CHAPTER V

THE GENERAL L^2-THEORY OF MAXIMAL FUNCTIONS ON $\mathbb{R}^n$ – II

Continuing our discussion in the previous chapter, we now deal with maximal functions associated to some surface measures on $\mathbf{R}^n$.

§5.1 Introduction: Main results

Let $\mathcal{H}$ be a smooth compact convex hypersurface of class C^∞ in $\mathbf{R}^n$ whose tangent lines have order of contact at most $m < \infty$. Suppose μ is a smooth surface measure on $\mathcal{H}$. Then, as shown by Cowling *et al* [10], for all $r \in \mathbf{R}^+,\ \sigma \in S^{n-1}$, we have

$$\hat{\mu}(r\sigma) = F(r,\sigma)\, e^{-irp(\sigma)\cdot\sigma} + F(r,-\sigma)\, e^{-irp(-\sigma)\cdot\sigma} + E(r,\sigma)$$

where

(a) $|p(\pm\sigma)\cdot\sigma| \le C$,

(b) $\left|\frac{\partial^k}{\partial r^k} F(r,\pm\sigma)\right| \le C_k\, r^{-k-\alpha}$ for $k = 0,1,2,\dots$, for some $\alpha > 0$.

(c) $\left|\frac{\partial^k}{\partial r^k} E(r,\sigma)\right| \le C_{k,l}\, r^{-l}$ for $k,l = 0,1,2,\dots$.

By rearranging F and E for $0 \le r \le 1$, we may assume that $F(0,\pm\sigma) = 0$ (or even $F(r,\pm\sigma) = 0$, for $0 \le r < s < 1$). Moreover, if for all $r \in \mathbf{R}^+,\ \sigma \in S^{n-1}$, we have (in place of (b)) (b′) $\left|\frac{\partial^k}{\partial r^k} F(r, \pm\sigma)\right| \le C_k\, r^{-k-\alpha}$ for $k = 0, 1, 2, \ldots$, for some $\alpha > \frac{1}{2}$,

we then say that μ is *of good decay type.*

We define the maximal function $\mathcal{M}_\mu f$ associated to μ by

$$\mathcal{M}_\mu f(x) = \sup_{r \in \mathbf{R}^+} |(\mu_r * f)(x)|, \quad x \in \mathbf{R}^n,$$

whenever $f \in \mathcal{S}(\mathbf{R}^n)$. We are then interested in the L^2-inequality

$$\|\mathcal{M}_\mu f\|_2 \le C\, \|f\|_2, \quad f \in \mathcal{S}(\mathbf{R}^n).$$

Our main results are the following.

Theorem 5.1A

Let $\mathcal{H}$ be a hypersurface in $\mathbf{R}^n$, and μ be a surface measure of good decay type on $\mathcal{H}$. Then the L^2-inequality holds.

Theorem 5.1B

Suppose that μ is a measure on $\mathbf{R}^n$, p is a bounded function on S^{n-1}, and $0 < \epsilon < 3$. If, for each $\sigma \in S^{n-1}$, there exists a measure $\nu = \nu_\sigma$ on $\mathbf{R}$ such that

(a) $\hat{\nu}(r) = \hat{\mu}(r\sigma)\, e^{irp(\sigma)\cdot\sigma}\, (1 + r^2)^{\frac{1+\epsilon}{4}}$ for $r \in \mathbf{R}^+$,

(b) $\int_{\mathbf{R}} (1 + s^2)\, |d\nu(s)| < C$, with C independent of σ,

then the L^2-inequality holds.

Remark. We shall show later that Theorem 5.1A is in fact a special case of Theorem 5.1B.

§5.2 A supporting lemma

Theorem 5.1B follows from the lemma below. We are indebted to Prof. Mauceri for help with its proof.

