\}, \quad x \in \mathbf{R},$$

and therefore we see that the *a priori* inequality holds only for the trivial case $p = \infty$. Indeed, for $p = \infty$, we have

$$\|\mathcal{M}f\|_\infty \le \|f\|_\infty.$$

On the other hand, for $1 \le p < \infty$, one may take as a counter example

$$f(x) = \begin{cases} \log|x|^{-1}, & \text{if } 0 < |x| < \frac{1}{2}, \\ 0, & \text{otherwise,} \end{cases}$$

and observe that $f \in L^p(\mathbf{R})$ but $\mathcal{M}f(x) = \infty$ everywhere.

For $n \ge 2$, the *a priori* inequality fails to hold for $1 \le p \le n/(n-1)$. To see this, take $F$ to be the function defined by

$$F(x) = \begin{cases} \frac{|x|^{1-n}}{\log|x|^{-1}}, & \text{if } 0 < |x| < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

We may observe that $F \in L^p(\mathbf{R}^n)$, whenever $1 \le p \le n/(n-1)$. In fact, for $p = 1$, we have

$$\begin{aligned} \int_{\mathbf{R}^n} |F(x)|\,dx &= \int_{0<|x|<\frac{1}{2}} \frac{|x|^{1-n}}{\log|x|^{-1}}\,dx \\ &= \int_0^{\frac{1}{2}} \int_{S^{n-1}} \frac{r^{1-n}}{\log r^{-1}}\, r^{n-1}\,d\sigma\,dr \\ &= \omega_{n-1} \int_0^{\frac{1}{2}} \frac{1}{\log r^{-1}}\,dr \\ &< \infty, \end{aligned}$$

and for $1 < p \leq n/(n-1)$, we have

$$\begin{aligned}\int_{\mathbf{R}^n} |F(x)|^p \, dx &= \int_{0<|x|<\frac{1}{2}} \frac{|x|^{(1-n)p}}{\log^p |x|^{-1}} \, dx \\ &= \int_0^{\frac{1}{2}} \int_{S^{n-1}} \frac{r^{(1-n)p}}{\log^p r^{-1}} \, r^{n-1} \, d\sigma \, dr \\ &= \omega_{n-1} \int_0^{\frac{1}{2}} \frac{r^{(1-n)(p-1)}}{\log^p r^{-1}} \, dr \\ &\leq \omega_{n-1} \int_0^{\frac{1}{2}} \frac{r^{-1}}{\log^p r^{-1}} \, dr \\ &= \omega_{n-1} \int_{\log 2}^{\infty} s^{-p} \, ds \\ &< \infty.\end{aligned}$$

On the other hand, we have $\sup_{r \in \mathbf{R}^+} |M_r F(x)| = \infty$ for all $x \in \mathbf{R}^n$. More precisely, for $x = 0$, we have

$$\begin{aligned}M_r F(0) &= \int_{S^{n-1}} F(-r\sigma) \, d\sigma \\ &= \frac{r^{1-n}}{\log r^{-1}} \\ &\to \infty, \quad \text{as } r \to 0.\end{aligned}$$

Now, for $x \neq 0$, recalling the fact that $d\sigma$ is rotationally invariant, we may assume that $x = (0, \ldots, 0, R)$, for some $R \in \mathbf{R}^+$, and then consider

$$M_R F(x) = \int_{S^{n-1}} F(x - R\sigma) \, d\sigma = R^n \int_{S_x(R)} F(y) \, dy.$$

We look at the part of the integral near the origin; where $y \in S_x(R) \cap B_0(\epsilon)$, for some $0 < \epsilon < \frac{1}{2}$, say. Here we have

$$\begin{aligned}M_R F(x) &\geq R^n \int_{S_x(R) \cap B_0(\epsilon)} F(y) \, dy \\ &= R^n \int_{S_x(R) \cap B_0(\epsilon)} F(y_1, \ldots, y_{n-1}, y_n) \, (1 + |\nabla y_n|^2)^{\frac{1}{2}} \, dy_1 \ldots dy_{n-1},\end{aligned}$$

where $y_n = R-(R^2-y_1^2-\cdots-y_{n-1}^2)^{\frac{1}{2}}$. By some dull calculation, one can show that $F(y_1,\ldots,y_{n-1},y_n) \geq C\,F(y_1,\ldots,y_{n-1},0)$ for all $(y_1,\ldots,y_n) \in B_0(\epsilon)$. Hence, we obtain

