CHAPTER IV

THE GENERAL L^2-THEORY OF MAXIMAL FUNCTIONS ON $\mathbf{R}^n$ – I

In this chapter, we present the general L^2-theory of maximal functions associated to some distributions on $\mathbf{R}^n$. The theory was originally developed by Cowling and Mauceri [13]. Here we add some similar results which we obtain by using the Mellin transform.

§4.1 Introduction: The results of Cowling and Mauceri

Let ϕ be a tempered distribution on $\mathbf{R}^n$ and ϕ_r $(r \in \mathbf{R}^+)$ be its dilate, defined by duality

$$\int_{\mathbf{R}^n} \phi_r(x)\, f(x)\, dx = \int_{\mathbf{R}^n} \phi(x)\, f(rx)\, dx, \quad f \in \mathcal{S}(\mathbf{R}^n).$$

We note that $\hat{\phi}_r(\xi) = \hat{\phi}(r\xi),\ \xi \in \mathbf{R}^n$. (One can also check that $\phi_r(x) = \phi(x/r)\, r^{-n},\ x \in \mathbf{R}^n$, when ϕ is a locally integrable function.)

For each $f \in \mathcal{S}(\mathbf{R}^n)$, we define the maximal function $\mathcal{M}_\phi f$ associated to ϕ by

$$\mathcal{M}_\phi f(x) = \sup_{r \in \mathbf{R}^+} |(\phi_r * f)(x)|, \quad x \in \mathbf{R}^n.$$

We then seek conditions on ϕ under which we have the L^2-inequality

$$\|\mathcal{M}_\phi f\|_2 \le C\, \|f\|_2,\ f \in \mathcal{S}(\mathbf{R}^n).$$

Cowling and Mauceri [13] obtained the following result.

THEOREM 4.1 (Cowling-Mauceri)

Suppose that ϕ is a compactly supported distribution on $\mathbf{R}^n$ and that

$$|\hat{\phi}(r\sigma)| \leq C\,(1+r)^{-\alpha}, \quad r \in \mathbf{R}^+,\ \sigma \in S^{n-1},$$

for some $\alpha > \frac{1}{2}$. Then the L^2-inequality holds.

The proof is based on a study of Riesz operators and an argument using g-functions; we refer the reader to [13] for the details. A different proof of this may be found in Rubio de Francia [24].

§4.2 More about the Riesz operators

As in [13], we introduce the Riesz operators as follows. For $a,\ b \in \mathbf{C}$, with $\mathrm{Re}(a) > 0$, the Riesz operator $R_{a,b}$ is defined on $\mathcal{S}(\mathbf{R}^n)$ by

$$R_{a,b}f(x) = \frac{2}{\Gamma(b)} \int_0^1 s^{a-1}\,(1-s^2)^{b-1}\,f(x/s)\,s^{-n}\,ds, \quad x \in \mathbf{R}^n.$$

On $\mathcal{S}'(\mathbf{R}^n)$, the Riesz operator $R_{a,b}$ may be defined by

$$(R_{a,b}\phi)\hat{}\,(\xi) = \frac{2}{\Gamma(b)} \int_0^1 s^{a-1}\,(1-s^2)^{b-1}\,\hat{\phi}(s\xi)\,ds, \quad \xi \in \mathbf{R}^n.$$

Either definition makes sense provided that $\mathrm{Re}(b) > 0$. By analytic continuation, however, it extends to the region $\mathrm{Re}(b) > -N$ for any $N \in \mathbf{Z}^+$.

One of the fundamental properties of the Riesz operators is given in the following lemma.

Lemma 4.2A (Cowling-Mauceri)

If $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(a-2b) > 0$, then the identity

$$R_{a-2b,b} R_{a,z} = R_{a-2b,b+z},$$

holds for all $z \in \mathbf{C}$.

Remark. We note that the identity of Lemma 4.2A above is in fact a generalization of that of Lemma 2.4C.

Now, generalizing Lemma 1.2 of [13], we have the following result.

Lemma 4.2B

Suppose ϕ is a compactly supported distribution on $\mathbf{R}^n$ and for $k = 0, 1, 2, \ldots$ there exist $\alpha_k > 0$ such that

$$\left| \frac{\partial^k}{\partial r^k} \hat{\phi}(r\sigma) \right| \le C_k \, (1+r)^{-\alpha_k}, \quad r \in \mathbf{R}^+, \ \sigma \in S^{n-1}.$$

Then for $\mathrm{Re}(a_k) + k > \alpha_k, \ k = 0, 1, 2, \ldots,$ and any $\delta > 0$, we have

$$\left| \frac{\partial^k}{\partial r^k} (R_{a_k,b} \phi)\hat{\ } (r\sigma) \right| \le C_k \, (1+r)^{-\alpha_k + m + \delta}, \quad r \in \mathbf{R}^+, \ \sigma \in S^{n-1},$$

where $m = \max\left(0, -\mathrm{Re}(b)\right)$.