Lemma 5.2

Suppose that μ is a measure on $\mathbf{R}^n$ such that $\hat\mu(0) = 0$ and $\nabla\hat\mu(0) = 0$, p is a bounded function on S^{n-1}, and $0 < \epsilon < 3$. Suppose also that for each $\sigma \in S^{n-1}$, there exists a measure $\nu = \nu_\sigma$ on $\mathbf{R}$ such that

(a) $\hat\nu(r) = \hat\mu(r\sigma)\, e^{irp(\sigma)\cdot\sigma}\,(1+r^2)^{\frac{1+\epsilon}{4}}$ for $r \in \mathbf{R}^+$,

(b) $\int_{\mathbf{R}}(1+s^2)\,|d\nu(s)| < C$, with C independent of σ.

Define $\psi = \psi_\sigma$ by

$$\psi(u) = \frac{1}{2\pi}\int_0^\infty \hat\mu(r\sigma)\, r^{-1-iu}\, dr, \quad u \in \mathbf{R}.$$

Then we have

$$|\psi(u)| \le C\,(1+|u|)^{-1-\frac{\epsilon}{2}}, \quad u \in \mathbf{R},$$

with $C = C(\epsilon, p)\int_{\mathbf{R}}(1+s^2)\,|d\nu(s)|$.

Proof. We note first that $\hat\nu \in C^2(\mathbf{R})$ because of (b). We also have $\hat\nu(0) = 0$ and $\hat\nu\,'(0) = 0$. And furthermore, for $0 \le r \le 1$, we have

$$\begin{aligned} |\hat\nu(r)| &= |\hat\nu(r) - \hat\nu(0)| \qquad (\text{as } \hat\nu(0) = 0) \\ &= \left| r\,\frac{\partial\hat\nu}{\partial r}(\rho)\right| \qquad (\text{for some } 0 < \rho < r) \\ &\le r\int_{\mathbf{R}} |s|\,|d\nu(s)| \\ &\le r\int_{\mathbf{R}} (1+s^2)\,|d\nu(s)| \\ &\le C\,r. \end{aligned}$$

Now, from (a), we have

$$\hat{\mu}(r\sigma) = \hat{\nu}(r)\, e^{-irq}\, (1+r^2)^{-\frac{1+\epsilon}{4}}, \quad r \in \mathbf{R}^+,$$

with $q = p(\sigma)\cdot\sigma$. Thus, whenever $u \in \mathbf{R}$,

$$\begin{aligned}
|\psi(u)| &\le \int_0^\infty |\hat{\mu}(r\sigma)|\, r^{-1}\, dr \\
&\le \int_0^1 |\hat{\nu}(r)|\, r^{-1}\, dr + \int_1^\infty |\hat{\nu}(r)|\, (1+r^2)^{-\frac{1+\epsilon}{4}}\, r^{-1}\, dr \\
&\le \int_0^1 C\, dr + \int_1^\infty \|\hat{\nu}\|_\infty\, r^{-\frac{3}{2}}\, dr \\
&\le C \int_{\mathbf{R}} (1+s^2)\, |d\nu(s)|.
\end{aligned}$$

We shall now examine the decay of $\psi(u)$ for $|u| > 1$.

Take a smooth function φ on $\mathbf{R}^+$ such that $\varphi(r) = 0$ if $0 \le r \le \frac{3}{2}$ and $\varphi(r) = 1$ if $r \ge 2$. We then write

$$\begin{aligned}
2\pi\, \psi(u) &= \int_0^\infty \hat{\nu}(r)\, e^{-irq}\, (1+r^2)^{-\frac{1+\epsilon}{4}}\, r^{-1-iu}\, dr \\
&= \int_0^\infty \hat{\nu}(r)\, e^{-irq}\, \{1-\varphi(r)\}\, (1+r^2)^{-\frac{1+\epsilon}{4}}\, r^{-1-iu}\, dr \\
&\quad + \int_0^\infty \hat{\nu}(r)\, e^{-irq}\, \varphi(r)\, \left\{(1+r^2)^{-\frac{1+\epsilon}{4}} - r^{-\frac{1+\epsilon}{2}}\right\} r^{-1-iu}\, dr \\
&\quad + \int_0^\infty \hat{\nu}(r)\, e^{-irq}\, \varphi(r)\, r^{-\frac{1+\epsilon}{2}}\, r^{-1-iu}\, dr \\
&= \psi_1(u) + \psi_2(u) + \psi_3(u), \quad \text{say.}
\end{aligned}$$