$$\begin{aligned}
M_R F(x) &\geq C\,R^n \int_{y_1^2+\cdots+y_{n-1}^2\leq\epsilon^2} F(y_1,\ldots,y_{n-1},0)\,dy_1\ldots dy_{n-1} \\
&= C\,R^n \int_{y_1^2+\cdots+y_{n-1}^2\leq\epsilon^2} \frac{(y_1^2+\cdots+y_{n-1}^2)^{(1-n)/2}}{\log(y_1^2+\cdots+y_{n-1}^2)^{-1/2}}\,dy_1\ldots dy_{n-1} \\
&= C\,R^n \int_0^\epsilon \int_{S^{n-2}} \frac{s^{1-n}}{\log s^{-1}}\,s^{n-2}\,d\tau\,ds \\
&= C\,R^n\,\omega_{n-2} \int_0^\epsilon \frac{s^{-1}}{\log s^{-1}}\,ds \\
&= C\,R^n\,\omega_{n-2} \int_{\log\epsilon^{-1}}^\infty t^{-1}\,dt \\
&= \infty,
\end{aligned}$$

as claimed.

For the remaining cases, we have the following results.

THEOREM 2.1 (Stein-Bourgain)

For $n \geq 2$, the *a priori* inequality

$$\|\mathcal{M}f\|_p \leq C_p\,\|f\|_p, \quad f \in \mathcal{S}(\mathbf{R}^n),$$

holds whenever $n/(n-1) < p \leq \infty$.

*Remark.* The result for $n \geq 3$ was proved earlier by Stein [34]. The case $n = 2$ was undecided until the work of Bourgain [3] in 1985. In this thesis, we shall only discuss the treatment for $n \geq 3$. Those who are interested in the result for $n = 2$ should see [3] (and also [4] for its extensions).

## §2.2 Variants of $M_r$

We need to develop the following in order to prove the theorem.

For each $\alpha > 0$, consider the function $m_\alpha$ given by

$$m_\alpha(x) = \begin{cases} \frac{(1-|x|^2)^{\alpha-1}}{\Gamma(\alpha)}, & \text{if } |x| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

Then, for each $r \in \mathbf{R}^+$, define the operator $M_{\alpha,r}$ on $\mathcal{S}(\mathbf{R}^n)$ by

$$M_{\alpha,r} f(x) = (m_{\alpha,r} * f)(x), \quad x \in \mathbf{R}^n,$$

where $m_{\alpha,r}(x) = m_\alpha(x/r)\, r^{-n},\ x \in \mathbf{R}^n$.

We have the following facts.

Fact 2.2A $\hat{m}_\alpha(\xi) = \pi^{\alpha+1}|\xi|^{-n/2-\alpha+1} J_{n/2+\alpha-1}(2\pi|\xi|),\ \xi \in \mathbf{R}^n$.

Proof. See [36], p. 171. □

Fact 2.2B $(M_{\alpha,r} f)\hat{\ }(\xi) = \hat{m}_\alpha(r\xi)\, \hat{f}(\xi),\ \xi \in \mathbf{R}^n$.

Proof. For all $\xi \in \mathbf{R}^n$, we have

$$\begin{aligned} \hat{m}_{\alpha,r}(\xi) &= \int_{\mathbf{R}^n} m_{\alpha,r}(x)\, e^{-2\pi i \xi \cdot x}\, dx \\ &= \int_{\mathbf{R}^n} m_\alpha(x/r)\, r^{-n}\, e^{-2\pi i \xi \cdot x}\, dx \\ &= \int_{\mathbf{R}^n} m_\alpha(y)\, e^{-2\pi i r \xi \cdot y}\, dy \\ &= \hat{m}_\alpha(r\xi), \end{aligned}$$

whence the fact. □

*Remark.* With these two facts, we may hereafter define $M_{\alpha,r} f$ for any complex number $\alpha$ via the relation $(M_{\alpha,r} f)\hat{\ }(\xi) = \hat{m}_\alpha(r\xi)\, \hat{f}(\xi),\ \xi \in \mathbf{R}^n$.

