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THE BOUNDEDNESS OF BESSEL-RIESZ OPERATORS ON GENERALIZED MORREY SPACES

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ABSTRACT. In this paper, we prove the boundedness of Bessel-Riesz operators on generalized Morrey spaces. The proof uses the usual dyadic decomposition, a Hedberg-type inequality for the operators, and the boundedness of Hardy-Littlewood maximal operator. Our results reveal that the norm of the operators is dominated by the norm of the kernels.

Key words and phrases: Bessel-Riesz operators, Hardy-Littlewood maximal operator, generalized Morrey spaces, Boundedness, Kernels.

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1. INTRODUCTION

We begin with the definition of Bessel-Riesz operators. For $\gamma \geq 0$ and $0 < \alpha < n$, we define

$$I_{\alpha,\gamma} f(x) := \int_{\mathbb{R}^n} K_{\alpha,\gamma}(x-y) f(y)\, dy,$$

for every $f \in L^p_{loc}(\mathbb{R}^n)$, $p \geq 1$, where $K_{\alpha,,\gamma}(x) := \frac{|x|^{\alpha-n}}{(1+|x|)^\gamma}$. Here, $I_{\alpha,\gamma}$ is called *Bessel-Riesz operator* and $K_{\alpha,\gamma}$ is called *Bessel-Riesz kernel*. The name of the kernel resembles the product of Bessel kernel and Riesz kernel [13]. While the Riesz kernel captures the local behaviour, the Bessel kernel take cares the global behaviour of the function. The Bessel-Riesz kernel is used in studying the behaviour of the solution of a Schrödinger type equation [8].

For $\gamma = 0$, we have $I_{\alpha,0} = I_\alpha$, known as *fractional integral operators* or *Riesz potentials*[12]. Around 1930, Hardy and Littlewood [5, 6] and Sobolev [11] have proved the boundedness of I_α on Lebesgue spaces via the inequality

$$\|I_\alpha f\|_{L^q} \leq C\, \|f\|_{L^p},$$

for every $f \in L^p(\mathbb{R}^n)$, where $1 < p < \frac{n}{\alpha}$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. Here C denotes a constant which may depend on α, p, q, and n, but not on f.

For $1 \leq p \leq q$, the *(classical) Morrey space* $L^{p,q}(\mathbb{R}^n)$ is defined by

$$L^{p,q}(\mathbb{R}^n) := \left\{ f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{L^{p,q}} < \infty \right\},$$

where $\|f\|_{L^{p,q}} := \sup\limits_{r>0, a\in\mathbb{R}^n} r^{n(1/q-1/p)} \left(\int_{|x-a|<r} |f(x)|^p dx \right)^{1/p}$. For these spaces, we have an inclusion property which is presented by the following theorem.

Theorem 1.1. *For $1 \leq p \leq q$, we have $L^q(\mathbb{R}^n) = L^{q,q}(\mathbb{R}^n) \subseteq L^{p,q}(\mathbb{R}^n) \subseteq L^{1,q}(\mathbb{R}^n)$.*

On Morrey spaces, Spanne [10] has shown that I_α is bounded form $L^{p_1,q_1}(\mathbb{R}^n)$ to $L^{p_2,q_2}(\mathbb{R}^n)$ for $1 < p_1 < q_1 < \frac{n}{\alpha}$, $\frac{1}{p_2} = \frac{1}{p_1} - \frac{\alpha}{n}$, and $\frac{1}{q_2} = \frac{1}{q_1} - \frac{\alpha}{n}$. Furthermore, Adams [1] and Chiarenza and Frasca [2] reproved it and obtained a stronger result which is presented below.

Theorem 1.2. *[Adams, Chiarenza-Frasca] If $0 < \alpha < n$, then we have*

$$\|I_\alpha f\|_{L^{p_2,q_2}} \leq C\, \|f\|_{L^{p_1,q_1}},$$

for every $f \in L^{p_1,q_1}(\mathbb{R}^n)$ where $1 < p_1 < q_1 < \frac{n}{\alpha}$, $\frac{1}{p_2} = \frac{1}{p_1}\left(1 - \frac{\alpha q_1}{n}\right)$, and $\frac{1}{q_2} = \frac{1}{q_1} - \frac{\alpha}{n}$.