Proof. The proof is essentially similar to that of Lemma 1.2 of [13]. For each $k = 0, 1, 2, \ldots,$ and for $r \in \mathbf{R}^+, \ \sigma \in S^{n-1}$, we have

$$\begin{aligned} \frac{\partial^k}{\partial r^k} (R_{a_k,b} \phi)\hat{\ } (r\sigma) &= \frac{2}{\Gamma(b)} \int_0^1 s^{a_k - 1} \, (1-s^2)^{b-1} \, \frac{\partial^k}{\partial r^k} \hat{\phi}(rs\sigma) \, ds \\ &= \frac{2}{\Gamma(b)} \int_0^1 s^{a_k + k - 1} \, (1-s^2)^{b-1} \, \frac{\partial^k}{\partial (rs)^k} \hat{\phi}(rs\sigma) \, ds. \end{aligned}$$

By our hypothesis on ϕ, it is easy to see that

$$\left|\frac{\partial^k}{\partial r^k}(R_{a_k,b}\phi)\hat{}\,(r\sigma)\right| \le C_k, \quad r \in \mathbf{R}^+,\ \sigma \in S^{n-1}.$$

Now, to examine the decay for $r > 1,\ \sigma \in S^{n-1}$, we rewrite

$$\begin{aligned}\frac{\partial^k}{\partial r^k}(R_{a_k,b}\phi)\hat{}\,(r\sigma) = \frac{2}{\Gamma(b)}\int_0^{1/(2r)} & s^{a_k+k-1}\,(1-s^2)^{b-1}\,\frac{\partial^k}{\partial(rs)^k}\hat{\phi}(rs\sigma)\,ds \\ &+ \frac{2}{\Gamma(b)}\int_{1/(2r)}^1 s^{a_k+k-1}\,(1-s^2)^{b-1}\,\frac{\partial^k}{\partial(rs)^k}\hat{\phi}(rs\sigma)\,ds.\end{aligned}$$

With the first integral, provided that $\mathrm{Re}(a_k) + k > \alpha_k$, we have

$$\begin{aligned}&\left|\int_0^{1/(2r)} s^{a_k+k-1}\,(1-s^2)^{b-1}\,\frac{\partial^k}{\partial(rs)^k}\hat{\phi}(rs\sigma)\,ds\right| \\ &\qquad \le C_k \int_0^{1/(2r)} s^{\mathrm{Re}(a_k)+k-1}\,ds \\ &\qquad = C_k\, r^{-(\mathrm{Re}(a_k)+k)} \\ &\qquad \le C_k\, r^{-\alpha_k}.\end{aligned}$$

Next, to estimate the second integral, we observe that if $\mathrm{Re}(b) \ge \delta > 0$, then

$$\begin{aligned}&\left|\int_{1/(2r)}^1 s^{a_k+k-1}\,(1-s^2)^{b-1}\,\frac{\partial^k}{\partial(rs)^k}\hat{\phi}(rs\sigma)\,ds\right| \\ &\qquad \le C\int_{1/(2r)}^1 s^{\mathrm{Re}(a_k)+k-1}\,(1-s^2)^{\delta-1}\,(rs)^{-\alpha_k}\,ds \\ &\qquad \le C_k\, r^{-\alpha_k}.\end{aligned}$$

Further, one may follow the argument used in [13] to show that if $\mathrm{Re}(b) \ge \delta - N$, for any $N \in \mathbf{Z}^+$, then

$$\left|\int_{1/(2r)}^1 s^{a_k+k-1}\,(1-s^2)^{b-1}\,\frac{\partial^k}{\partial(rs)^k}\hat{\phi}(rs\sigma)\,ds\right| \le C_k\, r^{-\alpha_k+N},$$

and then obtain the result by complex interpolation. □

§4.3 The Mellin transform argument

Using the Mellin transform, we may write

$$\hat{\phi}_r(\xi) = \int_{\mathbf{R}} \psi(u,\xi)\, r^{iu}\, du, \quad r \in \mathbf{R}^+,\ \xi \in \mathbf{R}^n$$

where

$$\psi(u,\xi) = \frac{1}{2\pi} \int_0^\infty \hat{\phi}_r(\xi)\, r^{-1-iu}\, dr, \quad u \in \mathbf{R},\ \xi \in \mathbf{R}^n.$$

