CHAPTER VI

A GENERALIZATION OF MAXIMAL FUNCTIONS ON COMPACT SEMISIMPLE LIE GROUPS

The study of maximal functions on compact Lie groups was recently developed by Cowling and Meaney [14]. We shall here generalize their result to a class of maximal functions on compact semisimple Lie groups.

§6.1 Introduction: The result of Cowling and Meaney

Let G be a compact Lie group of rank l with finite centre, and with its Haar measure normalized to have total mass 1. Let $\mathfrak{g}$ denote its Lie algebra, and let $\mathfrak{h}$ be a maximal toral subalgebra of $\mathfrak{g}$. We denote by Φ the root system of $(\mathfrak{g}^{\mathbf{C}}, \mathfrak{h}^{\mathbf{C}})$, and fix $\Delta = \{\alpha_j : j \in I\}$, where $I = \{1, \ldots, l\}$, to be a base of Φ (see [21], §10.1, for definition). With respect to Δ, we write Φ^+ for the set of positive roots, whose members are of the form

$$\alpha = \sum_{j \in I} n_j(\alpha)\alpha_j,$$

with $n_j(\alpha) \in \mathbf{Z}^+ \cup \{0\}$ for all $j \in I$, and also write Λ^+ for the set of dominant weights, which parametrizes the dual object of G. We then introduce the special element ρ where

$$\rho = \tfrac{1}{2} \sum_{\alpha \in \Phi^+} \alpha,$$

and the constant γ where

$$\gamma = \min_{j \in I} |\{\alpha \in \Phi^+ : n_j(\alpha) \geq 1\}|.$$

We equip the Lie algebra $\mathfrak{g}$ with the positive definite inner product $(\cdot\,,\cdot)$ derived from the Killing form. For each $\nu \in \mathfrak{h}^*$, we define $H_\nu \in \mathfrak{h}$ by

$$\nu(H) = (H_\nu, H), \quad H \in \mathfrak{h}.$$

We also transfer the inner product to $\mathfrak{h}^*$ via

$$(\nu, \nu') = (H_\nu, H_{\nu'}), \quad \nu, \nu' \in \mathfrak{h}^*.$$

The norm on $\mathfrak{h}^*$ and $\mathfrak{h}$, induced by these inner products, will then be denoted by $|\cdot|$.

Now fix $R > 0$ such that $\exp(rH_\rho)$ is regular in G for any $r \in (0, R)$. For a continuous function f on G, we define the maximal function $\mathcal{M}f$ by

$$\mathcal{M}f(x) = \sup_{r \in (0,R)} |(\mu_r * f)(x)|, \quad x \in G,$$

where μ_r is the $\mathrm{Ad}(G)$-invariant probability measure carried on the conjugacy class of $\exp(rH_\rho)$ in G, whose convolution with f is given by the formula

$$(\mu_r * f)(x) = \int_G f(xg \, \exp(rH_\rho) g^{-1}) \, dg, \quad x \in G.$$

We then have the following result of Cowling and Meaney [14].

Theorem 6.1 (Cowling-Meaney)

The *a priori* inequality

$$\|\mathcal{M}f\|_p \leq C_p \, \|f\|_p, \quad f \in C(G),$$

holds for every $p > 1 + (2\gamma)^{-1}$.

The keys to their proof are the g-function technique and the decay estimates on $\hat{\mu}_r$ (see [14] for the details of the proof).

§6.2 A more general result

In defining the maximal function $\mathcal{M}f$, we could actually replace H_ρ by any regular element $H \in \mathfrak{h}$, for which $\alpha(H) \neq 0$ for all $\alpha \in \Phi^+$. In this case, we have the more general maximal function $\mathcal{M}_H f$, defined by

$$\mathcal{M}_H f(x) = \sup_{r \in (0,R)} |(\mu_{rH} * f)(x)|, \quad x \in G,$$

where μ_{rH} is the $\mathrm{Ad}(G)$-invariant probability measure carried on the conjugacy class of $\exp(rH)$ in G.

When G is semisimple, we have the following results.

Theorem 6.2A

The estimates

$$\left| \left(\frac{\partial}{\partial r}\right)^k \hat{\mu}_{rH}(\lambda) \right| \leq C_k(H) \, \frac{(1+|\lambda|)^k}{(1+r|\lambda|)^\gamma}, \quad r \in (0,R), \ \lambda \in \Lambda^+,$$

holds for all $k = 0, 1, 2, \ldots$.

THEOREM 6.2B

The *a priori* inequality

$$\|\mathcal{M}_H f\|_p \leq C_p(H)\,\|f\|_p, \quad f \in C(G),$$

holds for every $p > 1 + (2\gamma)^{-1}$.

Remark. It is clear that Theorem 6.2A, together with the arguments of [14], imply the boundedness of $\mathcal{M}_H$ on $L^p(G)$ when $p > 1 + (2\gamma)^{-1}$, which is the assertion of Theorem 6.2B.

We prove Theorem 6.2A by handling first the case when G is simple, and then extend the result to the semisimple case. Our method is based on arguments of Lie group and representation theory, involving formulae for characters and dimensions, a study of root systems, the theory of weights, and properties of the Weyl group, which we develop in the next section.

We remark that Theorem 6.2A is sharp since the explicit expression used in [14] for the particular case in which $H = H_\rho$ shows no improvement is possible. In §6.6, we give an example which shows that Theorem 6.2B too is sharp at least in the case where $G = \mathbf{SU}(2)$.

§6.3 Lie group and representation theoretic arguments

Throughout this section, we shall assume that the Lie algebra $\mathfrak{g}$ is simple.

We start with some formulae for characters and dimensions of representations of G. To each $\lambda \in \Lambda^+$, we associate the representation π_λ, the set of weights ϖ_λ, the character χ_λ, and the dimension $d_\lambda = \chi_\lambda(\mathbf{1})$. For all $\lambda \in \Lambda^+$, we have (see [21], §22)

$$\chi_\lambda(\exp(H)) = \sum_{\lambda' \in \varpi_\lambda} m_\lambda(\lambda') \exp(i\lambda'(H)),$$

where $m_\lambda(\lambda') \in \mathbf{Z}^+$ is the multiplicity of λ' in π_λ. Accordingly,

$$d_\lambda = \sum_{\lambda' \in \varpi_\lambda} m_\lambda(\lambda') \,.$$

Let $\mathcal{W}$ be the Weyl group of $(\mathfrak{g}^{\mathbf{C}}, \mathfrak{h}^{\mathbf{C}})$, generated by the reflections σ_α corresponding to $\alpha \in \Delta$. For all $\lambda \in \Lambda^+$, the character and dimension formulae of Weyl read (see [21], §24.3)

$$\chi_\lambda(\exp(H)) = \frac{\sum_{\sigma \in \mathcal{W}} \det(\sigma) \exp(i\sigma(\lambda+\rho)(H))}{\prod_{\alpha \in \Phi^+} 2i \sin \frac{1}{2}\alpha(H)}$$

and

$$d_\lambda = \prod_{\alpha \in \Phi^+} \frac{(\lambda+\rho, \alpha)}{(\rho, \alpha)}.$$