For $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ and $1 \leq p < \infty$, we define the *generalized Morrey space*

$$L^{p,\phi}(\mathbb{R}^n) := \left\{ f \in L^p_{loc}(\mathbb{R}^n) : \|f\|_{L^{p,\phi}} < \infty \right\},$$

where $\|f\|_{L^{p,\phi}} := \sup\limits_{r>0, a\in\mathbb{R}^n} \frac{1}{\phi(r)} \left(\frac{1}{r^n} \int_{|x-a|<r} |f(x)|^p\, dx \right)^{1/p}$. Here we assume that ϕ is almost decreasing and $t^{n/p}\phi(t)$ is almost decreasing, so that ϕ satisfies the *doubling condition*, that is, there exists a constant C such that $\frac{1}{C} \leq \frac{\phi(r)}{\phi(v)} \leq C$ whenever $\frac{1}{2} \leq \frac{r}{v} \leq 2$.

In 1994, Nakai [9] obtained the boundedness of I_α from $L^{p_1,\phi}(\mathbb{R}^n)$ to $L^{p_2,\psi}(\mathbb{R}^n)$ where $1 < p_1 < \frac{n}{\alpha}$, $\frac{1}{p_2} = \frac{1}{p_1} - \frac{\alpha}{n}$ and $\int_r^\infty v^{\alpha-1}\phi(v)dv \leq Cr^\alpha\phi(r) \leq C\psi(r)$ for every $r > 0$. Nakai's result may be viewed as an extension of Spanne's. Later on, in 2009, Gunawan and Eridani [3] extended Adams-Chiarenza-Frasca's result.

Theorem 1.3. *[Gunawan-Eridani] If $\int_r^\infty \frac{\phi(v)}{v} dv \leq C\phi(r)$, and $\phi(r) \leq Cr^\beta$ for every $r > 0$, $-\frac{n}{p_1} \leq \beta < -\alpha$, $1 < p_1 < \frac{n}{\alpha}$, $0 < \alpha < n$, then we have*

$$\|I_\alpha f\|_{L^{p_2,\psi}} \leq C\, \|f\|_{L^{p_1,\phi}}$$

for every $f \in L^{p_1,\phi}(\mathbb{R}^n)$ where $p_2 = \frac{\beta p_1}{\alpha+\beta}$ and $\psi(r) = \phi(r)^{p_1/p_2}$, $r > 0$.

The proof of the boundedness of I_α on Lebesgue spaces, Morrey spaces, or generalized Morrey spaces, usually involves *Hardy-Littlewood maximal operator*, which is defined by

$$Mf(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|\, dy,$$

for every $f \in L^p_{loc}(\mathbb{R}^n)$ where $|B|$ denotes the Lebesgue measure of the ball $B = B(a,r)$ (centered at $a \in \mathbb{R}^n$ with radius $r > 0$). It is well known that the operator M is bounded on $L^p(\mathbb{R}^n)$ for $1 < p \le \infty$ [12, 13] and also on Morrey spaces $L^{p,q}$ for $1 < p \le q \le \infty$ [2].

Next, we know that $I_{\alpha,\gamma}$ is guaranteed to be bounded on generalized Morrey spaces because $K_{\alpha,\gamma}(x) \le K_\alpha(x)$ for every $x \in \mathbb{R}^n$. The boundedness of $I_{\alpha,\gamma}$ on Lebesgue spaces can also be proved by using Young's inequality, as shown in [7].

Theorem 1.4. [7] *For $\gamma > 0$ and $0 < \alpha < n$, we have $K_{\alpha,\gamma} \in L^t(\mathbb{R}^n)$ whenever $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$. Accordingly, we have*

$$\|I_{\alpha,\gamma} f\|_{L^q} \le \|K_{\alpha,\gamma}\|_{L^t} \|f\|_{L^p}$$

for every $f \in L^p(\mathbb{R}^n)$ where $1 \le p < t'$, $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$, $\frac{1}{q} + 1 = \frac{1}{p} + \frac{1}{t}$.