Note that ψ is homogeneous in ξ as for any $r \in \mathbf{R}^+$ we have

$$\psi(u, r\xi) = r^{iu}\, \psi(u,\xi), \quad u \in \mathbf{R},\ \xi \in \mathbf{R}^n.$$

Now, for all $f \in \mathcal{S}(\mathbf{R}^n)$, we have (invoking properties of the Fourier transform)

$$(\phi_r * f)\,\hat{} = \hat{\phi}_r\, \hat{f}$$

whence

$$\begin{aligned} \phi_r * f &= (\hat{\phi}_r\, \hat{f})\,\check{} \\ &= \left(\int_{\mathbf{R}} \psi(u,.)\, \hat{f}\, r^{iu}\, du \right)^{\check{}} \\ &= \int_{\mathbf{R}} (\check{\psi}(u,.) * f)\, r^{iu}\, du. \end{aligned}$$

We therefore find that

$$\mathcal{M}_\phi f(x) \le \int_{\mathbf{R}} |(\check{\psi}(u,.) * f)(x)|\, du, \quad x \in \mathbf{R}^n.$$

Applying Minkowski's inequality and Plancherel's theorem, we obtain

$$\begin{aligned} \|\mathcal{M}_\phi f\|_2 &\le \int_{\mathbf{R}} \|\check{\psi}(u,.) * f\|_2\, du \\ &= \int_{\mathbf{R}} \|\psi(u,.)\, \hat{f}\|_2\, du \\ &\le \int_{\mathbf{R}} \|\psi(u,.)\|_\infty\, \|\hat{f}\|_2\, du \\ &= \|f\|_2 \int_{\mathbf{R}} \|\psi(u,.)\|_\infty\, du \end{aligned}$$

where (by homogeneity of ψ)

$$\|\psi(u,.)\|_\infty = \sup_{\sigma\in S^{n-1}} |\psi(u,\sigma)|, \quad u \in \mathbf{R}.$$

To verify the L^2-inequality, it suffices to have

$$\int_{\mathbf{R}} \|\psi(u,.)\|_\infty \, du < \infty.$$

But this would be satisfied when

$$|\psi(u,\sigma)| \leq C\,(1+|u|)^{-1-\delta}, \quad u \in \mathbf{R},\ \sigma \in S^{n-1},$$

for some $\delta > 0$. With this approach, therefore, we attempt to find conditions on ϕ under which this inequality holds.

Our result is presented in the following section.

§4.4 A further result: The L^2-inequality

THEOREM 4.4

Suppose ϕ is a compactly supported distribution on $\mathbf{R}^n$ with $\hat{\phi}(0) = 0$ such that

$$|\hat{\phi}(r\sigma)| \leq C\,(1+r)^{-\epsilon}, \quad r \in \mathbf{R}^+,\ \sigma \in S^{n-1}$$

and

$$\left|\frac{\partial}{\partial r}\hat{\phi}(r\sigma)\right| \leq C\,(1+r)^{-1-\epsilon}, \quad r \in \mathbf{R}^+,\ \sigma \in S^{n-1},$$

for some $\epsilon > 0$. Then the L^2-inequality holds.

Remark. This result was also proved by Rubio de Francia [24] using a different method.

PROOF. As mentioned before, we need to show that

$$|\psi(u,\sigma)| \le C\,(1+|u|)^{-1-\delta}, \quad u \in \mathbf{R},\ \sigma \in S^{n-1}.$$

for some $\delta > 0$.

First, since $\hat{\phi}(0) = 0$ and $\hat{\phi}$ is differentiable, we have

$$|\hat{\phi}(r\sigma)| \le C\,r, \quad 0 \le r \le 1,$$

and thus we obtain

$$\begin{aligned} |\psi(u,\sigma)| &\le \int_0^\infty |\hat{\phi}(r\sigma)|\,r^{-1}\,dr \\ &\le \int_0^1 C\,dr + \int_1^\infty C\,r^{-1-\epsilon}\,dr \\ &\le C, \end{aligned}$$

for all $u \in \mathbf{R},\ \sigma \in S^{n-1}$.