For $\psi_1(u)$, we have

$$\begin{aligned}
-iu\, \psi_1(u) &= \int_0^\infty \hat{\nu}(r)\, e^{-irq}\, \{1-\varphi(r)\}\, (1+r^2)^{-\frac{1+\epsilon}{4}}\, \frac{\partial}{\partial r} r^{-iu}\, dr \\
&= -\int_0^\infty \frac{\partial}{\partial r}\Big[\hat{\nu}(r)\, e^{-irq}\, \{1-\varphi(r)\}\, (1+r^2)^{-\frac{1+\epsilon}{4}}\Big]\, r^{-iu}\, dr,
\end{aligned}$$

and similarly

$$-iu(1-iu)\, \psi_1(u) = \int_0^\infty \frac{\partial^2}{\partial r^2}\Big[\hat{\nu}(r)\, e^{-irq}\, \{1-\varphi(r)\}\, (1+r^2)^{-\frac{1+\epsilon}{4}}\Big]\, r^{1-iu}\, dr.$$

But $1-\varphi(r)=0$ for $r\geq 2$, and $\frac{\partial^2}{\partial r^2}\left[\hat\nu(r)\,e^{-irq}\,\{1-\varphi(r)\}\,(1+r^2)^{-\frac{1+\epsilon}{4}}\right]$ is in fact a linear combination of products of derivatives of the four factors, which are all bounded for $0\leq r\leq 2$ (in particular, the derivatives of $\hat\nu$ are dominated by $\int_{\mathbf{R}}(1+s^2)\,|d\nu(s)|$). Thus we obtain

$$|iu(1-iu)\,\psi_1(u)|\leq\int_0^2\left|\frac{\partial^2}{\partial r^2}\left[\hat\nu(r)\,e^{-irq}\,\{1-\varphi(r)\}\,(1+r^2)^{-\frac{1+\epsilon}{4}}\right]\right|\,r\,dr\leq C,$$

which gives

$$|\psi_1(u)|\leq C\,|u|^{-2}.$$

For $\psi_2(u)$, we also have

$$\begin{aligned}&-iu(1-iu)\,\psi_2(u)\\&\qquad=\int_0^\infty\frac{\partial^2}{\partial r^2}\left[\hat\nu(r)\,e^{-irq}\,\varphi(r)\left\{(1+r^2)^{-\frac{1+\epsilon}{4}}-r^{-\frac{1+\epsilon}{2}}\right\}\right]r^{1-iu}\,dr.\end{aligned}$$

Put

$$R(r)=(1+r^2)^{-\frac{1+\epsilon}{4}}-r^{-\frac{1+\epsilon}{2}}=r^{-\frac{1+\epsilon}{2}}\left\{(1+r^{-2})^{-\frac{1+\epsilon}{4}}-1\right\},\quad r\in\mathbf{R}^+.$$

For $r\geq\frac{3}{2}$, we can expand $(1+r^{-2})^{-\frac{1+\epsilon}{4}}$ as a convergent Taylor series to get

$$(1+r^{-2})^{-\frac{1+\epsilon}{4}}=1-\frac{1+\epsilon}{4}\,r^{-2}+\frac{1+\epsilon}{4}\,\frac{5+\epsilon}{4}\,\frac{r^{-4}}{2!}-\cdots.$$

Hence

$$R(r)=-\frac{1+\epsilon}{4}\,r^{-\frac{5+\epsilon}{2}}+\text{terms of a lower power of } r,\quad r\geq\tfrac{3}{2}.$$