Using the boundedness of Hardy-Littlewood maximal operator, we also know that $I_{\alpha,\gamma}$ is bounded on Morrey spaces.

Theorem 1.5. [7] *For $\gamma > 0$ and $0 < \alpha < n$, we have*

$$\|I_{\alpha,\gamma} f\|_{L^{p_2,q_2}} \le C \, \|K_{\alpha,\gamma}\|_{L^t} \|f\|_{L^{p_1,q_1}}$$

for every $f \in L^{p_1,q_1}(\mathbb{R}^n)$ where $1 < p_1 < q_1 < t'$, $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$, $\frac{1}{p_2} + 1 = \frac{1}{p_1} + \frac{1}{t}$, and $\frac{1}{q_2} + 1 = \frac{1}{q_1} + \frac{1}{t}$.

In 1999, Kurata *et al.* [8] proved the boundedness of $W \cdot I_{\alpha,\gamma}$ on generalized Morrey spaces where W is a multiplication operator. A similar result to Kurata's can be found in [3]. In the next section, we shall reprove the boundedness of $I_{\alpha,\gamma}$ on generalized Morrey spaces using a Hedberg-type inequality and the boundedness of Hardy-Littlewood maximal operator on these spaces.

Theorem 1.6. *(Nakai) For $1 < p \le \infty$, we have*

$$\|Mf\|_{L^{p,\phi}} \le C \, \|f\|_{L^{p,\phi}},$$

for every $f \in L^{p,\phi}(\mathbb{R}^n)$.

Our results show that the norm of Bessel-Riesz operators is dominated by the norm of their kernels on (generalized) Morrey spaces.

2. MAIN RESULTS

For $\gamma > 0$ and $0 < \alpha < n$, one may observe that the kernel $K_{\alpha,\gamma}$ belongs to Lebesgue spaces $L^t(\mathbb{R}^n)$ whenever $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$, where

$$\sum_{k \in \mathbb{Z}} \frac{(2^k R)^{(\alpha-n)t+n}}{(1+2^k R)^{\gamma t}} \sim \|K_{\alpha,\gamma}\|_{L^t}^t$$

(see [7]). With this in mind, we obtain the boundedness of $I_{\alpha,\gamma}$ on generalized Morrey spaces as in the following theorem.

Theorem 2.1. *Let* $\gamma > 0$ *and* $0 < \alpha < n$*. If* $\phi(r) \le Cr^{\beta}$ *for every* $r > 0$, $-\frac{\alpha t'}{p_1} \le \beta < -\alpha$, $1 < p_1 < t'$, *and* $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$*, then we have*

$$\|I_{\alpha,\gamma} f\|_{L^{p_2,\psi}} \le C\,\|K_{\alpha,\gamma}\|_{L^t}\|f\|_{L^{p_1,\phi}},$$

for every $f \in L^{p_1,\phi}(\mathbb{R}^n)$ *where* $p_2 = \frac{\beta p_1}{\alpha+\beta}$*, and* $\psi(r) = \phi(r)^{p_1/p_2}$.

Proof. Let $\gamma > 0$ and $0 < \alpha < n$. Suppose that $\phi(r) \le Cr^{\beta}$ for every $r > 0$, $-\frac{\alpha t'}{p_1} \le \beta < -\alpha$, $1 < p_1 < t'$, $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$. Take $f \in L^{p_1,\phi}(\mathbb{R}^n)$ and write

$$I_{\alpha,\gamma} f\,(x) := I_1\,(x) + I_2\,(x)\,,$$

for every $x \in \mathbb{R}^n$ where $I_1\,(x) := \int_{|x-y|<R} \frac{|x-y|^{\alpha-n} f(y)}{(1+|x-y|)^{\gamma}} dy$ and $I_2\,(x) := \int_{|x-y|\ge R} \frac{|x-y|^{\alpha-n} f(y)}{(1+|x-y|)^{\gamma}} dy$, $R > 0$.

