CHAPTER III

SPHERICAL MAXIMAL FUNCTIONS – II

The Mellin transform technique

The aim of this chapter is to present a different approach to the study of maximal functions, which is based on the use of the Mellin transform, due to Cowling and Mauceri [11].

§3.1 Introduction: The use of the Mellin transform

For each $u \in \mathbf{R}$, let K_u be the distribution on $\mathbf{R}^n$ whose Fourier transform is the function $\widehat{K}_u(\xi) = |\xi|^{-iu},\ \xi \in \mathbf{R}^n$. Following [33], p. 117, we have, formally, that

$$K_u(x) = C(u)\,|x|^{-n+iu}, \quad x \in \mathbf{R}^n,$$

where $C(u) = \pi^{-\frac{n}{2}+iu}\,\Gamma(\frac{n-iu}{2})\big/\Gamma(\frac{iu}{2})$. By considering the Fourier transform, it is clear that

$$\int_{\mathbf{R}^n} K_u(x)\,f(x)\,dx = \lim_{\epsilon\to 0^+}\int_{\mathbf{R}^n} K_{\epsilon,u}(x)\,f(x)\,dx, \quad f \in \mathcal{S}(\mathbf{R}^n),$$

where $K_{\epsilon,u}(x) = C(u)\,|x|^{-n+\epsilon+iu},\ x \in \mathbf{R}^n$.

Now let ϕ be the normalized surface measure on S^{n-1} and P_r be the Poisson kernel at $r \in \mathbf{R}^+$ (see for example [33], p. 61). Then, using the Mellin transform, we may write

$$(\phi - P_1)(x) = \int_{\mathbf{R}} A(u)\, K_u(x)\, du, \quad x \in \mathbf{R}^n.$$

As observed in [11], p. 80, the above relation holds if and only if

$$(\omega_{n-1}^{-1}\, \delta_1 - P_1)(s) = \int_{\mathbf{R}} A(u)\, C(u)\, s^{-n+iu}\, du, \quad s \in \mathbf{R}^+,$$

if and only if

$$(\omega_{n-1}^{-1}\, \delta_1 - P_1)(e^{2\pi t})\, e^{2\pi n t} = \int_{\mathbf{R}} A(u)\, C(u)\, e^{2\pi i t u}\, du, \quad t \in \mathbf{R},$$

if and only if

$$A(u)\, C(u) = \int_{\mathbf{R}} (\omega_{n-1}^{-1}\, \delta_1 - P_1)(e^{2\pi t})\, e^{2\pi(n-iu)t}\, dt, \quad u \in \mathbf{R},$$

if and only if

$$2\pi\, A(u)\, C(u) = \int_{\mathbf{R}^+} (\omega_{n-1}^{-1}\, \delta_1 - P_1)(s)\, s^{n-1-iu}\, ds, \quad u \in \mathbf{R},$$

where δ_1 denotes the point mass at 1. All these results hold in the distribution sense, with K_u defined to be the limit of $K_{\epsilon,u}$ as ϵ tends to 0^+.

§3.2 The behaviour of $A(u)$ and $C(u)$

We have here some facts about $A(u)$ and $C(u)$.

FACT 3.2A $\quad C(u) = O(1 + |u|)^{\frac{n}{2}},\ u \in \mathbf{R}$.

PROOF. Obvious (see p. 151 of [37] for details of the asymptotic behaviour of the gamma function). □

Fact 3.2B $\quad 4\pi^{\frac{n+3}{2}} A(u)\,C(u) = \Gamma(\frac{n}{2})\Gamma(\frac{1}{2}) - \Gamma(\frac{n-iu}{2})\Gamma(\frac{1+iu}{2})$.