It is well-known that $\mathfrak{g}^{\mathbf{C}}$ has the root space decomposition (see [38], p. 273)

$$\mathfrak{g}^{\mathbf{C}} = \mathfrak{h}^{\mathbf{C}} \oplus \bigoplus_{\alpha \in \Phi} \mathfrak{g}^{\mathbf{C}}_\alpha,$$

where $\mathfrak{g}^{\mathbf{C}}_\alpha$ denotes the root subspace of $\mathfrak{g}^{\mathbf{C}}$ corresponding to $\alpha \in \Phi$.

Assuming $l \geq 2$, we choose $j_0 \in I$, and then remove α_{j_0} from Δ to obtain

$$\Delta_0 = \{\alpha_j : j \in I_0\}, \quad \text{where } I_0 = I \setminus \{j_0\}.$$

Set $\Phi_0^+ = \{\alpha \in \Phi^+ : n_{j_0}(\alpha) = 0\}$, and put $\Phi_0 = \Phi_0^+ \cup -\Phi_0^+$. Clearly $\Phi_0 = -\Phi_0$ and $\sigma_\alpha \Phi_0 = \Phi_0$ for all σ_α $(\alpha \in \Delta_0)$. This shows that Φ_0 is a root system (see

[38], p. 370). Let $\mathfrak{h}_0$ be the subspace of $\mathfrak{h}$ spanned by H_α ($\alpha \in \Phi_0$). Then one may verify that

$$\mathfrak{g}_0^{\mathbf{C}} = \mathfrak{h}_0^{\mathbf{C}} \oplus \bigoplus_{\alpha \in \Phi_0} \mathfrak{g}_\alpha^{\mathbf{C}}$$

is a semisimple subalgebra of $\mathfrak{g}^{\mathbf{C}}$, with maximal toral subalgebra $\mathfrak{h}_0^{\mathbf{C}}$ (see [38], Ex. 30 of Ch. 4). Evidently Φ_0 is the root system of $(\mathfrak{g}_0^{\mathbf{C}}, \mathfrak{h}_0^{\mathbf{C}})$, Δ_0 is a base of Φ_0, and Φ_0^+ is the set of positive roots with respect to Δ_0.

Write Φ_0 as a disjoint union of irreducible root systems, say

$$\Phi_0 = \Phi_{01} \cup \cdots \cup \Phi_{0r}.$$

Let $q \in \{1, \ldots, r\}$. Denote by $\mathfrak{h}_{0q}$ the subspace of $\mathfrak{h}_0$ spanned by H_α ($\alpha \in \Phi_{0q}$). Then we find that

$$\mathfrak{g}_{0q}^{\mathbf{C}} = \mathfrak{h}_{0q}^{\mathbf{C}} \oplus \bigoplus_{\alpha \in \Phi_{0q}} \mathfrak{g}_\alpha^{\mathbf{C}}$$

is a simple ideal of $\mathfrak{g}_0^{\mathbf{C}}$, with maximal toral subalgebra $\mathfrak{h}_{0q}^{\mathbf{C}}$. We also note that

$$\mathfrak{h}_0^{\mathbf{C}} = \mathfrak{h}_{01}^{\mathbf{C}} \oplus \cdots \oplus \mathfrak{h}_{0r}^{\mathbf{C}}$$

and

$$\mathfrak{g}_0^{\mathbf{C}} = \mathfrak{g}_{01}^{\mathbf{C}} \oplus \cdots \oplus \mathfrak{g}_{0r}^{\mathbf{C}}.$$

Now denote by $(\cdot\,,\cdot)_0$ and $(\cdot\,,\cdot)_{0q}$ the inner products of $\mathfrak{g}_0$ and $\mathfrak{g}_{0q}$ respectively. Then we have (see [21], Lemma 5.1)

$$(\cdot\,,\cdot)_0|_{\mathfrak{g}_{0q} \times \mathfrak{g}_{0q}} = (\cdot\,,\cdot)_{0q},$$

and so

$$(X, Y)_0 = (X_1, Y_1)_{01} + \cdots + (X_r, Y_r)_{0r}$$

for all $X = X_1 + \cdots + X_r,\ Y = Y_1 + \cdots + Y_r \in \mathfrak{g}_0$, with $X_q, Y_q \in \mathfrak{g}_{0q}$. Further, since $\mathfrak{g}$ and $\mathfrak{g}_{0q}$ are simple, there exists a positive constant C_q satisfying (see [28], p. 242)

$$(X,Y)_{0q} = C_q\,(X,Y), \quad X, Y \in \mathfrak{g}_{0q}.$$

We transfer these inner products to the corresponding dual spaces in the usual way.

Let Λ_0^+ denote the set of dominant weights with respect to Δ_0. We need to determine the set of fundamental weights in Λ_0^+. Suppose $\{\omega_j : j \in I\}$ is the set of fundamental weights in Λ^+, for which (see [21], §13.1, for definition)

$$2\,\frac{(\omega_j, \alpha_k)}{(\alpha_k, \alpha_k)} = \delta_{jk}, \quad j, k \in I.$$

If we now set

$$\widetilde{\omega}_j = \omega_j - \mathrm{proj}_{\omega_{j_0}}(\omega_j), \quad j \in I,$$

then we have the following facts.

Fact 6.3A For each $j \in I_0,\ \widetilde{\omega}_j \in \mathfrak{h}_{0Q}^*$ whenever $\alpha_j \in \mathfrak{h}_{0Q}^*$.