Fact 2.2C $M_{0,r} f = \frac{1}{2} M_r f$.

Proof. Fix $x \in \mathbf{R}^n$. For each $\alpha > 0$, we have

$$\begin{aligned}
M_{\alpha,r} f(x) &= (m_{\alpha,r} * f)(x) \\
&= \int_{\mathbf{R}^n} m_{\alpha,r}(y)\, f(x-y)\, dy \\
&= \int_{\mathbf{R}^n} m_\alpha(y/r)\, r^{-n}\, f(x-y)\, dy \\
&= \int_{\mathbf{R}^n} m_\alpha(z)\, f(x-rz)\, dz \\
&= \int_0^\infty \int_{S^{n-1}} m_\alpha(s\sigma)\, f(x-rs\sigma)\, s^{n-1}\, d\sigma\, ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 \int_{S^{n-1}} (1-s^2)^{\alpha-1}\, f(x-rs\sigma)\, s^{n-1}\, d\sigma\, ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s^2)^{\alpha-1}\, G(s)\, ds \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s^2)^{\alpha-1}\, G(1)\, ds \\
&\qquad\qquad + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s^2)^{\alpha-1}\, \{G(s)-G(1)\}\, ds \\
&= \tfrac{1}{2}\, G(1)\, \frac{\Gamma(\frac{1}{2})}{\Gamma(\alpha+\frac{1}{2})} + \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s^2)^{\alpha-1}\, \{G(s)-G(1)\}\, ds,
\end{aligned}$$

where $G(s) = \int_{S^{n-1}} f(x-rs\sigma)\, s^{n-1}\, d\sigma$.

By continuity of $G$ at $s=1$, we have $|G(s)-G(1)| \le C\,(1-s),\ 0 \le s \le 1$, and so

$$\begin{aligned}
&\left| \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s^2)^{\alpha-1}\, \{G(s)-G(1)\}\, ds \right| \\
&\qquad \le \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s^2)^{\mathrm{Re}(\alpha)-1}\, |G(s)-G(1)|\, ds \\
&\qquad \le \frac{C}{\Gamma(\alpha)} \int_0^1 (1-s)^{\mathrm{Re}(\alpha)}\, (1+s)^{\mathrm{Re}(\alpha)-1}\, ds \\
&\qquad \to 0, \quad \text{as } \alpha \to 0.
\end{aligned}$$

Hence, we find that

$$M_{\alpha,r} f(x) \to \tfrac{1}{2}\, G(1), \quad \text{as } \alpha \to 0.$$

However, $G(1) = \int_{S^{n-1}} f(x - r\sigma)\, d\sigma = M_r f(x)$, and so we obtain

$$M_{\alpha,r} f(x) \to \tfrac{1}{2}\, M_r f(x), \quad \text{as } \alpha \to 0,$$

which proves the fact. □

## §2.3 Variants of $\mathcal{M}$

In analogy with $\mathcal{M}$, we define the operator $\mathcal{M}_\alpha$ on $\mathcal{S}(\mathbf{R}^n)$ by

$$\mathcal{M}_\alpha f(x) = \sup_{r \in \mathbf{R}^+} |M_{\alpha,r} f(x)|, \quad x \in \mathbf{R}^n,$$

for any complex number $\alpha$.

We have the following theorem.

Theorem 2.3 (Stein)

The inequality

$$\|\mathcal{M}_\alpha f\|_p \le C_{\alpha,p}\, \|f\|_p, \quad f \in \mathcal{S}(\mathbf{R}^n),$$

holds whenever

(a) $\mathrm{Re}(\alpha) > 1 - n + n/p$, for $1 < p \le 2$, and

(b) $\mathrm{Re}(\alpha) > (2 - n)/p$, for $2 \le p \le \infty$.

*Remark.* When $\mathrm{Re}(\alpha) = 0$, the inequality holds for $n/(n-1) < p \le \infty$. This and Fact 2.2C imply that Theorem 2.1 is in fact a special case of Theorem 2.3.