We therefore find that $|R(r)|$, $\left|\frac{\partial}{\partial r}R(r)\right|$ and $\left|\frac{\partial^2}{\partial r^2}R(r)\right|\leq C\,r^{-\frac{5+\epsilon}{2}}$ for $r\geq\frac{3}{2}$. And thus, since the derivatives of $\hat\nu$, e^{-irq} and φ are bounded, we obtain

$$|iu(1-iu)\,\psi_2(u)|\leq\int_{\frac{3}{2}}^\infty C\,r^{-\frac{5+\epsilon}{2}}\,dr\leq C,$$

which gives

$$|\psi_2(u)| \le C\,|u|^{-2}.$$

For $\psi_3(u)$, we write

$$\begin{aligned}
\psi_3(u) &= \int_0^\infty \hat{\nu}(r)\, e^{-irq}\, \varphi(r)\, r^{-\frac{3+\epsilon}{2}-iu}\, dr \\
&= \int_0^\infty \hat{\nu}(r)\, e^{-irq}\, \varphi(r)\, r^{z-1}\, dr \qquad (\text{with } z = -\tfrac{1+\epsilon}{2} - iu) \\
&= \int_{\mathbf{R}} \int_0^\infty \varphi(r)\, r^{z-1}\, e^{-ir(s+q)}\, dr\, d\nu(s) \qquad (\text{as } \varphi(r) = 0 \text{ if } 0 \le r \le \tfrac{3}{2}).
\end{aligned}$$

Then, for fixed $s \in \mathbf{R}$, consider

$$I(z) = \int_0^\infty \varphi(r)\, r^{z-1}\, e^{-ir(s+q)}\, dr, \quad z \in \mathbf{C}.$$

We observe that $I(z)$ continues analytically into $\mathrm{Re}(z) \le 1$. Furthermore, for $0 < \mathrm{Re}(z) < 1$, we write

$$\begin{aligned}
I(z) &= \int_0^\infty \{\varphi(r) - 1\}\, r^{z-1}\, e^{-ir(s+q)}\, dr + \int_0^\infty r^{z-1}\, e^{-ir(s+q)}\, dr \\
&= I_4(z) + I_5(z), \quad \text{say.}
\end{aligned}$$

Regarding I_4, we note in particular that

$$\int_{\mathbf{R}} I_4(z)\, d\nu(s) = \int_0^\infty \hat{\nu}(r)\, e^{-irq}\, \{\varphi(r) - 1\}\, r^{-\frac{3+\epsilon}{2}-iu}\, dr$$

continues analytically into $-2 < \mathrm{Re}(z) < 1$, since $\hat{\nu}(0) = 0$ and $\hat{\nu}\,'(0) = 0$.

Now, corresponding to I_4, we have

$$\begin{aligned}
\psi_4(u) &= \int_{\mathbf{R}} I_4(-\tfrac{1+\epsilon}{2} - iu)\, d\nu(s) \\
&= \int_0^\infty \hat{\nu}(r)\, e^{-irq}\, \{\varphi(r) - 1\}\, r^{-\frac{3+\epsilon}{2}-iu}\, dr \\
&= \int_0^\infty \left[\hat{\nu}(r)\, e^{-irq}\, \{\varphi(r) - 1\}\, r^{-\frac{1+\epsilon}{2}}\right] r^{-1-iu}\, dr.
\end{aligned}$$

Integrating by parts twice, we obtain

$$-iu(1-iu)\,\psi_4(u)=\int_0^\infty \frac{\partial^2}{\partial r^2}\Big[\hat\nu(r)\,e^{-irq}\,\{\varphi(r)-1\}\,r^{-\frac{1+\epsilon}{2}}\Big]\;r^{1-iu}\,dr.$$