Proof. For all $j, k \in I_0$, we have

$$\begin{aligned} 2\,\frac{(\widetilde{\omega}_j, \alpha_k)}{(\alpha_k, \alpha_k)} &= 2\,\frac{(\omega_j, \alpha_k)}{(\alpha_k, \alpha_k)} - 2\,\frac{(\mathrm{proj}_{\omega_{j_0}}(\omega_j), \alpha_k)}{(\alpha_k, \alpha_k)} \\ &= 2\,\frac{(\omega_j, \alpha_k)}{(\alpha_k, \alpha_k)} - 2\,\frac{(\omega_j, \omega_{j_0})}{(\omega_{j_0}, \omega_{j_0})}\,\frac{(\omega_{j_0}, \alpha_k)}{(\alpha_k, \alpha_k)} \\ &= 2\,\frac{(\omega_j, \alpha_k)}{(\alpha_k, \alpha_k)} - 0 \\ &= \delta_{jk}. \end{aligned}$$

Now take $j \in I_0$, and let $Q \in \{1, \ldots, r\}$ such that $\alpha_j \in \mathfrak{h}_{0Q}^*$. Clearly

$$\widetilde{\omega}_j \perp \mathfrak{h}_{0q}^*, \quad q \neq Q.$$

Writing $\widetilde{\omega}_j = \widetilde{\omega}_{j1} + \cdots + \widetilde{\omega}_{jr}$, with $\widetilde{\omega}_{jq} \in \mathfrak{h}_{0q}^*$ for all $q \in \{1, \ldots, r\}$, we find that

$$\widetilde{\omega}_{jq} = 0, \quad q \neq Q.$$

We therefore have

$$\widetilde{\omega}_j = \widetilde{\omega}_{jQ} \in \mathfrak{h}_{0Q}^*,$$

as stated. □

Fact 6.3B $\{\widetilde{\omega}_j : j \in I_0\}$ is the set of fundamental weights in Λ_0^+.

Proof. Take $j, k \in I_0$. Suppose that $\widetilde{\omega}_j \in \mathfrak{h}_{0q}^*$ and $\alpha_k \in \mathfrak{h}_{0q'}^*$ for some $q, q' \in \{1, \ldots, r\}$. If $q \neq q'$, then clearly $(\widetilde{\omega}_j, \alpha_k)_0 = 0$; otherwise we have

$$2\,\frac{(\widetilde{\omega}_j, \alpha_k)_0}{(\alpha_k, \alpha_k)_0} = 2\,\frac{(\widetilde{\omega}_j, \alpha_k)_{0q}}{(\alpha_k, \alpha_k)_{0q}} = 2\,\frac{(\widetilde{\omega}_j, \alpha_k)}{(\alpha_k, \alpha_k)} = \delta_{jk}.$$

Using Fact 6.3A, the assertion follows. □

Fact 6.3C Suppose $\lambda = \sum_{j \in I} n_j\, \omega_j \in \Lambda^+$. Then λ can be rewritten as

$$\lambda = \lambda_0 + \lambda_1$$

where $\lambda_0 = \sum_{j \in I_0} n_j\, \widetilde{\omega}_j \in \Lambda_0^+$ (with the same n_j's) and $\lambda_1 = \mathrm{proj}_{\omega_{j_0}}(\lambda)$.

Proof. Note first that $\widetilde{\omega}_{j_0} = 0$. Now,

$$\begin{aligned}
\lambda &= \sum_{j\in I} n_j\,\omega_j \\
&= \sum_{j\in I} n_j\,\widetilde{\omega}_j + \sum_{j\in I} n_j\,\mathrm{proj}_{\omega_{j_0}}(\omega_j) \\
&= \sum_{j\in I_0} n_j\,\widetilde{\omega}_j + \sum_{j\in I} n_j\,\frac{(\omega_j,\omega_{j_0})}{(\omega_{j_0},\omega_{j_0})}\omega_{j_0} \\
&= \sum_{j\in I_0} n_j\,\widetilde{\omega}_j + \frac{(\sum_{j\in I} n_j\,\omega_j,\omega_{j_0})}{(\omega_{j_0},\omega_{j_0})}\omega_{j_0} \\
&= \sum_{j\in I_0} n_j\,\widetilde{\omega}_j + \frac{(\lambda,\omega_{j_0})}{(\omega_{j_0},\omega_{j_0})}\omega_{j_0} \\
&= \sum_{j\in I_0} n_j\,\widetilde{\omega}_j + \mathrm{proj}_{\omega_{j_0}}(\lambda) \\
&= \lambda_0 + \lambda_1,
\end{aligned}$$

as claimed. □

Remark. It is well-known that the special element ρ is a dominant weight in Λ^+. Indeed, $\rho = \sum_{j\in I}\omega_j$ (see [21], Lemma 13.3A). By Fact 6.3C, we may rewrite $\rho = \rho_0 + \rho_1$ where $\rho_0 = \sum_{j\in I_0}\widetilde{\omega}_j \in \Lambda_0^+$ and $\rho_1 = \mathrm{proj}_{\omega_{j_0}}(\rho)$. But then $\rho_0 = \frac{1}{2}\sum_{\alpha\in\Phi_0^+}\alpha$, giving $\rho_1 = \frac{1}{2}\sum_{\alpha\in\Phi_1^+}\alpha$, where $\Phi_1^+ = \Phi^+ \setminus \Phi_0^+$. As another consequence, we also have $\rho_1 = c\,\omega_{j_0}$ for some $c > 0$. But we know that $2\,\frac{(\omega_{j_0},\alpha_{j_0})}{(\alpha_{j_0},\alpha_{j_0})} = 1$, and so we find $c = 2\,\frac{(\rho_1,\alpha_{j_0})}{(\alpha_{j_0},\alpha_{j_0})}$. Hence we determine $\omega_{j_0} = \frac{1}{2}\,\frac{(\alpha_{j_0},\alpha_{j_0})}{(\rho_1,\alpha_{j_0})}\,\rho_1$, with $\rho_1 = \frac{1}{2}\sum_{\alpha\in\Phi_1^+}\alpha$. This offers a method of finding the fundamental weight ω_{j_0} for any given $j_0 \in I$.

Introduce $\mathfrak{h}_1 = \{H \in \mathfrak{h} : \alpha(H) = 0,\ \alpha \in \Delta_0\}$. Obviously $\mathfrak{h}_1$ is a subalgebra of $\mathfrak{h}$, which is spanned by H_{ρ_1} (by the above remark). Moreover, we have (like Fact 6.3C)

Fact 6.3D Every $H \in \mathfrak{h}$ can be written as

$$H = H_0 + H_1$$

where $H_0 \in \mathfrak{h}_0$ and $H_1 \in \mathfrak{h}_1$.

Remark. $H_0 \in \mathfrak{h}_0$ means that $H_0 = H_{\nu_0}$, where $\nu_0 \in \mathrm{span}(\Delta_0)$, while $H_1 \in \mathfrak{h}_1$ means that $H_1 = H_{\nu_1}$, where $\nu_1 = s\,\rho_1$ for some $s \in \mathbf{R}$. Thus clearly $\mathfrak{h}_0 \perp \mathfrak{h}_1$, and so Fact 6.3D actually states that $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}_1$.