## §2.4 The $g$-function argument and the $L^2$-inequality

For each $\alpha \in \mathbf{C}$, let $\phi_\alpha$ be a smooth compactly supported function such that $\hat{\phi}_\alpha(0) = \hat{m}_\alpha(0)$, and define the function $g_\alpha f$ by

$$g_\alpha f(x) = \left\{ \int_0^\infty |M_{\alpha,r} f(x) - (\phi_{\alpha,r} * f)(x)|^2 \, r^{-1} \, dr \right\}^{\frac{1}{2}}, \quad x \in \mathbf{R}^n,$$

with $\phi_{\alpha,r}(x) = \phi_\alpha(x/r)\, r^{-n},\ x \in \mathbf{R}^n$.

We have the following lemmas.

Lemma 2.4A (Stein)

The inequality

$$\|g_\alpha f\|_2 \le C_\alpha \, \|f\|_2, \quad f \in \mathcal{S}(\mathbf{R}^n),$$

holds whenever $\mathrm{Re}(\alpha) > (1-n)/2$.

Proof. By Fubini's and Plancherel's theorems, we have

$$\begin{aligned}
\|g_\alpha f\|_2^2 &= \int_{\mathbf{R}^n} \int_0^\infty |M_{\alpha,r} f(x) - (\phi_{\alpha,r} * f)(x)|^2 \, r^{-1} \, dr \, dx \\
&= \int_0^\infty \int_{\mathbf{R}^n} |M_{\alpha,r} f(x) - (\phi_{\alpha,r} * f)(x)|^2 \, dx \, r^{-1} \, dr \\
&= \int_0^\infty \|M_{\alpha,r} f - \phi_{\alpha,r} * f\|_2^2 \, r^{-1} \, dr \\
&= \int_0^\infty \|(M_{\alpha,r} f)\hat{\ } - (\phi_{\alpha,r} * f)\hat{\ }\|_2^2 \, r^{-1} \, dr \\
&= \int_0^\infty \|\hat{m}_{\alpha,r} \, \hat{f} - \hat{\phi}_{\alpha,r} \, \hat{f}\|_2^2 \, r^{-1} \, dr \\
&= \int_0^\infty \int_{\mathbf{R}^n} |\hat{m}_\alpha(r\xi) - \hat{\phi}_\alpha(r\xi)|^2 \, |\hat{f}(\xi)|^2 \, d\xi \, r^{-1} \, dr \\
&= \int_{\mathbf{R}^n} \int_0^\infty |\hat{m}_\alpha(r\xi) - \hat{\phi}_\alpha(r\xi)|^2 \, r^{-1} \, dr \ |\hat{f}(\xi)|^2 \, d\xi.
\end{aligned}$$

Thus we need to show that $\displaystyle\int_0^\infty |\hat{m}_\alpha(r\xi) - \hat{\phi}_\alpha(r\xi)|^2 \, r^{-1} \, dr \le C_\alpha,\ \xi \in \mathbf{R}^n$.

To do so, let us assume that $|\xi| = 1$ and split the integral into two parts, the first deals with $r \leq 1$ while the second deals with $r > 1$. For the first one, we have

$$\int_0^1 |\hat{m}_\alpha(r\xi) - \hat{\phi}_\alpha(r\xi)|^2 \, r^{-1} \, dr \leq \int_0^1 C \, dr \leq C,$$

by the smoothness of $\hat{m}_\alpha$ and $\hat{\phi}_\alpha$ and the assumption that $\hat{m}_\alpha(0) = \hat{\phi}_\alpha(0)$. Next, to tackle the second one, we use Fact 2.2A together with the fact that $J_z(r) = O(r^{-\frac{1}{2}})$, $r > 1$, whenever $\mathrm{Re}(z) > -\frac{1}{2}$ (see [36], p. 158), and the properties of $\phi_\alpha$ to have

$$|\hat{m}_\alpha(r\xi) - \hat{\phi}_\alpha(r\xi)| \leq C_\alpha \, r^{-n/2-\mathrm{Re}(\alpha)+\frac{1}{2}}, \quad r > 1,$$

provided $\mathrm{Re}(\alpha) > (1-n)/2$. It then follows that

$$\int_1^\infty |\hat{m}_\alpha(r\xi) - \hat{\phi}_\alpha(r\xi)|^2 \, r^{-1} \, dr \leq C_\alpha \int_1^\infty r^{-n-2\mathrm{Re}(\alpha)} \, dr \leq C_\alpha,$$

whenever $\mathrm{Re}(\alpha) > (1-n)/2$.