But $\varphi(r)-1=0$ for $r\geq 2$, and $\big|\frac{\partial^2}{\partial r^2}\big[\hat\nu(r)\,e^{-irq}\,\{\varphi(r)-1\}\,r^{-\frac{1+\epsilon}{2}}\big]\big|\leq C\,r^{-\frac{1+\epsilon}{2}}$ for $0\leq r\leq 2$ since again the derivatives of $\hat\nu$, e^{-irq} and φ are bounded. Hence we find that

$$|iu(1-iu)\,\psi_4(u)|\leq\int_0^2 C\,r^{\frac{1-\epsilon}{2}}\,dr<\infty \qquad (\text{as } 0<\epsilon<3),$$

whence

$$|\psi_4(u)|\leq C\,|u|^{-2}.$$

It remains to estimate $\psi_5(u)=\int_{\mathbf{R}} I_5(-\frac{1+\epsilon}{2}-iu)\,d\nu(s)$. In order to do so, let us consider the function h defined on $\mathbf{C}$ by

$$h(w)=w^{z-1}\,e^{-wt},\quad \text{where } z=-\tfrac{1+\epsilon}{2}-iu,\ t\in\mathbf{R}.$$

By integrating h around the contour γ_1 (see `Figure 1`) for $t>0$, or around the contour γ_2 (see `Figure 2`) for $t<0$, one can observe that, as $R\to\infty$,

$$\int_0^\infty r^{z-1}\,e^{-irt}\,dr=(it)^{-z}\,\Gamma(z),\quad t\neq 0.$$

`Figure 1`. The contour γ_1

`Figure 2`. The contour γ_2

We therefore find that

$$\begin{aligned}
|\psi_5(u)| &= \left| \int_{\mathbf{R}} I_5(z)\, d\nu(s) \right| \qquad (\text{with } z = -\tfrac{1+\epsilon}{2} - iu) \\
&= \left| \int_{\mathbf{R}} \int_0^\infty r^{z-1}\, e^{-ir(s+q)}\, dr\, d\nu(s) \right| \\
&= \left| \int_{\mathbf{R}} \{i(s+q)\}^{-z}\, \Gamma(z)\, d\nu(s) \right| \\
&\le e^{-\frac{\pi}{2}u}\, |\Gamma(-\tfrac{1+\epsilon}{2} - iu)| \int_{\mathbf{R}} |s+q|^{\frac{1+\epsilon}{2}}\, |d\nu(s)| \\
&\le C\, |u|^{-1-\frac{\epsilon}{2}} \int_{\mathbf{R}} |s+q|^{\frac{1+\epsilon}{2}}\, |d\nu(s)| \qquad (\text{as } |u| > 1) \\
&\le C\, |u|^{-1-\frac{\epsilon}{2}} \int_{\mathbf{R}} \{1 + (s+q)^2\}\, |d\nu(s)| \qquad (\text{as } 0 < \epsilon < 3) \\
&\le C\, |u|^{-1-\frac{\epsilon}{2}}\, (1+q^2) \int_{\mathbf{R}} (1+s^2)\, |d\nu(s)| \\
&\le C\, |u|^{-1-\frac{\epsilon}{2}},
\end{aligned}$$

with $C = C(\epsilon, q) \int_{\mathbf{R}} (1+s^2)\, |d\nu(s)|$.

Combining this with the previous estimates, we obtain the result.

The proof of the lemma is therefore complete. □

§5.3 Invoking a result of Hörmander

To prove Theorem 5.1A, we invoke a result of Hörmander [20], pp. 121-122, which we modify slightly as follows.

Lemma 5.3

Suppose that Φ is a smooth function on $\mathbf{R}$, and that for $k = 0, 1, 2, \ldots$

$$\left| \frac{\partial^k}{\partial r^k} \Phi(r) \right| \le C_k\, (1 + |r|)^{-k-\epsilon}, \quad r \in \mathbf{R},$$

for some $\epsilon > 0$. It then follows that $\check{\Phi} \in L^1(\mathbf{R})$. In general, $s^k\, \check{\Phi}(s) \in L^1(\mathbf{R})$, for $k = 0, 1, 2, \ldots$.