Suppose we are in $(\mathfrak{g}_0, \mathfrak{h}_0)$. To each $\lambda_0 \in \Lambda_0^+$, we associate the representation $\widetilde{\pi}_{\lambda_0}$, the set of weights $\widetilde{\varpi}_{\lambda_0}$, the character $\widetilde{\chi}_{\lambda_0}$, and the dimension $\widetilde{d}_{\lambda_0}$. For all $\lambda_0 \in \Lambda_0^+$ and $H_0 \in \mathfrak{h}_0$, we have

$$\widetilde{\chi}_{\lambda_0}(\exp(H_0)) = \sum_{\lambda' \in \widetilde{\varpi}_{\lambda_0}} \widetilde{m}_{\lambda_0}(\lambda')\, \exp(i\lambda'(H_0))$$

and

$$\widetilde{d}_{\lambda_0} = \sum_{\lambda' \in \widetilde{\varpi}_{\lambda_0}} \widetilde{m}_{\lambda_0}(\lambda')\,,$$

with $\widetilde{m}_{\lambda_0}(\lambda') \in \mathbf{Z}^+$ being the multiplicity of λ' in $\widetilde{\pi}_{\lambda_0}$.

Let $\mathcal{W}_0$ (or $\mathcal{W}[\Delta_0]$ if necessary) denote the subgroup of $\mathcal{W}$ generated by σ_α $(\alpha \in \Delta_0)$. The Weyl formulae then read

$$\widetilde{\chi}_{\lambda_0}(\exp(H_0)) = \frac{\sum_{\tau \in \mathcal{W}_0} \det(\tau)\, \exp(i\tau(\lambda_0 + \rho_0)(H_0))}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\alpha(H_0)}$$

and

$$\widetilde{d}_{\lambda_0} = \prod_{\alpha \in \Phi_0^+} \frac{(\lambda_0 + \rho_0, \alpha)}{(\rho_0, \alpha)}.$$

We should note that the inner product in the expression above is really the inner product of $\mathfrak{g}$. Indeed, we may calculate

$$
\begin{aligned}
\widetilde{d}_{\lambda_0} &= \lim_{r\to 0} \widetilde{\chi}_{\lambda_0}(\exp(rH_{\rho_0})) \\
&= \lim_{r\to 0} \frac{\sum_{\tau\in\mathcal{W}_0} \det(\tau)\, \exp(i\tau(\lambda_0+\rho_0)(rH_{\rho_0}))}{\prod_{\alpha\in\Phi_0^+} 2i\,\sin\frac{1}{2}\alpha(rH_{\rho_0})} \\
&= \lim_{r\to 0} \frac{\sum_{\tau\in\mathcal{W}_0} \det(\tau)\, \exp(i\tau\rho_0(rH_{\lambda_0+\rho_0}))}{\prod_{\alpha\in\Phi_0^+} 2i\,\sin\frac{1}{2}\alpha(rH_{\rho_0})} \\
&= \lim_{r\to 0} \frac{\prod_{\alpha\in\Phi_0^+} 2i\,\sin\frac{1}{2}\alpha(rH_{\lambda_0+\rho_0})}{\prod_{\alpha\in\Phi_0^+} 2i\,\sin\frac{1}{2}\alpha(rH_{\rho_0})} \\
&= \prod_{\alpha\in\Phi_0^+} \frac{\alpha(H_{\lambda_0+\rho_0})}{\alpha(H_{\rho_0})} \\
&= \prod_{\alpha\in\Phi_0^+} \frac{(\lambda_0+\rho_0,\alpha)}{(\rho_0,\alpha)}
\end{aligned}
$$

(see [39], p. 106, for clarification).

Allowing $\mathcal{W}$ to act, one may observe that all the above facts still hold for the system constituted by $\sigma\Phi_0$ ($\sigma\in\mathcal{W}$), as well as for that by Φ_0. Moreover, the two facts below explain the connection between one system and another.

Fact 6.3E $\sigma\mathcal{W}[\Delta_0]\sigma^{-1} = \mathcal{W}[\sigma\Delta_0]$ for any $\sigma\in\mathcal{W}$.

Proof. Obvious (see [21], Lemma 9.2, for justification). □

Fact 6.3F $\widetilde{\chi}_{\sigma\lambda_0}(\exp(H_{\sigma\nu_0})) = \widetilde{\chi}_{\lambda_0}(\exp(H_{\nu_0}))$ for any $\sigma\in\mathcal{W}$.

PROOF. For any $\sigma \in \mathcal{W}$, we have (by Fact 6.3E)

$$\begin{aligned}
\widetilde{\chi}_{\sigma\lambda_0}(\exp(H_{\sigma\nu_0})) &= \frac{\sum_{\tau \in \mathcal{W}[\sigma\Delta_0]} \det(\tau) \exp(i\tau(\sigma\lambda_0 + \sigma\rho_0)(H_{\sigma\nu_0}))}{\prod_{\alpha \in \sigma\Phi_0^+} 2i \sin \frac{1}{2}\alpha(H_{\sigma\nu_0})} \\
&= \frac{\sum_{\tau \in \sigma\mathcal{W}[\Delta_0]\sigma^{-1}} \det(\tau) \exp(i\tau\sigma(\lambda_0 + \rho_0)(H_{\sigma\nu_0}))}{\prod_{\alpha \in \sigma\Phi_0^+} 2i \sin \frac{1}{2}\alpha(H_{\sigma\nu_0})} \\
&= \frac{\sum_{\tau \in \mathcal{W}[\Delta_0]} \det(\sigma\tau\sigma^{-1}) \exp(i\sigma\tau(\lambda_0 + \rho_0)(H_{\sigma\nu_0}))}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\sigma\alpha(H_{\sigma\nu_0})} \\
&= \frac{\sum_{\tau \in \mathcal{W}[\Delta_0]} \det(\tau) \exp(i\tau(\lambda_0 + \rho_0)(H_{\nu_0}))}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\alpha(H_{\nu_0})} \\
&= \widetilde{\chi}_{\lambda_0}(\exp(H_{\nu_0})),
\end{aligned}$$

as claimed. □

§6.4 The proof of the theorem

The outline of the proof is as follows. We first look for an estimate for all $r \in (0, R)$, then examine the decay for large r, and finally combine the results. The result obtained is valid under the assumption that G is simple, but then it extends to every semisimple Lie group G.