For arbitrary $\xi \neq 0$, one may substitute $\sigma = \xi/|\xi|$ and still obtain the same result. The lemma is therefore proved. □

Lemma 2.4B (Stein)

The inequality

$$\left\| \sup_{r \in \mathbf{R}^+} \left\{ r^{-1} \int_0^r |M_{\alpha,s} f(\,\cdot\,)|^2 \, ds \right\}^{\frac{1}{2}} \right\|_2 \leq C_\alpha \, \|f\|_2, \quad f \in \mathcal{S}(\mathbf{R}^n),$$

holds whenever $\mathrm{Re}(\alpha) > (1-n)/2$.

Proof. Fix $x \in \mathbf{R}^n$. For each $r \in \mathbf{R}^+$, we have

$$\left\{ r^{-1} \int_0^r |M_{\alpha,s}f(x)|^2\, ds \right\}^{\frac{1}{2}} \le \left\{ r^{-1} \int_0^r |M_{\alpha,s}f(x) - (\phi_{\alpha,s} * f)(x)|^2\, ds \right\}^{\frac{1}{2}} + \left\{ r^{-1} \int_0^r |(\phi_{\alpha,s} * f)(x)|^2\, ds \right\}^{\frac{1}{2}}.$$

However, we observe that

$$\begin{aligned} r^{-1} \int_0^r |M_{\alpha,s}f(x) - (\phi_{\alpha,s} * f)(x)|^2\, ds &\le \int_0^r |M_{\alpha,s}f(x) - (\phi_{\alpha,s} * f)(x)|^2\, s^{-1}\, ds \\ &\le \int_0^\infty |M_{\alpha,s}f(x) - (\phi_{\alpha,s} * f)(x)|^2\, s^{-1}\, ds \\ &= g_\alpha^2 f(x), \end{aligned}$$

and

$$\begin{aligned} r^{-1} \int_0^r |(\phi_{\alpha,s} * f)(x)|^2\, ds &\le r^{-1} \int_0^r \left\{ \sup_{s \in \mathbf{R}^+} |(\phi_{\alpha,s} * f)(x)| \right\}^2 ds \\ &= \left\{ \sup_{s \in \mathbf{R}^+} |(\phi_{\alpha,s} * f)(x)| \right\}^2. \end{aligned}$$

So, taking the supremum in $r$, we get

$$\sup_{r \in \mathbf{R}^+} \left\{ r^{-1} \int_0^r |M_{\alpha,s}f(x)|^2\, ds \right\}^{\frac{1}{2}} \le g_\alpha f(x) + \sup_{s \in \mathbf{R}^+} |(\phi_{\alpha,s} * f)(x)|,$$

whence (by Lemma 2.4A and Theorem 1.5A)

$$\begin{aligned} \left\| \sup_{r \in \mathbf{R}^+} \left\{ r^{-1} \int_0^r |M_{\alpha,s}f(\,\cdot\,)|^2\, ds \right\}^{\frac{1}{2}} \right\|_2 &\le \|g_\alpha f\|_2 + \left\| \sup_{s \in \mathbf{R}^+} |(\phi_{\alpha,s} * f)(\,\cdot\,)| \right\|_2 \\ &\le C_\alpha\, \|f\|_2, \end{aligned}$$

whenever $\mathrm{Re}(\alpha) > (1-n)/2$. □

Lemma 2.4C (Stein)

If $\mathrm{Re}(\alpha) > \mathrm{Re}(\alpha')$, then for every $x \in \mathbf{R}^n$ we have the identity

$$M_{\alpha,r}f(x) = \frac{2}{\Gamma(\alpha - \alpha')} \int_0^1 M_{\alpha',rs}f(x)\, (1-s^2)^{\alpha-\alpha'-1}\, s^{n+2\alpha'-1}\, ds.$$

Proof. Fix $x \in \mathbf{R}^n$. With $H(t) = \int_{S^{n-1}} f(x - rt\sigma)\, d\sigma$, we may express

$$M_{\alpha,r} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-t^2)^{\alpha-1}\, t^{n-1}\, H(t)\, dt.$$

Let us now put

$$I(x) = \frac{2}{\Gamma(\alpha-\alpha')} \int_0^1 M_{\alpha',rs} f(x)\, (1-s^2)^{\alpha-\alpha'-1}\, s^{n+2\alpha'-1}\, ds.$$