Next, we invoke the identity (of Lemma 4.2A)

$$\phi = R_{a+2b,-b} R_{a,b} \phi$$

where $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(a+2b) > 0$, to have

$$\frac{\partial}{\partial r}\hat{\phi}(r\sigma) = \frac{2}{\Gamma(-b)} \int_0^1 s^{a+2b-1}\,(1-s^2)^{-b-1}\,\frac{\partial}{\partial r}(R_{a,b}\phi)\hat{}\,(rs\sigma)\,ds,$$

for all $r \in \mathbf{R}^+,\ \sigma \in S^{n-1}$. Applying Fubini's theorem, we obtain

$$
\begin{aligned}
&2\pi\, iu\, \psi(u,\sigma)\\
&= -\int_0^\infty \hat{\phi}(r\sigma)\,(-iu)\, r^{-1-iu}\, dr\\
&= -\int_0^\infty \hat{\phi}(r\sigma)\, \frac{\partial}{\partial r} r^{-iu}\, dr\\
&= \int_0^\infty \frac{\partial}{\partial r}\hat{\phi}(r\sigma)\ r^{-iu}\, dr\\
&= \int_0^\infty \frac{2}{\Gamma(-b)} \int_0^1 s^{a+2b-1}\,(1-s^2)^{-b-1}\, \frac{\partial}{\partial r}(R_{a,b}\phi)\hat{\ }(rs\sigma)\, ds\ r^{-iu}\, dr\\
&= \frac{2}{\Gamma(-b)} \int_0^1 \int_0^\infty \frac{\partial}{\partial (rs)}(R_{a,b}\phi)\hat{\ }(rs\sigma)(rs)^{-iu} d(rs)\, s^{a+2b-1+iu}(1-s^2)^{-b-1} ds\\
&= \frac{2}{\Gamma(-b)} \int_0^\infty \frac{\partial}{\partial t}(R_{a,b}\phi)\hat{\ }(t\sigma)\ t^{-iu}\, dt\ \int_0^1 s^{a+2b-1+iu}\,(1-s^2)^{-b-1}\, ds\\
&= \frac{\Gamma(\frac{a+2b+iu}{2})}{\Gamma(\frac{a+iu}{2})} \int_0^\infty \frac{\partial}{\partial t}(R_{a,b}\phi)\hat{\ }(t\sigma)\ t^{-iu}\, dt.
\end{aligned}
$$

So, for $\mathrm{Re}(a) > \epsilon$ and $b = -\frac{\epsilon}{4}$, we have

$$
|u\,\psi(u,\sigma)| \le \left| \frac{\Gamma(-\frac{\epsilon}{4} + \frac{a+iu}{2})}{\Gamma(\frac{a+iu}{2})} \right| \int_0^\infty \left| \frac{\partial}{\partial t}(R_{a,-\frac{\epsilon}{4}}\phi)\hat{\ }(t\sigma) \right| dt \le C\,|u|^{-\frac{\epsilon}{4}}, \quad |u| > 1,
$$

using the facts that $|\Gamma(z)| \sim C\, e^{-\frac{\pi}{2}|\mathrm{Im}(z)|}\, |\mathrm{Im}(z)|^{\mathrm{Re}(z)-\frac{1}{2}}$ as $|\mathrm{Im}(z)| \to \infty$ (see [37], p. 151) and $\left|\frac{\partial}{\partial t}(R_{a,-\frac{\epsilon}{4}}\phi)\hat{\ }(t\sigma)\right| \le C\,(1+t)^{-1-\frac{\epsilon}{2}}$ for $t \in \mathbf{R}^+$ (by Lemma 4.2B, with $\delta = \frac{\epsilon}{4}$). We therefore find that

$$
|\psi(u,\sigma)| \le C\,|u|^{-1-\frac{\epsilon}{4}}, \quad |u| > 1,\ \sigma \in S^{n-1}.
$$

Combining this with the previous estimate, we obtain

$$
|\psi(u,\sigma)| \le C\,(1+|u|)^{-1-\frac{\epsilon}{4}}, \quad u \in \mathbf{R},\ \sigma \in S^{n-1},
$$

as desired. □

Remark. The condition $\hat{\phi}(0) = 0$ in the theorem can in fact be removed. When ϕ does not satisfy this condition, we can set $\chi = \phi - \varphi$, where $\varphi \in \mathcal{S}(\mathbf{R}^n)$ with $\hat{\varphi}(0) = \hat{\phi}(0)$. Thus χ satisfies the hypothesis, and hence the conclusion holds for $\mathcal{M}_\chi f$. But this implies that the conclusion also holds for $\mathcal{M}_\phi f$ since $\mathcal{M}_\varphi f$ is known to be dominated by the Hardy-Littlewood maximal function (by Theorem 1.5A). Moreover, the theorem is also true for a distribution ϕ which is not compactly supported but such that $\hat{\phi}$ is continuously differentiable.