First, for all $r \in (0, R),\ \lambda \in \Lambda^+$, we have (see [14], p. 813)

$$\hat{\mu}_{rH}(\lambda) = \frac{\chi_\lambda(\exp(rH))}{d_\lambda}.$$

Using the multiplicity formulae, we write

$$\hat{\mu}_{rH}(\lambda) = \frac{\sum_{\lambda' \in \varpi_\lambda} m_\lambda(\lambda') \exp(i\lambda'(rH))}{\sum_{\lambda' \in \varpi_\lambda} m_\lambda(\lambda')}.$$

It follows from this expression that

$$\left|\left(\frac{\partial}{\partial r}\right)^k \hat{\mu}_{rH}(\lambda)\right| \leq \frac{\sum_{\lambda' \in \varpi_\lambda} m_\lambda(\lambda') \left|\left(\frac{\partial}{\partial r}\right)^k \exp(i\lambda'(rH))\right|}{\sum_{\lambda' \in \varpi_\lambda} m_\lambda(\lambda')}$$

$$\leq |H|^k \, |\lambda|^k$$

$$= C_k(H) \, |\lambda|^k,$$

for all $k = 0, 1, 2, \ldots$.

Now, by the Weyl formulae, for all $r \in (0, R)$, $\lambda \in \Lambda^+$, we have

$$\hat{\mu}_{rH}(\lambda) = \frac{\sum_{\sigma \in \mathcal{W}} \det(\sigma) \, \exp(i(\lambda + \rho)(rH))}{\prod_{\alpha \in \Phi^+} 2i \, \sin \frac{1}{2}\alpha(rH)} \prod_{\alpha \in \Phi^+} \frac{(\rho, \alpha)}{(\lambda + \rho, \alpha)}.$$

In the case $l = 1$, one can easily obtain

$$\left|\left(\frac{\partial}{\partial r}\right)^k \hat{\mu}_{rH}(\lambda)\right| \leq C_k(H) \, \frac{|\lambda + \rho|^k}{r|\lambda + \rho|},$$

for all $k = 0, 1, 2, \ldots$. So assume, hereafter, that $l \geq 2$.

For each $\lambda \in \Lambda^+$, choose $j_0 \in I$ for which $(\lambda + \rho, \alpha_{j_0})$ is maximal. As before, we write $\Delta_0 = \Delta \setminus \{\alpha_{j_0}\}$, $\Phi_0^+ = \{\alpha \in \Phi^+ : n_{j_0}(\alpha) = 0\}$, and $\Phi_1^+ = \{\alpha \in \Phi^+ : n_{j_0}(\alpha) \geq 1\}$. (Note that $\Phi_1^+ = \Phi^+ \setminus \Phi_0^+$, and that Φ_1^+ depends on the choice of j_0, and so depends on λ.) Clearly, if $\alpha \in \Phi_0^+$, then

$$(\lambda + \rho, \alpha) \;\geq\; (\rho, \alpha) \;\geq\; C,$$

and if $\alpha \in \Phi_1^+$, then (by the choice of j_0)

$$(\lambda + \rho, \alpha) \;\geq\; n_{j_0}(\alpha) \, (\lambda + \rho, \alpha_{j_0}) \;\geq\; C \, |\lambda + \rho|.$$

Moreover, we have

$$\gamma = \min_{j \in I} | \, \{\alpha \in \Phi^+ : n_j(\alpha) \geq 1\} \, | \leq |\Phi_1^+|.$$

Recall that $\mathcal{W}_0$ is the subgroup of $\mathcal{W}$ generated by σ_α ($\alpha \in \Delta_0$). For an appropriate $\mathcal{S} \subset \mathcal{W}$, we may decompose $\mathcal{W}$ as

$$\mathcal{W} = \bigcup_{\sigma \in \mathcal{S}} \sigma \mathcal{W}_0 \quad \text{(disjoint union)}.$$

We then obtain

$$\hat{\mu}_{rH}(\lambda) = \sum_{\sigma \in \mathcal{S}} \left(\frac{\sum_{\tau \in \mathcal{W}_0} \det(\sigma\tau)\, \exp(i\sigma\tau(\lambda+\rho)(rH))}{\prod_{\alpha \in \Phi^+} 2i \sin \frac{1}{2}\alpha(rH)} \prod_{\alpha \in \Phi^+} \frac{(\rho,\alpha)}{(\lambda+\rho,\alpha)} \right).$$

For each reflection $\sigma_\alpha \in \mathcal{W}$, we know that $\det(\sigma_\alpha) = -1$, $\sigma_\alpha \alpha = -\alpha$, and $\sigma_\alpha(\Phi^+ \setminus \{\alpha\}) = \Phi^+ \setminus \{\alpha\}$ (see [21], Lemma 10.2B). Thus, for any $\sigma \in \mathcal{W}$, we have

$$\prod_{\alpha \in \Phi^+} 2i \sin \tfrac{1}{2}\alpha(rH) = \det(\sigma) \prod_{\alpha \in \Phi^+} 2i \sin \tfrac{1}{2}\sigma\alpha(rH).$$

It follows that

$$\hat{\mu}_{rH}(\lambda) = \sum_{\sigma \in \mathcal{S}} \left(\frac{\sum_{\tau \in \mathcal{W}_0} \det(\tau)\, \exp(i\sigma\tau(\lambda+\rho)(rH))}{\prod_{\alpha \in \Phi_0^+} 2i \sin \frac{1}{2}\sigma\alpha(rH)} \prod_{\alpha \in \Phi_0^+} \frac{(\rho,\alpha)}{(\lambda+\rho,\alpha)} \right) \times$$
$$\left(\prod_{\alpha \in \Phi_1^+} \frac{1}{2i \sin \frac{1}{2}\sigma\alpha(rH)} \frac{(\rho,\alpha)}{(\lambda+\rho,\alpha)} \right).$$

Now fix $\sigma \in \mathcal{S}$. We write $H = H_{\sigma\nu}$, with $\nu = \nu_0 + \nu_1$, where $\nu_0 \in \operatorname{span}(\Delta_0)$ and $\nu_1 = s\,\rho_1$ for some $s \in \mathbf{R}$. Then put $H_0 = H_{\nu_0}$ and $H_1 = H_{\nu_1}$. Next recall that Λ_0^+ is the set of dominant weights corresponding to Φ_0^+. For each $\lambda \in \Lambda^+$, we write $\lambda = \lambda_0 + \lambda_1$, where $\lambda_0 \in \Lambda_0^+$ and $\lambda_1 = c\,\rho_1$ for some $c \in \mathbf{R}^+$. Hence, for all $\alpha \in \Phi_0^+$, we have

$$(\rho,\alpha) = (\rho_0,\alpha)$$

and

$$(\lambda+\rho,\alpha) = (\lambda_0+\rho_0,\alpha).$$

Furthermore, for all $\alpha \in \Phi_0^+$, we have

$$\begin{aligned}
\sigma\alpha(H) &= (\sigma\alpha, \sigma\nu) \\
&= (\alpha, \nu) \\
&= (\alpha, \nu_0 + \nu_1) \\
&= (\alpha, \nu_0) \qquad \text{(as } \nu_1 \perp \alpha) \\
&= \alpha(H_{\nu_0}) \\
&= \alpha(H_0),
\end{aligned}$$