Using dyadic decomposition, we have the following estimate for I_1:

$$\begin{aligned}
|I_1(x)| &\le \sum_{k=-\infty}^{-1} \int_{2^kR\le|x-y|<2^{k+1}R} \frac{|x-y|^{\alpha-n}\,|f\,(y)|}{(1+|x-y|)^{\gamma}} dy \\
&\le C_1 \sum_{k=-\infty}^{-1} \frac{\left(2^kR\right)^{\alpha-n}}{(1+2^kR)^{\gamma}} \int_{2^kR\le|x-y|<2^{k+1}R} |f\,(y)|\,dy \\
&\le C_2\, Mf\,(x) \sum_{k=-\infty}^{-1} \frac{\left(2^kR\right)^{\alpha-n+n/t}\left(2^kR\right)^{n/t'}}{(1+2^kR)^{\gamma}}.
\end{aligned}$$

We then use Hölder's inequality to get

$$|I_1\,(x)| \le C_2 Mf\,(x) \left(\sum_{k=-\infty}^{-1} \frac{\left(2^kR\right)^{(\alpha-n)t+n}}{(1+2^kR)^{\gamma t}}\right)^{1/t} \left(\sum_{k=-\infty}^{-1} \left(2^kR\right)^{n}\right)^{1/t'}.$$

Because we have

$$\left(\sum_{k=-\infty}^{-1} \frac{\left(2^kR\right)^{(\alpha-n)t+n}}{(1+2^kR)^{\gamma t}}\right)^{1/t} \le \left(\sum_{k\in\mathbb{Z}} \frac{\left(2^kR\right)^{(\alpha-n)t+n}}{(1+2^kR)^{\gamma t}}\right)^{1/t} \sim \|K_{\alpha,\gamma}\|_{L^t}\,,$$

we obtain $|I_1(x)| \le C_3\,\|K_{\alpha,\gamma}\|_{L^t}\,Mf\,(x)\,R^{n/t'}$.

To estimate I_2, we use Hölder's inequality again:

$$\begin{aligned}
|I_2\,(x)| &\le C_4 \sum_{k=0}^{\infty} \frac{\left(2^kR\right)^{\alpha-n}}{(1+2^kR)^{\gamma}} \int_{2^kR\le|x-y|<2^{k+1}R} |f\,(y)|\,dy \\
&\le C_4 \sum_{k=0}^{\infty} \frac{\left(2^kR\right)^{\alpha-n}}{(1+2^kR)^{\gamma}} \left(2^kR\right)^{n/p_1'} \left(\int_{2^kR\le|x-y|<2^{k+1}R} |f\,(y)|^{p_1}\,dy\right)^{1/p_1}.
\end{aligned}$$

It follows that

$$\begin{aligned}
|I_2\,(x)| &\le C_5\,\|f\|_{L^{p_1,\phi}} \sum_{k=0}^{\infty} \frac{\left(2^kR\right)^{\alpha-n+n/t}}{(1+2^kR)^{\gamma}} \phi\left(2^kR\right)\left(2^kR\right)^{n/t'} \\
&\le C_6\,\|f\|_{L^{p_1,\phi}} \sum_{k=0}^{\infty} \frac{\left(2^kR\right)^{\alpha-n+n/t}}{(1+2^kR)^{\gamma}} \left(2^kR\right)^{\beta+n/t'}.
\end{aligned}$$

Another use of Hölder's inequality gives

$$|I_2(x)| \le C_6 \|f\|_{L^{p_1,\phi}} \left(\sum_{k=0}^{\infty} \frac{\left(2^k R\right)^{(\alpha-n)t+n}}{\left(1+2^k R\right)^{\gamma t}} \right)^{1/t} \left(\sum_{k=0}^{\infty} \left(2^k R\right)^{\beta t'+n} \right)^{1/t'}.$$

Because $\beta t' + n < 0$ and $\sum_{k=0}^{\infty} \frac{\left(2^k R\right)^{(\alpha-n)t+n}}{\left(1+2^k R\right)^{\gamma t}} \lesssim \|K_{\alpha,\gamma}\|_{L^t}^t$, we obtain

$$|I_2(x)| \le C_7 \|K_{\alpha,\gamma}\|_{L^t} \|f\|_{L^{p_1,\phi}} R^\beta R^{n/t'}.$$