Proof. First of all (see [20], p. 121), there exists a smooth function φ on $\mathbf{R}$ supported in $\{r \in \mathbf{R} : \frac{1}{2} < |r| < 2\}$ such that

$$\sum_{j=-\infty}^{\infty} \varphi(2^{-j} r) = 1, \quad r \neq 0.$$

With this function φ, we have

$$\sum_{j=1}^{\infty} \varphi(2^{-j} r) = 0, \quad |r| \leq \tfrac{1}{2},$$

and

$$\sum_{j=1}^{\infty} \varphi(2^{-j} r) = 1, \quad |r| \geq 2.$$

Let us now put

$$\varphi_0(r) = \begin{cases} 1 - \sum_{j=1}^{\infty} \varphi(2^{-j} r), & \text{if } r \neq 0, \\ 1, & \text{if } r = 0. \end{cases}$$

It is clear that φ_0 is smooth and supported in $\{r \in \mathbf{R} : |r| < 2\}$, and that

$$\varphi_0(r) + \sum_{j=1}^{\infty} \varphi(2^{-j} r) = 1, \quad r \in \mathbf{R}.$$

This enables us to decompose Φ into

$$\Phi = \Phi_0 + \sum_{j=1}^{\infty} \Phi_j$$

where $\Phi_0 = \varphi_0 \, \Phi$ and $\Phi_j(r) = \varphi(2^{-j} r) \, \Phi(r),\ r \in \mathbf{R}$ $(j = 1, 2, 3, \dots)$.

We first observe that since $\Phi_0 = \varphi_0 \, \Phi$ is smooth and compactly supported on $\mathbf{R}$, we have

$$|\check{\Phi}_0(s)| \leq \int_{\mathbf{R}} |\Phi_0(r)| \, dr \leq C, \quad s \in \mathbf{R},$$

and

$$4\pi^2 s^2\, |\check{\Phi}_0(s)| \le \int_{\mathbf{R}} \Big| \frac{\partial^2}{\partial r^2} \Phi_0(r) \Big|\, dr \le C, \quad s \in \mathbf{R}.$$

Hence

$$|\check{\Phi}_0(s)| \le C\,(1+|s|)^{-2}, \quad s \in \mathbf{R},$$

yielding

$$\int_{\mathbf{R}} |\check{\Phi}_0(s)|\, ds < \infty.$$

Now, for $j = 1, 2, 3, \ldots$, we have

$$\begin{aligned}\int_{\mathbf{R}} |\Phi_j(r)|^2\, dr &= \int_{\mathbf{R}} |\varphi(2^{-j} r)\, \Phi(r)|^2\, dr \\ &\le C \int_{2^{j-1} \le |r| \le 2^{j+1}} |\Phi(r)|^2\, dr \\ &\le C \int_{2^{j-1} \le |r| \le 2^{j+1}} |r|^{-2\epsilon}\, dr \\ &\le C\, 2^{(1-2\epsilon) j},\end{aligned}$$

and similarly

$$\begin{aligned}&\int_{\mathbf{R}} \Big| 2^j \,\frac{\partial}{\partial r} \Phi_j(r) \Big|^2\, dr \\ &\qquad = \int_{\mathbf{R}} \Big| \frac{\partial \varphi}{\partial r}(2^{-j} r)\, \Phi(r) + 2^j\, \varphi(2^{-j} r)\, \frac{\partial}{\partial r} \Phi(r) \Big|^2\, dr \\ &\qquad \le C \int_{2^{j-1} \le |r| \le 2^{j+1}} \Big\{ |\Phi(r)|^2 + 2^j\, |\Phi(r)|\, \Big| \frac{\partial}{\partial r} \Phi(r) \Big| + 2^{2j}\, \Big| \frac{\partial}{\partial r} \Phi(r) \Big|^2 \Big\}\, dr \\ &\qquad \le C \int_{2^{j-1} \le |r| \le 2^{j+1}} \Big\{ |r|^{-2\epsilon} + 2^j\, |r|^{-1-2\epsilon} + 2^{2j}\, |r|^{-2-2\epsilon} \Big\}\, dr \\ &\qquad \le C\, 2^{(1-2\epsilon) j}.\end{aligned}$$