and (whenever $\tau \in \mathcal{W}_0$)

$$\begin{aligned}
\sigma\tau(\lambda+\rho)(H) &= (\sigma\tau(\lambda+\rho), \sigma\nu) \\
&= (\tau(\lambda+\rho), \nu) \\
&= (\tau(\lambda_0+\rho_0) + \tau(\lambda_1+\rho_1), \nu_0+\nu_1) \\
&= (\tau(\lambda_0+\rho_0) + (\lambda_1+\rho_1), \nu_0+\nu_1) \qquad \text{(as } \tau \in \mathcal{W}_0) \\
&= (\tau(\lambda_0+\rho_0), \nu_0) + (\lambda_1+\rho_1, \nu_1) \qquad \text{(by orthogonality)} \\
&= \tau(\lambda_0+\rho_0)(H_{\nu_0}) + (\lambda_1+\rho_1)(H_{\nu_1}) \\
&= \tau(\lambda_0+\rho_0)(H_0) + (\lambda_1+\rho_1)(H_1).
\end{aligned}$$

It turns out that

$$\frac{\sum_{\tau\in\mathcal{W}_0}\det(\tau)\,\exp(i\sigma\tau(\lambda+\rho)(rH))}{\prod_{\alpha\in\Phi_0^+}2i\,\sin\frac{1}{2}\sigma\alpha(rH)}\prod_{\alpha\in\Phi_0^+}\frac{(\rho,\alpha)}{(\lambda+\rho,\alpha)}$$

$$=\exp(i(\lambda_1+\rho_1)(rH_1))\,\frac{\sum_{\tau\in\mathcal{W}_0}\det(\tau)\,\exp(i\tau(\lambda_0+\rho_0)(rH_0))}{\prod_{\alpha\in\Phi_0^+}2i\,\sin\frac{1}{2}\alpha(rH_0)}\times\prod_{\alpha\in\Phi_0^+}\frac{(\rho_0,\alpha)}{(\lambda_0+\rho_0,\alpha)}$$

$$=\exp(i(\lambda_1+\rho_1)(rH_1))\,\frac{\widetilde{\chi}_{\lambda_0}(\exp(rH_0))}{\widetilde{d}_{\lambda_0}}$$

$$=\exp(i(\lambda_1+\rho_1)(rH_1))\,\frac{\sum_{\lambda'\in\widetilde{\varpi}_{\lambda_0}}\widetilde{m}_{\lambda_0}(\lambda')\,\exp(i\lambda'(rH_0))}{\sum_{\lambda'\in\widetilde{\varpi}_{\lambda_0}}\widetilde{m}_{\lambda_0}(\lambda')}$$

$$=\frac{\sum_{\lambda'\in\widetilde{\varpi}_{\lambda_0}}\widetilde{m}_{\lambda_0}(\lambda')\,\exp(i(\lambda'+\lambda_1+\rho_1)(rH))}{\sum_{\lambda'\in\widetilde{\varpi}_{\lambda_0}}\widetilde{m}_{\lambda_0}(\lambda')}\qquad\text{(by orthogonality).}$$

Thus we find that

$$\hat{\mu}_{rH}(\lambda)=\sum_{\sigma\in\mathcal{S}}\left(\frac{\sum_{\lambda'\in\widetilde{\varpi}_{\lambda_0}}\widetilde{m}_{\lambda_0}(\lambda')\,\exp(i(\lambda'+\lambda_1+\rho_1)(rH))}{\sum_{\lambda'\in\widetilde{\varpi}_{\lambda_0}}\widetilde{m}_{\lambda_0}(\lambda')}\right)\times\left(\prod_{\alpha\in\Phi_1^+}\frac{1}{\sigma\alpha(rH)}\,\frac{\sigma\alpha(rH)}{2i\,\sin\frac{1}{2}\sigma\alpha(rH)}\,\frac{(\rho,\alpha)}{(\lambda+\rho,\alpha)}\right).$$

Now, for all $k=0,1,2,\ldots$, we have the estimates

$$\left|\left(\frac{\partial}{\partial r}\right)^k\frac{\sum_{\lambda'\in\widetilde{\varpi}_{\lambda_0}}\widetilde{m}_{\lambda_0}(\lambda')\,\exp(i(\lambda'+\lambda_1+\rho_1)(rH))}{\sum_{\lambda'\in\widetilde{\varpi}_{\lambda_0}}\widetilde{m}_{\lambda_0}(\lambda')}\right|\le|H|^k\,|\lambda+\rho|^k,$$

$$\left|\left(\frac{\partial}{\partial r}\right)^k\prod_{\alpha\in\Phi_1^+}\sigma\alpha(rH)^{-1}\right|\le C_k(H)\,r^{-k-|\Phi_1^+|},$$

$$\left|\left(\frac{\partial}{\partial r}\right)^k\prod_{\alpha\in\Phi_1^+}\frac{\sigma\alpha(rH)}{2i\,\sin\frac{1}{2}\sigma\alpha(rH)}\right|\le C_k\qquad\text{(by Leibnitz' rule), and}$$

$$\left|\prod_{\alpha\in\Phi_1^+}\frac{(\rho,\alpha)}{(\lambda+\rho,\alpha)}\right|\le C\,|\lambda+\rho|^{-|\Phi_1^+|}\qquad(\text{as }(\lambda+\rho,\alpha)\ge C\,|\lambda+\rho|\text{ for }\alpha\in\Phi_1^+).$$

Therefore, by Leibnitz' rule for the derivatives of products, we obtain

$$\begin{aligned}
&\left|\left(\frac{\partial}{\partial r}\right)^k \hat{\mu}_{rH}(\lambda)\right| \\
&\le \sum_{\sigma\in\mathcal{S}} \sum_{k_1+k_2+k_3=k} C_{k_1,k_2,k_3} \left|\left(\frac{\partial}{\partial r}\right)^{k_1}(1^{\text{st}}\text{ term})\right| \left|\left(\frac{\partial}{\partial r}\right)^{k_2}(2^{\text{nd}}\text{ term})\right| \times \\
&\qquad\qquad \left|\left(\frac{\partial}{\partial r}\right)^{k_3}(3^{\text{rd}}\text{ term})\right| \left|4^{\text{th}}\text{ term}\right| \\
&\le \sum_{\sigma\in\mathcal{S}} \sum_{k_1+k_2+k_3=k} C_{k_1,k_2,k_3}(H)\,|H|^{k_1}\,|\lambda+\rho|^{k_1}\,r^{-k_2}\,(r|\lambda+\rho|)^{-|\Phi_1^+|} \\
&\le C_k(H)\,(1+|H|)^k\,\frac{|\lambda+\rho|^k}{(r|\lambda+\rho|)^{|\Phi_1^+|}} \qquad (\text{provided } r|\lambda+\rho|>1) \\
&\le C_k(H)\,\frac{|\lambda+\rho|^k}{(r|\lambda+\rho|)^{\gamma}} \qquad (\text{as } \gamma\le|\Phi_1^+|),
\end{aligned}$$

for all $k=0,1,2,\ldots,$ as desired.