Summing the two estimates, we obtain

$$|I_{\alpha,\gamma} f(x)| \le C_8 \|K_{\alpha,\gamma}\|_{L^t} \left(Mf(x) R^{n/t'} + \|f\|_{L^{p_1,\phi}} R^{n/t'+\beta} \right),$$

for every $x \in \mathbb{R}^n$. Now, for each $x \in \mathbb{R}^n$, choose $R > 0$ such that $R^\beta = \frac{Mf(x)}{\|f\|_{L^{p_1,\phi}}}$. Hence we get a Hedberg-type inequality for $I_{\alpha,\gamma} f$, namely

$$|I_{\alpha,\gamma} f(x)| \le C_9 \|K_{\alpha,\gamma}\|_{L^t} \|f\|_{L^{p_1,\phi}}^{-\alpha/\beta} Mf(x)^{1+\alpha/\beta}.$$

Now put $p_2 := \frac{\beta p_1}{\alpha+\beta}$. For arbitrary $a \in \mathbb{R}^n$ and $r > 0$, we have

$$\left(\int_{|x-a|<r} |I_{\alpha,\gamma} f(x)|^{p_2} dx \right)^{1/p_2} \le C_9 \|K_{\alpha,\gamma}\|_{L^t} \|f\|_{L^{p_1,\phi}}^{1-p_1/p_2} \left(\int_{|x-a|<r} |Mf(x)|^{p_1} dx \right)^{1/p_2}.$$

We divide both sides by $\phi(r)^{p_1/p_2} r^{n/p_2}$ to get

$$\frac{\left(\int_{|x-a|<r} |I_{\alpha,\gamma} f(x)|^{p_2} \, dx \right)^{1/p_2}}{\psi(r) r^{n/p_2}} \le C_9 \|K_{\alpha,\gamma}\|_{L^t} \|f\|_{L^{p_1,\phi}}^{1-p_1/p_2} \frac{\left(\int_{|x-a|<r} |Mf(x)|^{p_1} dx \right)^{(1/p_2)}}{\phi(r)^{p_1/p_2} r^{n/p_2}},$$

where $\psi(r) := \phi(r)^{p_1/p_2}$. Taking the supremum over $a \in \mathbb{R}^n$ and $r > 0$, we obtain

$$\|I_{\alpha,\gamma} f\|_{L^{p_2,\psi}} \le C_{10} \|K_{\alpha,\gamma}\|_{L^t} \|f\|_{L^{p_1,\phi}}^{1-p_1/p_2} \|Mf\|_{L^{p_1,\phi}}^{p_1/p_2}.$$

By the boundedness of the maximal operator on generalized Morrey spaces (Nakai's Theorem), the desired result follows: $\|I_{\alpha,\gamma} f\|_{L^{p_2,\psi}} \le C \|K_{\alpha,\gamma}\|_{L^t} \|f\|_{L^{p_1,\phi}}$. ∎

We note that from the inclusion property of Morrey spaces, we have

$$\|K_{\alpha,\gamma}\|_{L^{s,t}} \le \|K_{\alpha,\gamma}\|_{L^{t,t}} = \|K_{\alpha,\gamma}\|_{L^t}$$

whenever $1 \le s \le t$, $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$. Now we wish to obtain a more general result for the boundedness of $I_{\alpha,\gamma}$ by using the fact that the kernel $K_{\alpha,\gamma}$ belongs to Morrey spaces.

Theorem 2.2. *Let $\gamma > 0$ and $0 < \alpha < n$. If $\phi(r) \le C r^\beta$ for every $r > 0$, $-\frac{\alpha t'}{p_1} \le \beta < -\alpha$, $1 < p_1 < t'$, $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$, then we have*

$$\|I_{\alpha,\gamma} f\|_{L^{p_2,\psi}} \le C \|K_{\alpha,\gamma}\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}},$$

for every $f \in L^{p_1,\phi}(\mathbb{R}^n)$ where $1 \le s \le t$, $p_2 = \frac{\beta p_1}{\alpha+\beta}$, and $\psi(v) = \phi(v)^{p_1/p_2}$.