Proof. Since $\int_{\mathbf{R}^+} g(s)\,\delta_1(s)\,ds = g(1)$ whenever g is continuous, we have

$$\int_{\mathbf{R}^+} \omega_{n-1}^{-1}\,\delta_1(s)\,s^{n-1-iu}\,ds = \omega_{n-1}^{-1} = \tfrac{1}{2}\,\pi^{-\frac{n}{2}}\,\Gamma(\tfrac{n}{2}).$$

Now, recalling that $P_1(s) = \kappa_n\,(1+s^2)^{-\frac{n+1}{2}}$, with $\kappa_n = \pi^{-\frac{n+1}{2}}\,\Gamma(\frac{n+1}{2})$ (see again [33], p. 61), and making the substitution $1-t = (1+s^2)^{-1}$, we have

$$\begin{aligned}\int_{\mathbf{R}^+} P_1(s)\,s^{n-1-iu}\,ds &= \int_{\mathbf{R}^+} \kappa_n\,(1+s^2)^{-\frac{n+1}{2}}\,s^{n-1-iu}\,ds \\ &= \tfrac{1}{2}\,\kappa_n \int_0^1 (1-t)^{-\frac{1-iu}{2}}\,t^{\frac{n-2-iu}{2}}\,dt \\ &= \tfrac{1}{2}\,\kappa_n\,\Gamma(\tfrac{n-iu}{2})\Gamma(\tfrac{1+iu}{2})/\Gamma(\tfrac{n+1}{2}) \\ &= \tfrac{1}{2}\,\pi^{-\frac{n+1}{2}}\,\Gamma(\tfrac{n-iu}{2})\Gamma(\tfrac{1+iu}{2}).\end{aligned}$$

It then follows that

$$2\pi\,A(u)\,C(u) = \tfrac{1}{2}\,\pi^{-\frac{n+1}{2}}\,\{\Gamma(\tfrac{n}{2})\Gamma(\tfrac{1}{2}) - \Gamma(\tfrac{n-iu}{2})\Gamma(\tfrac{1+iu}{2})\},$$

verifying Fact 3.2B. □

Fact 3.2C $\quad A(u) = O(1+|u|)^{-\frac{n}{2}},\ u \in \mathbf{R}$.

Proof. This follows immediately from Fact 3.2A and Fact 3.2B above. □

§3.3 The behaviour of the kernels K_u

First of all, recall that $K_u(x) = C(u)\,|x|^{-n+iu},\ x \in \mathbf{R}^n$, where $C(u) = \pi^{-\frac{n}{2}+iu}\,\Gamma(\frac{n-iu}{2})/\Gamma(\frac{iu}{2})$, and that $\widehat{K}_u(\xi) = |\xi|^{-iu},\ \xi \in \mathbf{R}^n$. Also note that the function $x \mapsto |x|^{-n+iu}$ is locally integrable away from 0.

Moreover, we have the following fact regarding K_u.

FACT 3.3A $\int_{|x|\geq 2|y|} |K_u(x-y) - K_u(x)|\, dx \leq C\,(1+|u|)^{\frac{n}{2}}\,\log(4+|u|)$.

PROOF. We split the integral into two parts: the first is over the region $2|y| \leq |x| \leq (3+|u|)|y|$, and the second deals with the complement. Then, using Fact 3.2A, we have

$$\begin{aligned}
&\int_{2|y|\leq|x|\leq(3+|u|)|y|} |K_u(x-y) - K_u(x)|\, dx \\
&\qquad = \int_{2|y|\leq|x|\leq(3+|u|)|y|} |C(u)|\, \left||x-y|^{-n+iu} - |x|^{-n+iu}\right| dx \\
&\qquad \leq C\,(1+|u|)^{\frac{n}{2}} \int_{|y|\leq|x|\leq(4+|u|)|y|} |x|^{-n}\, dx \\
&\qquad = C\,(1+|u|)^{\frac{n}{2}}\,\omega_{n-1}\,\log(4+|u|) \\
&\qquad \leq C\,(1+|u|)^{\frac{n}{2}}\,\log(4+|u|),
\end{aligned}$$

and

$$\begin{aligned}
&\int_{|x|>(3+|u|)|y|} |K_u(x-y) - K_u(x)|\, dx \\
&\qquad = \int_{|x|>(3+|u|)|y|} |\nabla K_u(x - t_x y)|\, |y|\, dx \qquad (0 < t_x < 1) \\
&\qquad = \int_{|x|>(3+|u|)|y|} |C(u)|\, |n - iu|\, |y|\, |x - t_x y|^{-n-1}\, dx \\
&\qquad \leq C\,(1+|u|)^{\frac{n}{2}} \int_{|x|>(3+|u|)|y|} |n - iu|\, |y|\, \big(|x| - |y|\big)^{-n-1}\, dx \\
&\qquad = C\,(1+|u|)^{\frac{n}{2}} \int_{|x|>(2+|u|)|y|} |n - iu|\, |y|\, |x|^{-n-1}\, dx \\
&\qquad = C\,(1+|u|)^{\frac{n}{2}}\, |n - iu|\, (2+|u|)^{-1}\, \omega_{n-1} \\
&\qquad \leq C\,(1+|u|)^{\frac{n}{2}},
\end{aligned}$$