Combining this with the previous estimate, we obtain the result. □

§6.5 An extension to the semisimple case

We shall here extend our result to every semisimple Lie group G. The key is to verify the validity of Fact 6.3B.

We write Φ as a disjoint union of irreducible root systems (see [29], p. 36)

$$\Phi = \Phi^{(1)} \cup \cdots \cup \Phi^{(n)},$$

and split Δ into

$$\Delta = \Delta^{(1)} \cup \cdots \cup \Delta^{(n)},$$

with $\Delta^{(m)} = \Delta \cap \Phi^{(m)}$ being a base of $\Phi^{(m)}$ for each $m \in \{1,\ldots,n\}$. The Lie algebra $\mathfrak{g}^{\mathbf{C}}$ is now a direct sum of simple ideals

$$\mathfrak{g}^{\mathbf{C}} = \mathfrak{g}^{(1)\mathbf{C}} \oplus \cdots \oplus \mathfrak{g}^{(n)\mathbf{C}}.$$

As before, we choose $j_0 \in I$ and remove α_{j_0} from Δ to obtain

$$\Delta_0 = \Delta \setminus \{\alpha_{j_0}\}.$$

But $\alpha_{j_0} \in \Delta^{(M)}$ for some $M \in \{1, \dots, n\}$, and so

$$\Delta_0 = \Delta^{(1)} \cup \cdots \cup \Delta_0^{(M)} \cup \cdots \Delta^{(n)},$$

with $\Delta_0^{(M)} = \Delta^{(M)} \setminus \{\alpha_{j_0}\}$. The Lie algebra $\mathfrak{g}_0$ (as in §6.3) then decomposes into

$$\mathfrak{g}_0^{\mathbf{C}} = \mathfrak{g}^{(1)\mathbf{C}} \oplus \cdots \oplus \mathfrak{g}_0^{(M)\mathbf{C}} \oplus \cdots \oplus \mathfrak{g}^{(n)\mathbf{C}},$$

where $\mathfrak{g}_0^{(M)\mathbf{C}}$ is the Lie subalgebra corresponding to $\Delta_0^{(M)}$.

Now let $K,\ K_0,\ K^{(m)}$, and $K_0^{(M)}$ denote the Killing forms of $\mathfrak{g}$, $\mathfrak{g}_0$, $\mathfrak{g}^{(m)}$, and $\mathfrak{g}_0^{(M)}$ respectively. Then, for each $m \in \{1, \dots, n\},\ m \neq M$, we have

$$K_0|_{\mathfrak{g}^{(m)} \times \mathfrak{g}^{(m)}} = K^{(m)} = K|_{\mathfrak{g}^{(m)} \times \mathfrak{g}^{(m)}};$$

while for $m = M$, the connection between $K^{(M)}$ and $K_0^{(M)}$ is explained in §6.3. So we find that Fact 6.3B still holds.

The extension is therefore clear. $\square$

§6.6 An example: the sharpness of the estimate

We shall here consider an example concerning the sharpness of the L^p-estimate of Theorem 6.2B.

Let $G = \mathbf{SU}(2)$, the Lie group consisting of 2×2 complex matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

with $|\alpha|^2 + |\beta|^2 = 1$. Its Lie algebra $\mathfrak{g}$ then contains all matrices of the form

$$\begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix}$$

with $a \in \mathbf{R},\ b \in \mathbf{C}$. Here $\gamma = 1$ and the special element is

$$H_\rho = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

In $\mathfrak{g}$, one may define the norm $|\cdot|$ by

$$\left| \begin{pmatrix} ia & b \\ -\bar{b} & -ia \end{pmatrix} \right| = (a^2 + |b|^2)^{\frac{1}{2}}, \quad a \in \mathbf{R},\ b \in \mathbf{C}.$$

For any $y \in G,\ X \in \mathfrak{g}$, one may observe that

$$X' = yXy^{-1} \in \mathfrak{g},$$

with $|X'| = |X|$. Conversely, for any $X,\ X' \in \mathfrak{g}$ with $|X| = |X'|$, one can find $y \in G$ such that $X' = yXy^{-1}$.

Denote by $B_0(\pi)$ the ball in $\mathfrak{g}$ which has centre 0 and radius π. It is then evident that the map $\exp : B_0(\pi) \to G$ is injective. Indeed, for each $x \in G$, there exists a unique $X \in B_0(\pi)$ for which $x = \exp(X)$. Diagonalizing such an X, one has

$$x = y\, \exp(\omega H_\rho)\, y^{-1}, \quad \text{where } \omega = |X|,$$

for some $y \in G$. It is seen here that $\text{trace}(x) = 2\, \cos \omega$.

As suggested in [34], let us consider the function $f : G \to \mathbf{R}^+$ given by

$$f(\exp(X)) = \begin{cases} \frac{|X|^{-2}}{\log |X|^{-1}}, & \text{if } 0 < |X| < \frac{1}{2}, \\ 0, & \text{otherwise.} \end{cases}$$

One may observe that $f \in L^p(G)$, whenever $1 \leq p \leq \frac{3}{2}$. On the other hand, regarding the maximal function $\mathcal{M}f = \mathcal{M}_{H_\rho} f$, we claim that $\mathcal{M}f(x) = \infty$ for all $x \in G$.