Then clearly $I(x) =$

$$\frac{2}{\Gamma(\alpha-\alpha')\,\Gamma(\alpha')} \int_0^1 \int_0^1 (1-t^2)^{\alpha'-1}\, t^{n-1}\, H(st)\, dt\, (1-s^2)^{\alpha-\alpha'-1}\, s^{n+2\alpha'-1}\, ds.$$

Now consider the bijection $(s,t) \mapsto (u,v)$ from $[0,1]^2$ to $[0,1]^2$ given by

$$u = st, \quad v = s(1-t^2)^{\frac{1}{2}}(1-s^2t^2)^{-\frac{1}{2}}.$$

With $h(u,v) = u^2 + v^2 - u^2v^2$, we observe that

$$s = h^{\frac{1}{2}}, \quad t = uh^{-\frac{1}{2}},$$

$$1 - s^2 = (1-u^2)(1-v^2), \quad 1 - t^2 = (1-u^2)v^2h^{-1},$$

and

$$\frac{\partial s}{\partial u} = \frac{1}{2}\, h^{-\frac{1}{2}}\, \frac{\partial h}{\partial u}, \quad \frac{\partial s}{\partial v} = \frac{1}{2}\, h^{-\frac{1}{2}}\, \frac{\partial h}{\partial v},$$

$$\frac{\partial t}{\partial u} = h^{-\frac{1}{2}} - \frac{1}{2}\, uh^{-\frac{3}{2}}\, \frac{\partial h}{\partial u}, \quad \frac{\partial t}{\partial v} = -\frac{1}{2}\, uh^{-\frac{3}{2}}\, \frac{\partial h}{\partial v},$$

giving

$$\left| \frac{\partial(s,t)}{\partial(u,v)} \right| = \frac{1}{2}\, h^{-1}\, \frac{\partial h}{\partial v} = (1-u^2)vh^{-1}.$$

Using this change of variables, we rewrite $I(x) =$

$$\frac{2}{\Gamma(\alpha-\alpha')\,\Gamma(\alpha')} \int_0^1 \int_0^1 v^{2\alpha'-1}\, (1-v^2)^{\alpha-\alpha'-1}\, dv\ (1-u^2)^{\alpha-1}\, u^{n-1}\, H(u)\, du.$$

But, on substituting $w = v^2$, we have

$$\int_0^1 v^{2\alpha'-1}\,(1-v^2)^{\alpha-\alpha'-1}\,dv = \frac{1}{2}\int_0^1 w^{\alpha'-1}\,(1-w)^{\alpha-\alpha'-1}\,dw$$
$$= \frac{\Gamma(\alpha')\,\Gamma(\alpha-\alpha')}{2\,\Gamma(\alpha)}.$$

We therefore find that

$$I(x) = \frac{1}{\Gamma(\alpha)}\int_0^1 (1-u^2)^{\alpha-1}\,u^{n-1}\,H(u)\,du = M_{\alpha,r}f(x),$$

whence the identity. □

Lemma 2.4B and Lemma 2.4C lead to the $L^2$-inequality.

Corollary 2.4D (Stein)

The inequality

$$\|\mathcal{M}_\alpha f\|_2 \le C_\alpha\,\|f\|_2, \quad f \in \mathcal{S}(\mathbf{R}^n),$$

holds whenever $\mathrm{Re}(\alpha) > 1 - n/2$.

Proof. Fix $x \in \mathbf{R}^n$. For each $r \in \mathbf{R}^+$, we have (by Lemma 2.4C and the Cauchy-Schwarz inequality)

$$
\begin{aligned}
&|M_{\alpha,r}f(x)| \\
=&\left|\frac{2}{\Gamma(\alpha-\alpha')}\int_0^1 M_{\alpha',rs}f(x)\,(1-s^2)^{\alpha-\alpha'-1}\,s^{n+2\alpha'-1}\,ds\right| \\
\le&\frac{2}{\Gamma(\alpha-\alpha')}\int_0^1 |M_{\alpha',rs}f(x)|\,(1-s^2)^{\alpha-\alpha'-1}\,s^{n+2\alpha'-1}\,ds \\
\le&\frac{2}{\Gamma(\alpha-\alpha')}\left\{\int_0^1 |M_{\alpha',rs}f(x)|^2\,ds\right\}^{\frac{1}{2}}\left\{\int_0^1 (1-s^2)^{2(\alpha-\alpha'-1)}\,s^{2(n+2\alpha'-1)}\,ds\right\}^{\frac{1}{2}} \\
\le&C_\alpha\left\{r^{-1}\int_0^r |M_{\alpha',t}f(x)|^2\,dt\right\}^{\frac{1}{2}},
\end{aligned}
$$