whence the estimate. □

For each $f \in \mathcal{S}(\mathbf{R}^n)$, one may define the function $I_u f$ by

$$I_u f(x) = K_u * f(x), \quad x \in \mathbf{R}^n.$$

This function makes sense for all $u \neq 0$ via the relation

$$I_u f(x) = \lim_{\epsilon \to 0^+} K_{\epsilon,u} * f(x), \quad x \in \mathbf{R}^n,$$

where $K_{\epsilon,u}(x) = C(u)\,|x|^{-n+\epsilon+iu}$, $x \in \mathbf{R}^n$, or via the formula

$$(I_u f)\hat{} = \widehat{K}_u\,\hat{f};$$

(see [33], pp. 51 and 117 for more details).

We have the following lemma which gives an estimate for $I_u f$.

Lemma 3.3B (Cowling-Mauceri)

Fix $\epsilon > 0$, sufficiently small. Then, for $1 + \epsilon \leq p \leq 2$, we have

$$\|I_u f\|_p \leq C_{p,\epsilon}(u)\,\|f\|_p, \quad f \in L^p(\mathbf{R}^n),$$

where $C_{p,\epsilon}(u) = \frac{C}{\epsilon}\,(1 + |u|)^{\frac{1+\epsilon}{1-\epsilon}(n/p-n/2)}\,\log^{\frac{1+\epsilon}{1-\epsilon}(2/p-1)}(4 + |u|)$.

Remark. Cowling and Mauceri [11] actually obtained a stronger version of this lemma. Its proof, however, requires certain techniques of Coifman and Weiss [7] and Fefferman and Stein [15].

We first verify that I_u is of weak-type (1,1) and of type (2,2), and then later prove the lemma by interpolation.

FACT 3.3C $\quad \|I_u f\|_2 = \|f\|_2,\ f \in S(\mathbf{R}^n)$.

PROOF. The proof follows straight from the definition. Indeed, since $|\widehat{K}_u(\xi)| = 1$ for all $\xi \in \mathbf{R}^n$, we have (by Plancherel's theorem)

$$\|I_u f\|_2 = \|(I_u f)\hat{}\|_2 = \|\widehat{K}_u \hat{f}\|_2 = \|\hat{f}\|_2 = \|f\|_2, \quad f \in \mathcal{S}(\mathbf{R}^n),$$

as stated. $\square$

Remark. As a consequence of Fact 3.3C, we find that I_u extends to an isometry on $L^2(\mathbf{R}^n)$. In fact, if $f \in L^2(\mathbf{R}^n)$ is supported in a cube Q, then $I_u f(x) = \int_{\mathbf{R}^n} K_u(x-y)\, f(y)\, dy,\ x \in \mathbf{R}^n \setminus Q$, where K_u is defined pointwise, because the singularity of $K_u(x-y)$ occurs when $f(y) = 0$. This is evident if $f \in \mathcal{S}(\mathbf{R}^n)$, and the formula follows by a limiting process for square integrable f.

FACT 3.3D $\quad m\{x : |I_u f(x)| > \alpha\} \le \frac{C}{\alpha}\,(1+|u|)^{\frac{n}{2}} \log(4+|u|)\,\|f\|_1,$

$f \in L^1(\mathbf{R}^n) \cap L^2(\mathbf{R}^n)$.