Then (by Plancherel's theorem)

$$\int_{\mathbf{R}} |\check{\Phi}_j(s)|^2\, ds = \int_{\mathbf{R}} |\Phi_j(r)|^2\, dr \le C\, 2^{(1-2\epsilon) j}$$

and

$$\int_{\mathbf{R}} 2^{2j}\, 4\pi^2 s^2\, |\check{\Phi}_j(s)|^2\, ds = \int_{\mathbf{R}} \Big| 2^j\, \frac{\partial}{\partial r} \Phi_j(r) \Big|^2\, dr \le C\, 2^{(1-2\epsilon) j}.$$

These give

$$\int_{\mathbf{R}} 2^{-j}\,(1+2^{2j}\,4\pi^2 s^2)\,|\check{\Phi}_j(s)|^2\,ds \le C\,2^{-2\epsilon j}.$$

Applying the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
&\int_{\mathbf{R}} |\check{\Phi}_j(s)|\,ds \\
&\quad\le \left\{\int_{\mathbf{R}} 2^{-j}\,(1+2^{2j}\,4\pi^2 s^2)\,|\check{\Phi}_j(s)|^2\,ds\right\}^{\frac12}\left\{\int_{\mathbf{R}} 2^{j}\,(1+2^{2j}\,4\pi^2 s^2)^{-1}\,ds\right\}^{\frac12} \\
&\quad\le C\,2^{-\epsilon j}\left\{\int_{|s|\le 2^{-j}} 2^{j}\,ds+\int_{|s|\ge 2^{-j}} 2^{-j}\,(2\pi s)^{-2}\,ds\right\}^{\frac12} \\
&\quad\le C\,2^{-\epsilon j}.
\end{aligned}$$

Since $\check{\Phi} = \check{\Phi}_0 + \sum_{j=1}^{\infty} \check{\Phi}_j$, we therefore find that

$$\|\check{\Phi}\|_1 \le \|\check{\Phi}_0\|_1 + \sum_{j=1}^{\infty} \|\check{\Phi}_j\|_1 \le C(\epsilon) < \infty,$$

which implies that $\check{\Phi} \in L^1(\mathbf{R})$.

In general, the proof also applies to $\frac{\partial^k}{\partial r^k}\Phi(r)$, and so we have $s^k\,\check{\Phi}(s) \in L^1(\mathbf{R})$ for all $k = 0, 1, 2, \ldots$. □

§5.4 The proof of the theorem

We now come to the proof of Theorem 5.1A.

We recall that for $r \in \mathbf{R}^+,\ \sigma \in S^{n-1}$, we have

$$\hat{\mu}(r\sigma) = F(r,\sigma)\,e^{-irp(\sigma)\cdot\sigma} + F(r,-\sigma)\,e^{-irp(-\sigma)\cdot\sigma} + E(r,\sigma),$$

with p, F, and E as prescribed. We assume here that $F(r,\pm\sigma) = 0$, for $0 \le r < s < 1$, to have $\frac{\partial^k}{\partial r^k}F(r,\sigma)\big|_{r=0} = 0$, for all $k = 0, 1, 2, \ldots$.

For $u \in \mathbf{R},\ \sigma \in S^{n-1}$, consider

$$\begin{aligned}
\psi(u,\sigma) &= \frac{1}{2\pi}\int_0^\infty \hat{\mu}(r\sigma)\, r^{-1-iu}\, dr \\
&= \frac{1}{2\pi}\int_0^\infty F(r,\sigma)\, e^{-irp(\sigma)\cdot\sigma}\, r^{-1-iu}\, dr \\
&\qquad + \frac{1}{2\pi}\int_0^\infty F(r,-\sigma)\, e^{-irp(-\sigma)\cdot\sigma}\, r^{-1-iu}\, dr \\
&\qquad + \frac{1}{2\pi}\int_0^\infty E(r,\sigma)\, r^{-1-iu}\, dr \\
&= \psi_1(u,\sigma) + \psi_2(u,\sigma) + \psi_3(u,\sigma), \quad \text{say.}
\end{aligned}$$