Proof. Let $\gamma > 0$ and $0 < \alpha < n$. Suppose that $\phi(r) \le Cr^{\beta}$ for every $r > 0$, $-\frac{\alpha t'}{p_1} \le \beta < -\alpha$, $1 < p_1 < t'$, $\frac{n}{n+\gamma-\alpha} < t < \frac{n}{n-\alpha}$. As in the proof of Theorem 2.1, we have $I_{\alpha,\gamma}f(x) := I_1(x) + I_2(x)$ for every $x \in \mathbb{R}^n$. Now, we estimate I_1 using dyadic decomposition as follow:

$$\begin{aligned} |I_1(x)| &\le \sum_{k=-\infty}^{-1} \int_{2^k R \le |x-y| < 2^{k+1}R} \frac{|x-y|^{\alpha-n} |f(y)|}{(1+|x-y|)^{\gamma}} dy \\ &\le C_1 \sum_{k=-\infty}^{-1} \frac{\left(2^k R\right)^{\alpha-n}}{(1+2^k R)^{\gamma}} \int_{2^k R \le |x-y| < 2^{k+1}R} |f(y)|\, dy \\ &= C_2 Mf(x) \sum_{k=-\infty}^{-1} \frac{\left(2^k R\right)^{\alpha-n+n/s} \left(2^k R\right)^{n/s'}}{(1+2^k R)^{\gamma}}, \end{aligned}$$

where $1 \le s \le t$. By Hölder's inequality,

$$|I_1(x)| \le C_2 Mf(x) \left(\sum_{k=-\infty}^{-1} \frac{\left(2^k R\right)^{(\alpha-n)s+n}}{(1+2^k R)^{\gamma s}} \right)^{1/s} \left(\sum_{k=-\infty}^{-1} \left(2^k R\right)^{n} \right)^{1/s'}.$$

We also have $\sum_{k=-\infty}^{-1} \frac{\left(2^k R\right)^{(\alpha-n)s+n}}{\left(1+2^k R\right)^{\gamma s}} \lesssim \int_{0<|x|<R} K_{\alpha,\gamma}^s(x)\, dx$, so that

$$|I_1(x)| \le C_3 Mf(x) \left(\int_{0<|x|<R} K_{\alpha,\gamma}^s(x)\, dx \right)^{\frac{1}{s}} R^{n/s'} \le C_3 \left\| K_{\alpha,\gamma} \right\|_{L^{s,t}} Mf(x) R^{n/t'}.$$

Next, we estimate I_2 by using Hölder's inequality. As in the proof of Theorem 2.1, we obtain

$$|I_2(x)| \le C_4 \sum_{k=0}^{\infty} \frac{\left(2^k R\right)^{\alpha-n}}{(1+2^k R)^{\gamma}} \left(2^k R\right)^{n/p_1'} \left(\int_{2^k R \le |x-y| < 2^{k+1}R} |f(y)|^{p_1} dy \right)^{1/p_1}.$$

It thus follows that

$$\begin{aligned} |I_2(x)| &\le C_5 \|f\|_{L^{p_1,\phi}} \sum_{k=0}^{\infty} \frac{\left(2^k R\right)^{\alpha} \phi(2^k R)}{(1+2^k R)^{\gamma}} \frac{\left(\int_{2^k R \le |x-y| < 2^{k+1}R} dy \right)^{1/s}}{(2^k R)^{n/s}} \\ &\le C_6 \|f\|_{L^{p_1,\phi}} \sum_{k=0}^{\infty} \phi(2^k R) \left(2^k R\right)^{n/t'} \frac{\left(\int_{2^k R \le |x-y| < 2^{k+1}R} K_{\alpha,\gamma}^s(x-y) dy \right)^{1/s}}{(2^k R)^{n/s-n/t}} \end{aligned}$$