PROOF. As suggested in Ch. II of [33], given $\alpha > 0$, we decompose $\mathbf{R}^n$ into F and Ω, so that $\mathbf{R}^n = F \cup \Omega$, and $F \cap \Omega = \emptyset$, and so that

(a) $|f(x)| \le \alpha$, for almost every $x \in F$,

(b) $\Omega = \bigcup_{j=1}^{\infty} Q_j$, with $Q_j^\circ \cap Q_k^\circ = \emptyset$, whenever $j \ne k$,

(c) $m(\Omega) \le \frac{C}{\alpha}\,\|f\|_1$, and

(d) $\frac{1}{m(Q_j)} \int_{Q_j} |f(x)|\, dx \le C\,\alpha$.

We then define

$$g(x) = \begin{cases} f(x), & \text{if } x \in F, \\ \frac{1}{m(Q_j)} \int_{Q_j} f(x)\, dx, & \text{if } x \in Q_j^\circ, \end{cases}$$

and put $h = f - g$. Now $f,\ g$ and h lie in $L^2(\mathbf{R}^n)$, on which I_u is well-defined. We see that $h(x) = 0$ if $x \in F$, and that $\int_{Q_j} h(x)\,dx = 0$ for each cube Q_j. Moreover, since $I_u f = I_u g + I_u h$, we have

$$m\{x : |I_u f(x)| > \alpha\} \le m\{x : |I_u g(x)| > \alpha/2\} + m\{x : |I_u h(x)| > \alpha/2\}.$$

We shall first estimate $I_u g$. Observe that

$$\begin{aligned}
\|g\|_2^2 &= \int_{\mathbf{R}^n} |g(x)|^2\,dx \\
&= \int_F |g(x)|^2\,dx + \int_\Omega |g(x)|^2\,dx \\
&\le \int_F \alpha\,|f(x)|\,dx + C\,\alpha^2\,m(\Omega) \\
&\le \alpha\,\|f\|_1 + C\,\alpha\,\|f\|_1 \\
&= C\,\alpha\,\|f\|_1.
\end{aligned}$$

So, applying Fact 3.3C to g, we obtain $\|I_u g\|_2 = \|g\|_2$, and hence (as remarked in Ch. I, §1.4)

$$m\{x : |I_u g(x)| > \alpha/2\} \le \frac{4}{\alpha^2}\,\|I_u g\|_2^2 = \frac{4}{\alpha^2}\,\|g\|_2^2 \le \frac{4C}{\alpha}\,\|f\|_1.$$

It now remains to estimate $I_u h$. For each $j = 1, 2, 3, \ldots,$ define

$$h_j(x) = \begin{cases} h(x), & \text{if } x \in Q_j, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly $h(x) = \sum_{j=1}^{\infty} h_j(x)$ and $I_u h(x) = \sum_{j=1}^{\infty} I_u h_j(x),\ x \in \mathbf{R}^n$. Now, since $\int_{Q_j} h_j(y)\,dy = 0$, we have

$$I_u h_j(x) = \int_{Q_j} \{K_u(x-y) - K_u(x-y_j)\}\,h_j(y)\,dy, \quad x \in \mathbf{R}^n \setminus Q_j,$$

with y_j being the centre of the cube Q_j.

For each cube Q_j, let Q_j^* be the cube which has the same centre y_j but is expanded $2n^{\frac{1}{2}}$ times. Then put $\Omega^* = \bigcup Q_j^*$ and $F^* = \mathbf{R}^n \setminus \Omega^*$. Clearly $\Omega \subset \Omega^*$, with $m(\Omega^*) \leq (2n^{\frac{1}{2}})^n\, m(\Omega)$, and $F^* \subset F$. Moreover, if $x \notin Q_j^*$, then $|x - y_j| \geq 2|y - y_j|$, for all $y \in Q_j$. It thus follows from Fact 3.3A that

$$\int_{F^*} |I_u h(x)|\, dx \leq C\,(1+|u|)^{\frac{n}{2}}\, \log(4+|u|)\, \|f\|_1.$$

We therefore obtain

$$m\{x \in F^* : |I_u h(x)| > \alpha/2\} \leq \frac{2C}{\alpha}\,(1+|u|)^{\frac{n}{2}}\, \log(4+|u|)\, \|f\|_1.$$

But since $m(\mathbf{R}^n \setminus F^*) = m(\Omega^*) \leq (2n^{\frac{1}{2}})^n\, m(\Omega) \leq \frac{C}{\alpha}\, \|f\|_1$, we conclude

$$m\{x : |I_u h(x)| > \alpha/2\} \leq \frac{C}{\alpha}\,(1+|u|)^{\frac{n}{2}}\, \log(4+|u|)\, \|f\|_1,$$

completing the proof of Fact 3.3D. □

We shall now prove the lemma using interpolation theorems.