Before verifying our claim, we remark that $f(-\exp(X)) = f(\exp(X'))$ where $|X'| = \pi - |X|$. Moreover, $f(yxy^{-1}) = f(x)$ for all $x,\ y \in G$. In fact, for all $x,\ y \in G$, we have

$$\begin{aligned} f(yxy^{-1}) &= f(y\, \exp(X)\, y^{-1}) \qquad \text{(for some } X \in \mathfrak{g}) \\ &= f(\exp(yXy^{-1})) \\ &= f(\exp(X')) \qquad \text{(where } |X'| = |X|) \\ &= f(\exp(X)) \\ &= f(x). \end{aligned}$$

Similarly, we observe that $\mathcal{M}f(yxy^{-1}) = \mathcal{M}f(x)$ for all $x,\ y \in G$. To be precise, for all $x,\ y \in G$, we have

$$\begin{aligned} \mathcal{M}f(yxy^{-1}) &= \sup_{r\in(0,\pi)} \int_G f(yxy^{-1}g\, \exp(rH_\rho)\, g^{-1})\, dg \\ &= \sup_{r\in(0,\pi)} \int_G f(xy^{-1}g\, \exp(rH_\rho)\, g^{-1}y)\, dg \\ &= \sup_{r\in(0,\pi)} \int_G f(xg'\, \exp(rH_\rho)\, g'^{-1})\, dg' \qquad (g' = y^{-1}g) \\ &= \mathcal{M}f(x). \end{aligned}$$

We shall now verify our claim. First, for $x = \pm\mathbf{1}$, we have

$$\begin{aligned}\mathcal{M}f(\pm\mathbf{1}) &= \sup_{r\in(0,\pi)} \int_G f(\pm g\,\exp(rH_\rho)\,g^{-1})\,dg\\ &= \sup_{r\in(0,\pi)} \int_G f(\pm\exp(rH_\rho))\,dg\\ &= \sup_{r\in(0,\pi)} f(\pm\exp(rH_\rho))\\ &= \sup_{r\in(0,\frac{1}{2})} \frac{r^{-2}}{\log r^{-1}}\\ &= \infty.\end{aligned}$$

Next, for $x \neq \pm\mathbf{1}$, we may assume that $x = \exp(\frac{t}{2}H_\rho)$ for some $0 < t < 2\pi$, and hence

$$\begin{aligned}\mathcal{M}f(x) &= \mathcal{M}f(\exp(\tfrac{t}{2}H_\rho))\\ &= \sup_{r\in(0,\pi)} \int_G f(\exp(\tfrac{t}{2}H_\rho)\,g\,\exp(rH_\rho)\,g^{-1})\,dg\\ &\geq \int_G f(\exp(\tfrac{t}{2}H_\rho)\,g\,\exp(\tfrac{t}{2}H_\rho)\,g^{-1})\,dg.\end{aligned}$$

Writing each $g \in G$ as $g = h_\theta k_\phi h_{\theta'}$, where $h_\theta = \exp(\frac{\theta}{2}H_\rho)$ and k_ϕ is the matrix of rotation with angle $\frac{\phi}{2}$, we have (see [41], pp. 99-100)

$$\begin{aligned}\mathcal{M}f(x) &\geq \frac{1}{16\pi^2}\int_{-2\pi}^{2\pi}\int_0^{2\pi}\int_0^{\pi} f(h_t h_\theta k_\phi h_{\theta'} h_t h_{-\theta'} k_{-\phi} h_{-\theta})\,\sin\phi\,d\phi\,d\theta\,d\theta'\\ &= \frac{1}{16\pi^2}\int_{-2\pi}^{2\pi}\int_0^{2\pi}\int_0^{\pi} f(h_t h_\theta k_\phi h_t k_{-\phi} h_{-\theta})\,\sin\phi\,d\phi\,d\theta\,d\theta'\\ &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(h_t h_\theta k_\phi h_t k_{-\phi} h_{-\theta})\,\sin\phi\,d\phi\,d\theta\\ &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(h_{-\theta} h_t h_\theta k_\phi h_t k_{-\phi})\,\sin\phi\,d\phi\,d\theta\\ &= \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(h_t k_\phi h_t k_{-\phi})\,\sin\phi\,d\phi\,d\theta\\ &= \frac{1}{2}\int_0^{\pi} f(h_t k_\phi h_t k_{-\phi})\,\sin\phi\,d\phi.\end{aligned}$$

Let us now investigate the integrand. Multiplying out, we get

$$h_t k_\phi h_t k_{-\phi} = \begin{pmatrix} e^{it}\cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2} & \cos\frac{\phi}{2}\sin\frac{\phi}{2}(1-e^{it}) \\ -\cos\frac{\phi}{2}\sin\frac{\phi}{2}(1-e^{-it}) & e^{-it}\cos^2\frac{\phi}{2} + \sin^2\frac{\phi}{2} \end{pmatrix}.$$

As seen before, this matrix is similar to $\exp(\omega H_\rho)$, where $\omega = \cos^{-1}(\sin^2\frac{\phi}{2} + \cos^2\frac{\phi}{2}\cos t)$. By observation (thanks to John Cornwall for making it easier), there exists a constant $C = C_t \in (0,1)$ such that

$$\cos(\pi - \phi) \le \sin^2\frac{\phi}{2} + \cos^2\frac{\phi}{2}\cos t \le \cos C(\pi - \phi), \quad \phi \in (\pi - \tfrac{1}{2}, \pi),$$

and accordingly

$$0 < C(\pi - \phi) \le \omega \le \pi - \phi < \tfrac{1}{2}, \quad \phi \in (\pi - \tfrac{1}{2}, \pi).$$

Hence we find that

$$\begin{aligned} f(h_t k_\phi h_t k_{-\phi}) &= f(\exp(\omega H_\rho)) \\ &= \frac{\omega^{-2}}{\log\omega^{-1}} \\ &\ge \frac{(\pi-\phi)^{-2}}{\log\{C(\pi-\phi)\}^{-1}}, \end{aligned}$$

for all $\phi \in (\pi - \frac{1}{2}, \pi)$. It therefore follows that

$$\begin{aligned} \mathcal{M}f(x) &\ge \frac{1}{2}\int_{\pi-\frac{1}{2}}^{\pi} \frac{(\pi-\phi)^{-2}}{\log\{C(\pi-\phi)\}^{-1}} \sin\phi\, d\phi \\ &\ge \frac{1}{4}\int_{\pi-\frac{1}{2}}^{\pi} \frac{(\pi-\phi)^{-1}}{\log\{C(\pi-\phi)\}^{-1}}\, d\phi \\ &= \frac{1}{4}\int_{0}^{C/2} \frac{\varphi^{-1}}{\log\varphi^{-1}}\, d\varphi \\ &= \frac{1}{4}\int_{\log(2/C)}^{\infty} \frac{d\psi}{\psi} \\ &= \infty, \end{aligned}$$

as claimed. □

§6.7 Table of $|\Phi^+|$ and γ

The following table is due to Cowling and Meaney [14]. The reader should consult the paper for the method of finding the constant γ.

Root system	$\|\Phi^+\|$	γ
A_l $(l \geq 1)$	$l(l+1)/2$	l
B_l $(l \geq 2)$	l^2	$2l-1$
C_l $(l \geq 3)$	l^2	$2l-1$
D_l $(l \geq 4)$	$l(l-1)$	$2l-2$
E_6	36	16
E_7	63	27
E_8	120	57
F_4	24	15
G_2	6	5

Table 1. $|\Phi^+|$ and γ