Because $\phi(r) \le Cr^{\beta}$ and $\frac{\left(\int_{2^k R \le |x-y| < 2^{k+1}R} K_{\alpha,\gamma}^s(x-y) dy\right)^{1/s}}{\left(2^k R\right)^{n/s-n/t}} \lesssim \left\| K_{\alpha,\gamma} \right\|_{L^{s,t}}$ for every $k = 0, 1, 2, \ldots$, we get

$$\begin{aligned} |I_2(x)| &\le C_7 \left\| K_{\alpha,\gamma} \right\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}} \sum_{k=0}^{\infty} \left(2^k R\right)^{\beta+n/t'} \\ &\le C_8 \left\| K_{\alpha,\gamma} \right\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}} R^{\beta} R^{n/t'}. \end{aligned}$$

From the two estimates, we obtain

$$|I_{\alpha,\gamma}f(x)| \le C_9 \left\| K_{\alpha,\gamma} \right\|_{L^{s,t}} \left(Mf(x) R^{n/t'} + \|f\|_{L^{p_1,\phi}} R^{n/t'+\beta} \right),$$

for every $x \in \mathbb{R}^n$. Now, for each $x \in \mathbb{R}^n$, choose $R > 0$ such that $R^{\beta} = \frac{Mf(x)}{\|f\|_{L^{p_1,\phi}}}$. Hence we get

$$|I_{\alpha,\gamma}f(x)| \le C_9 \left\| K_{\alpha,\gamma} \right\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}}^{-\alpha/\beta} Mf(x)^{1+\alpha/\beta}.$$

Put $p_2 := \frac{\beta p_1}{\alpha+\beta}$. For arbitrary $a \in \mathbb{R}^n$ and $r > 0$, we have

$$\left(\int_{|x-a|<r} |I_{\alpha,\gamma} f(x)|^{p_2} dx\right)^{1/p_2} \leq C_9 \|K_{\alpha,\gamma}\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}}^{1-p_1/p_2} \left(\int_{|x-a|<r} |Mf(x)|^{p_1} dx\right)^{1/p_2}.$$

Divide both sides by $\phi(r)^{p_1/p_2} r^{n/p_2}$ and take the supremum over $a \in \mathbb{R}^n$ and $r > 0$ to get

$$\|I_{\alpha,\gamma} f\|_{L^{p_2,\psi}} \leq C_{10} \|K_{\alpha,\gamma}\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}}^{1-p_1/p_2} \|Mf\|_{L^{p_1,\phi}}^{p_1/p_2},$$

where $\psi(r) := \phi(r)^{p_1/p_2}$. With the boundedness of the maximal operator on generalized Morrey spaces (Nakai's Theorem), we obtain

$$\|I_{\alpha,\gamma} f\|_{L^{p_2,\psi}} \leq C_{p_1,\phi} \|K_{\alpha,\gamma}\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}},$$

as desired. ∎

Note that by Theorem 2.2 and the inclusion of Morrey spaces, we recover Theorem 2.1:

$$\|I_{\alpha,\gamma} f\|_{L^{p_2,\psi}} \leq C\ \|K_{\alpha,\gamma}\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}} \leq C\ \|K_{\alpha,\gamma}\|_{L^{t}} \|f\|_{L^{p_1,\phi}}.$$

We still wish to obtain a better estimate. The following lemma presents that the Bessel-Riesz kernels belong to generalized Morrey space $L^{s,\sigma}(\mathbb{R}^n)$ for some $s \geq 1$ and a suitable function σ.

Lemma 2.3. *Suppose that $\gamma > 0$ and $0 < \alpha < n$. If $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies*

$$\int_{0<r\leq R} r^{(\alpha-n)s+n-1} dr \leq C\sigma^s(R) R^n$$

for every $R > 0$, then $K_{\alpha,\gamma} \in L^{s,\sigma}(\mathbb{R}^n)$.