Let us fix $\sigma \in S^{n-1}$ hereafter. By Theorem 4.4, we have

$$|\psi_3(u,\sigma)| \le C\,(1+|u|)^{-1-\delta}, \quad u \in \mathbf{R},$$

for some $\delta > 0$, assuming that $E(0,\sigma) = 0$.

It then remains to tackle ψ_1 and ψ_2. But since they are similar, it suffices to work with one of them, ψ_1 say. We put

$$\hat{\mu}_1(r\sigma) = F(r,\sigma)\, e^{-irp(\sigma)\cdot\sigma},$$

and define $\hat{\nu}$ on $\mathbf{R}$ by

$$\hat{\nu}(r) = \begin{cases} \hat{\mu}_1(r\sigma)\, e^{irp(\sigma)\cdot\sigma}\,(1+r^2)^{\frac{1+\epsilon}{4}}, & \text{if } r \ge 0, \\ \hat{\mu}_1((-r)(-\sigma))\, e^{irp(-\sigma)\cdot\sigma}\,(1+r^2)^{\frac{1+\epsilon}{4}}, & \text{if } r < 0, \end{cases}$$

for some $0 < \epsilon < \min(3, \alpha - \frac{1}{2})$. Thus

$$\hat{\nu}(r) = \begin{cases} F(r,\sigma)\,(1+r^2)^{\frac{1+\epsilon}{4}}, & \text{if } r \ge 0, \\ F(-r,-\sigma)\,(1+r^2)^{\frac{1+\epsilon}{4}}, & \text{if } r < 0. \end{cases}$$

Writing

$$\hat{\nu}(r) = \begin{cases} F(r,\sigma)\, r^{\frac{1+\epsilon}{2}}\,(1+r^{-2})^{\frac{1+\epsilon}{4}}, & \text{if } r \ge 0, \\ F(-r,-\sigma)\,(-r)^{\frac{1+\epsilon}{2}}\,(1+r^{-2})^{\frac{1+\epsilon}{4}}, & \text{if } r < 0, \end{cases}$$

and applying Leibnitz's formula, we obtain that for $k = 0, 1, 2, \ldots,$

$$\left| \frac{\partial^k}{\partial r^k} \hat{\nu}(r) \right| \le C, \quad |r| \le 1$$

and

$$\left| \frac{\partial^k}{\partial r^k} \hat{\nu}(r) \right| \le C\, |r|^{-k-\frac{\epsilon}{2}}, \quad |r| > 1.$$

Hence, for all $k = 0, 1, 2, \ldots,$ we have

$$\left| \frac{\partial^k}{\partial r^k} \hat{\nu}(r) \right| \le C\, (1 + |r|)^{-k-\frac{\epsilon}{2}}, \quad r \in \mathbf{R}.$$

Following Lemma 5.3, we have $\nu = (\hat{\nu})\check{} \in L^1(\mathbf{R})$ (which assures us that ν defines a measure on $\mathbf{R}$), and moreover $s^2\, \nu(s) \in L^1(\mathbf{R})$. So we find that there exists a measure ν on $\mathbf{R}$ which satisfies the hypothesis of Lemma 5.2.

This therefore concludes the proof of Theorem 5.1A. $\square$

Remark. The result extends to surface measures of class C^K for some finite K; K is sufficiently large so that the hypothesis

$$\left| \frac{\partial^k}{\partial r^k} F(r, \pm\sigma) \right| \le C_k\, r^{-k-\alpha}, \quad r \in \mathbf{R}^+,\ \sigma \in S^{n-1},$$

is satisfied for $k = 0, 1, 2, 3$ and 4.