provided $\mathrm{Re}(\alpha) > \mathrm{Re}(\alpha') + \frac{1}{2}$. Thus, taking the supremum in $r$, we obtain

$$\mathcal{M}_\alpha f(x) \leq C_\alpha \sup_{r\in\mathbf{R}^+} \left\{ r^{-1} \int_0^r |M_{\alpha',t} f(x)|^2 \, dt \right\}^{\frac{1}{2}},$$

whence (by Lemma 2.4B)

$$\|\mathcal{M}_\alpha f\|_2 \leq C_\alpha \, \|f\|_2,$$

provided $\mathrm{Re}(\alpha) > 1 - n/2$. □

## §2.5 The $L^p$-inequality

Before verifying the $L^p$-inequality, we note that besides the $L^2$-estimate we also have the following two.

FACT 2.5A  For $p > 1$, and $\mathrm{Re}(\alpha) \geq 1$, $\|\mathcal{M}_\alpha f\|_p \leq C_p \, \|f\|_p, \; f \in \mathcal{S}(\mathbf{R}^n)$.

PROOF. When $\mathrm{Re}(\alpha) \geq 1$, we see that $|m_\alpha(x)| \leq 1$ for all $x \in \mathbf{R}^n$. Hence $\mathcal{M}_\alpha f$ is essentially majorized by the Hardy-Littlewood maximal function $\mathcal{M} f$, and so the estimate follows. □

FACT 2.5B  For $\mathrm{Re}(\alpha) > 0$, $\|\mathcal{M}_\alpha f\|_\infty \leq C_\alpha \, \|f\|_\infty, \; f \in \mathcal{S}(\mathbf{R}^n)$.

PROOF. When $\mathrm{Re}(\alpha) > 0$, we have $\int_{\mathbf{R}^n} |m_\alpha(x)| \, dx \leq C_\alpha$. The estimate then follows immediately from this. □

We shall now verify the $L^p$-inequality by using the Stein's analytic interpolation theorem involving complex $\alpha$ (see [36], p. 205).

Set $p_0 = 1+\epsilon,\ p_1 = 2,\ a_0 = 1$, and $a_1 = 1-n/2$. Then, if $a = a_0(1-t)+a_1 t$ and $1/p = (1-t)/p_0 + t/p_1$, with $0 < t < 1$, then the inequality

$$\|\mathcal{M}_\alpha f\|_p \le C_{\alpha,p}\,\|f\|_p, \quad f \in \mathcal{S}(\mathbf{R}^n),$$

holds whenever $\mathrm{Re}(\alpha) > 1-n+n/p+\delta(p,\epsilon)$, for $1+\epsilon < p \le 2$. By continuity, the inequality holds whenever $\mathrm{Re}(\alpha) > 1-n+n/p$, for $1 < p \le 2$.

Similarly, set $p_0 = 2,\ p_1 = \infty,\ a_0 = 1-n/2$, and $a_1 = 0$. If $a = a_0(1-t)$ and $1/p = (1-t)/2$, with $0 < t < 1$, then the inequality

$$\|\mathcal{M}_\alpha f\|_p \le C_{\alpha,p}\,\|f\|_p, \quad f \in \mathcal{S}(\mathbf{R}^n),$$

holds whenever $\mathrm{Re}(\alpha) > (2-n)/p$, for $2 \le p \le \infty$. □

## §2.6 An extension from $\mathcal{S}(\mathbf{R}^n)$ to $L^p(\mathbf{R}^n)$

We shall here extend the result, which we have restricted to the class $\mathcal{S}(\mathbf{R}^n)$, to all $L^p(\mathbf{R}^n)$ functions.

Lemma 2.6A

Let $E \subset \mathbf{R}^n$ be an open set with finite measure and set $f = \chi_E$. Then $\mathcal{M}f$ is measurable and satisfies the $L^p$-inequality.