Proof (of Lemma 3.3B). We wish to show that I_u can be defined on $L^1(\mathbf{R}^n) + L^2(\mathbf{R}^n)$, so as to be subadditive, of weak-type (1,1) and of type (2,2). If we can do this, then we will be able to use the Marcinkiewicz theorem.

Suppose that $f \in L^1(\mathbf{R}^n) + L^2(\mathbf{R}^n)$, that is, there exist $f_1 \in L^1(\mathbf{R}^n)$ and $f_2 \in L^2(\mathbf{R}^n)$ such that $f = f_1 + f_2$. Let S be any finite measurable subset of $\mathbf{R}^n$ such that $\chi_{\mathbf{R}^n \setminus S} f$ lies in $L^2(\mathbf{R}^n)$. Then $f = \chi_S f + \chi_{\mathbf{R}^n \setminus S} f$ is a good $L^1(\mathbf{R}^n) + L^2(\mathbf{R}^n)$ decomposition. Next, for each $k \in \mathbf{Z}^+$, define

$$f_{1k} = \begin{cases} f_1(x), & \text{if } |f_1(x)| \leq k, \\ k\, f_1(x)/|f_1(x)|, & \text{otherwise.} \end{cases}$$

Then clearly $f_1 = \lim_{k\to\infty} f_{1k}$, in L^1-norm. Moreover, since for any $\alpha > 0$, $m\{x : |I_u f_{1k}(x) - I_u f_{1k'}(x)| > \alpha\} \leq \frac{C}{\alpha} \|f_{1k} - f_{1k'}\|_1$, we find that $I_u f_{1k}$ converges in measure (as $k \to \infty$) to a function we call $I_u f_1$, and

$$m\{x : |I_u f_1(x)| > \alpha\} \leq \frac{C}{\alpha} \|f_1\|_1.$$

We now define $I_u f$ to be $I_u f_1 + I_u f_2$. It is not hard to see that $I_u f$ is actually well-defined (that is, if $f = f_3 + f_4$, then $I_u f = I_u f_3 + I_u f_4$), linear, and of weak-type (1,1) and of type (2,2). Hence, by the Marcinkiewicz interpolation theorem, we find that I_u is of type (q, q), whenever $1 < q < 2$. In particular, when q is close to 1, we have

$$\|I_u f\|_q \leq C_q(u) \|f\|_q, \quad f \in L^q(\mathbf{R}^n),$$

with $C_q(u) = \frac{C}{q-1} (1 + |u|)^{\frac{n}{2}} \log(4 + |u|)$.

If we now fix $q = 1 + \epsilon$ for some small $\epsilon > 0$, and then apply the M. Riesz convexity theorem (see [36], p. 179), we then obtain for all $1 + \epsilon \leq p \leq 2$

$$\|I_u f\|_p \leq C_{p,\epsilon}(u) \|f\|_p, \quad f \in L^p(\mathbf{R}^n),$$

where $C_{p,\epsilon} = \frac{C}{\epsilon} (1 + |u|)^{\frac{1+\epsilon}{1-\epsilon}(n/p - n/2)} \log^{\frac{1+\epsilon}{1-\epsilon}(2/p-1)} (4 + |u|)$.

This concludes the proof of the lemma. $\square$

§3.4 The proof of Stein's theorem

Let ϕ be the normalized surface measure on S^{n-1}. For a smooth function f on $\mathbf{R}^n$, we define the maximal function $\mathcal{M}_\phi f$ by

$$\mathcal{M}_\phi f(x) = \sup_{r \in \mathbf{R}^+} |(\phi_r * f)(x)|, \quad x \in \mathbf{R}^n,$$

where ϕ_r denotes the dilate of ϕ defined by duality

$$\int_{\mathbf{R}^n} \phi_r(x)\, f(x)\, dx = \int_{\mathbf{R}^n} \phi(x)\, f(rx)\, dx, \quad f \in \mathcal{S}(\mathbf{R}^n).$$

We rewrite Stein's result for the spherical maximal function as follows.

Theorem 3.4 (Stein-Cowling-Mauceri)

For $n \geq 3$, the *a priori* inequality

$$\|\mathcal{M}_\phi f\|_p \leq C_p\, \|f\|_p, \quad f \in \mathcal{S}(\mathbf{R}^n),$$

holds for all $n/(n-1) < p \leq \infty$.

Proof. With the notation established earlier, we may write, at least formally,

$$\phi(x) = P_1(x) + \int_{\mathbf{R}} A(u)\, K_u(x)\, du, \quad x \in \mathbf{R}^n.$$

Accordingly, for every $r \in \mathbf{R}^+$, we have

$$\phi_r(x) = P_r(x) + \int_{\mathbf{R}} A(u)\, K_u(x)\, r^{-iu}\, du, \quad x \in \mathbf{R}^n.$$

Now, whenever $f \in \mathcal{S}(\mathbf{R}^n)$, we have

$$(\phi_r * f)(x) = (P_r * f)(x) + \int_{\mathbf{R}} A(u)\, I_u f(x)\, r^{-iu}\, du, \quad x \in \mathbf{R}^n,$$

and consequently

$$|(\phi_r * f)(x)| \leq |(P_r * f)(x)| + \int_{\mathbf{R}} |A(u)|\, |I_u f(x)|\, du, \quad x \in \mathbf{R}^n.$$

Hence we find that

$$\mathcal{M}_\phi f(x) \leq \mathcal{M}_{P_1} f(x) + \int_{\mathbf{R}} |A(u)|\, |I_u f(x)|\, du, \quad x \in \mathbf{R}^n,$$

whence (by Minkowski's inequality),

$$\|\mathcal{M}_\phi f\|_p \le \|\mathcal{M}_{P_1} f\|_p + \int_{\mathbf{R}} |A(u)| \, \|I_u f\|_p \, du.$$

By Theorem 1.5A (with $\phi = P_1$), we see that $\mathcal{M}_{P_1} f$ is majorized by the Hardy-Littlewood maximal function. Thus, to prove the L^p-inequality for $1 < p \le 2$, it suffices to show that

$$\int_{\mathbf{R}} |A(u)| \, \|I_u f\|_p \, du \le C_p \, \|f\|_p$$

or, by Lemma 3.3B, to show that

$$\int_{\mathbf{R}} |A(u)| \, C_{p,\epsilon}(u) \, du < \infty$$

for $1 + \epsilon \le p \le 2$, with $C_{p,\epsilon}(u)$ as in the lemma.

By simple observation, one has $\log(4 + |u|) \le \frac{C}{\epsilon} (1 + |u|)^\epsilon$, $u \in \mathbf{R}^n$, for any $\epsilon > 0$, and so it follows that

$$C_{p,\epsilon}(u) \le C_{p,\epsilon} \, (1 + |u|)^{n/p - n/2 + \delta(p,\epsilon)},$$

with $\delta(p, \epsilon) \to 0$, as $\epsilon \to 0$. Now, recalling Fact 3.2C, we obtain

$$\int_{\mathbf{R}} |A(u)| \, C_{p,\epsilon}(u) \, du \le C_{p,\epsilon} \int_{\mathbf{R}} (1 + |u|)^{n/p - n + \delta(p,\epsilon)} \, du < \infty,$$

provided that $n/p - n + \delta(p, \epsilon) < -1$ or $p > n/(n - 1 - \delta(p, \epsilon))$.

This therefore verifies the L^p-inequality for $n/(n - 1) < p \le 2$.

We conclude the proof by noting that the L^p-inequality for $2 < p < \infty$ follows from the Marcinkiewicz interpolation theorem (which we may apply since we trivially have $\|\mathcal{M}_\phi f\|_\infty \le C \, \|f\|_\infty$ whenever $f \in L^\infty(\mathbf{R}^n)$). □

Remark. This proof can be used to deal with L^p-functions f (see [11] for details). The basic fact is that the formal expressions above can be justified as vector-valued integrals in general.