Proof. Recalling the definition of $M_r f$ and $\mathcal{M}f$, we note that $M_r f$ is defined for all $r \in \mathbf{R}^+$, and so is $\mathcal{M}f$. Now take a sequence of smooth functions $\{f_k\}$ increasing to $f$. By the monotone convergence theorem,

$$\lim_{k\to\infty} M_r f_k(x) = M_r f(x), \quad x \in \mathbf{R}^n,$$

for every $r \in \mathbf{R}^+$. Hence,

$$\liminf_{k\to\infty} \mathcal{M}f_k(x) \geq M_r f(x), \quad x \in \mathbf{R}^n,$$

for every $r \in \mathbf{R}^+$, giving

$$\liminf_{k\to\infty} \mathcal{M}f_k(x) \geq \mathcal{M}f(x), \quad x \in \mathbf{R}^n.$$

On the other hand, we have

$$\mathcal{M}f_k(x) \leq \mathcal{M}f(x), \quad x \in \mathbf{R}^n,$$

for each $k = 1, 2, 3, \ldots$. So we conclude that

$$\mathcal{M}f(x) = \lim_{k\to\infty} \mathcal{M}f_k(x), \quad x \in \mathbf{R}^n.$$

Therefore, by the dominated convergence theorem, $\mathcal{M}f$ is measurable and

$$\|\mathcal{M}f\|_p = \| \lim_{k\to\infty} \mathcal{M}f_k\|_p = \lim_{k\to\infty} \|\mathcal{M}f_k\|_p \leq C \lim_{k\to\infty} \|f_k\|_p = C\,\|f\|_p,$$

whenever $n/(n-1) < p \leq \infty$. □

LEMMA 2.6B

Let $F \subset \mathbf{R}^n$ be a set of measure zero and set $g = \chi_F$. Then $\mathcal{M}g$ is measurable and $\mathcal{M}g(x) = 0$ almost everywhere.

PROOF. By regularity, there exists a sequence of open sets $\{E_k\}$ satisfying $E_{k+1} \subseteq E_k$, $m(E_k) < \frac{1}{k}$, and $F \subseteq \bigcap E_k$. Now put $f_k = \chi_{E_k}$ and $f = \chi_E$, where $E = \bigcap E_k$. It then suffices to show that $\mathcal{M}f(x) = 0$ for almost every $x \in \mathbf{R}^n$. Since $\{f_k\}$ decreases to $f$, we have

$$0 \leq \mathcal{M}f(x) \leq \liminf_{k\to\infty} \mathcal{M}f_k(x), \quad x \in \mathbf{R}^n.$$

However, when $n/(n-1) < p \leq \infty$, we have

$$\| \liminf_{k\to\infty} \mathcal{M} f_k \|_p \leq \liminf_{k\to\infty} \|\mathcal{M} f_k\|_p \leq C \liminf_{k\to\infty} \|f_k\|_p \leq C \liminf_{k\to\infty} \tfrac{1}{k} = 0,$$

which implies that $\liminf_{k\to\infty} \mathcal{M} f_k(x) = 0$ almost everywhere. We therefore conclude that $\mathcal{M} f(x) = 0$ almost everywhere. □

*Remark.* Lemma 2.6B means that if $f(x) = g(x)$ almost everywhere, then $\mathcal{M} f(x) = \mathcal{M} g(x)$ almost everywhere.

Corollary 2.6C

If $f$ is the characteristic function of a measurable set $E$, then $\mathcal{M} f$ is measurable and satisfies the $L^p$-inequality.

Corollary 2.6D

If $f$ is a simple measurable function, then $\mathcal{M} f$ is measurable and satisfies the $L^p$-inequality.

Corollary 2.6E

If $f$ is a non-negative measurable function, then $\mathcal{M} f$ is measurable and satisfies the $L^p$-inequality.

Corollary 2.6F

If $f$ is a measurable function, then $\mathcal{M} f$ is measurable and satisfies the $L^p$-inequality.

*Remark.* Stein and Wainger [35] show that this implies that $\lim_{r\to 0} M_r f(x) = f(x)$ almost everywhere, for any $f \in L^p(\mathbf{R}^n)$, with $n/(n-1) < p \leq \infty$.