§3.5 The pointwise convergence result

We shall now show how the Mellin transform technique also leads to the pointwise convergence result, that is

$$\lim_{r\to 0}(\phi_r * f)(x) = f(x), \quad \text{a.e.},$$

whenever $f \in L^p(\mathbf{R}^n)$.

We have already seen that whenever $f \in \mathcal{S}(\mathbf{R}^n)$, we have

$$(\phi_r * f)(x) = (P_r * f)(x) + \int_{\mathbf{R}} A(u)\, I_u f(x)\, r^{-iu}\, du, \quad x \in \mathbf{R}^n,$$

which also makes sense as a vector-valued integral for $f \in L^p(\mathbf{R}^n)$. However, by Theorem 1.5B(b), we see that

$$\lim_{r\to 0}(P_r * f)(x) = f(x), \quad \text{a.e.},$$

and thus we only need to show that

$$\lim_{r\to 0}\int_{\mathbf{R}} A(u)\, I_u f(x)\, r^{-iu}\, du = 0, \quad \text{a.e.}$$

To do so, we fix a subset S of $\mathbf{R}^n$ which has finite measure. Then

$$(\phi_r * f - P_r * f)(x)\,\chi_S(x) = \int_{\mathbf{R}} A(u)\, I_u f(x)\,\chi_S(x)\, r^{-iu}\, du, \quad x \in \mathbf{R}^n,$$

and (by Fubini's theorem and Hölder's inequality)

$$\begin{aligned}
&\int_{\mathbf{R}^n}\int_{\mathbf{R}} |A(u)\, I_u f(x)\, \chi_S(x)|\, du\, dx \\
&\qquad = \int_{\mathbf{R}}\int_{\mathbf{R}^n} |A(u)\, I_u f(x)\, \chi_S(x)|\, dx\, du \\
&\qquad = \int_{\mathbf{R}}\int_{S} |A(u)\, I_u f(x)|\, dx\, du \\
&\qquad \le \int_{\mathbf{R}} \left\{ \int_S |A(u)\, I_u f(x)|^p\, dx \right\}^{\frac{1}{p}} \left\{ \int_S dx \right\}^{\frac{1}{p'}} du \\
&\qquad \le \{m(S)\}^{\frac{1}{p'}} \int_{\mathbf{R}} |A(u)|\, \|I_u f\|_p\, du \\
&\qquad \le C_p\, \|f\|_p \\
&\qquad = C^*, \quad \text{say},
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{p'} = 1,\ p \ge n/(n-1)$. Thus the map $(u, x) \mapsto A(u)\, I_u f(x)\, \chi_S(x)$ lies in $L^1(\mathbf{R} \times S)$. As is well-known, this implies that there exist functions $h_j \in L^1(\mathbf{R})$ and $k_j \in L^1(S)$ $(j = 1, 2, 3, \ldots)$ such that

$$\sum_{j=1}^{\infty} \|h_j\|_1\, \|k_j\|_1 \le 2C^*$$

and

$$A(u)\, I_u f(x)\, \chi_S(x) = \sum_{j=1}^{\infty} h_j(u)\, k_j(x), \quad \text{a.e.}$$

Consequently,

$$\begin{aligned}
(\phi_r * f - P_r * f)(x) &= \sum_{j=1}^{\infty} \int_{\mathbf{R}} h_j(u)\, k_j(x)\, r^{-iu}\, du \\
&= \sum_{j=1}^{\infty} k_j(x) \int_{\mathbf{R}} h_j(u)\, r^{-iu}\, du,
\end{aligned}$$

for almost every $x \in S$. However, $\lim_{r \to 0} \int_{\mathbf{R}} h_j(u)\, r^{-iu}\, du = 0$ (by the Riemann-Lebesgue lemma), and $\left| \int_{\mathbf{R}} h_j(u)\, r^{-iu}\, du \right| \le \|h_j\|_1$, for each $j = 1, 2, 3, \ldots$, and so it follows that

$$(\phi_r * f - P_r * f)(x) \to 0, \quad \text{as } r \to 0,$$

for almost every $x \in S$. Since S is an arbitrary finite subset of $\mathbf{R}^n$, we conclude that

$$(\phi_r * f - P_r * f)(x) \to 0, \quad \text{as } r \to 0,$$

almost everywhere. □