Proof. Suppose that the hypothesis holds. It is sufficient to evaluate the integral around 0. We observe that

$$\int_{|x|\leq R} K^s_{\alpha,\gamma}(x)dx \;=\; \int_{|x|\leq R} \frac{|x|^{(\alpha-n)s}}{(1+|x|)^{\gamma s}} dx \leq C \int_{0<r\leq R} r^{(\alpha-n)s+n-1} dr \leq C\sigma^s(R) R^n.$$

We divide both sides of the inequality by $\sigma^s(R) R^n$ and take s^{th}-root to obtain

$$\frac{\left(\int_{|x|\leq R} K^s_{\alpha,\gamma}(x)dx\right)^{1/s}}{\sigma(R) R^{n/s}} \leq C^{1/s}.$$

Now, taking the supremum over $R > 0$, we have

$$\sup_{R>0} \frac{\left(\int_{|x|\leq R} K^s_{\alpha,\gamma}(x)dx\right)^{1/s}}{\sigma(R) R^{n/s}} < \infty.$$

Hence $K_{\alpha,\gamma} \in L^{s,\sigma}(\mathbb{R}^n)$. ∎

By the hypothesis of Lemma 2.3 we also obtain $\frac{\left(\int_{2^k R<|x|\leq 2^{k+1}R} K^s_{\rho,\gamma}(x)dx\right)^{1/s}}{\sigma(2^k R)\left(2^k R\right)^{n/s}} \lesssim \|K_{\alpha,\gamma}\|_{L^{s,\sigma}}$ for every integer k and $R > 0$. Moreover, $\frac{\left(\sum_{k=-1}^{-\infty} K^s_{\alpha,\gamma}\left(2^k R\right)\left(2^k R\right)^n\right)^{1/s}}{\sigma(R)(R)^{n/s}} \lesssim \|K_{\alpha,\gamma}\|_{L^{s,\sigma}}$ holds for every $R > 0$. One may observe that $1 \leq s \leq \frac{n \ln R_1}{-\ln \sigma(R_1)}$ for every $R_1 > 1$. For $\sigma(R) = R^{-n/t}$, this inequality reduces to $1 \leq s \leq t$.

We shall now use the lemma to prove the following theorem.

Theorem 2.4. *Suppose that* $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ *satisfies the doubling condition and* $\sigma(r) \le Cr^{-\alpha}$ *for every* $r > 0$*, so that* $K_{\alpha,\gamma} \in L^{s,\sigma}(\mathbb{R}^n)$ *for* $1 \le s < \frac{n}{n-\alpha}$*, where* $0 < \alpha < n$ *and* $\gamma > 0$*. If* $\phi(r) \le Cr^\beta$ *for every* $r > 0$*, where* $-\frac{n}{p_1} < \beta < -\alpha$*, then we have*

$$\|I_{\alpha,\gamma} f\|_{L^{p_2,\psi}} \le C_{p_1,\phi} \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} \|f\|_{L^{p_1,\phi}}$$

for every $f \in L^{p_1,\phi}(\mathbb{R}^n)$*, where* $1 < p_1 < \frac{n}{\alpha}$*,* $p_2 = \frac{\beta p_1}{\beta+n-\alpha}$ *and* $\psi(r) = \phi(r)^{p_1/p_2}$*.*

Proof. Let $\gamma > 0$ and $0 < \alpha < n$. Suppose that $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the doubling condition and $\sigma(r) \le Cr^{-\alpha}$ for every $r > 0$, such that $K_{\alpha,\gamma} \in L^{s,\sigma}(\mathbb{R}^n)$ for $1 \le s < \frac{n}{n-\alpha}$. Suppose also that $\phi(r) \le Cr^\beta$ for every $r > 0$, where $-\frac{n}{p_1} < \beta < -\alpha$, $1 < p_1 < \frac{n}{\alpha}$. As in the proof of Theorem 2.1, we write $I_{\alpha,\gamma} f(x) := I_1(x) + I_2(x)$ for every $x \in \mathbb{R}^n$. As usual, we estimate I_1 by using dyadic decomposition:

$$\begin{aligned} |I_1(x)| &\le \sum_{k=-\infty}^{-1} \int_{2^kR \le |x-y| < 2^{k+1}R} \frac{|x-y|^{\alpha-n} |f(y)|}{(1+|x-y|)^\gamma} dy \\ &\le C_1 \sum_{k=-\infty}^{-1} \frac{(2^kR)^{\alpha-n}}{(1+2^kR)^\gamma} \int_{2^kR \le |x-y| < 2^{k+1}R} |f(y)|\, dy \\ &= C_2 Mf(x) \sum_{k=-\infty}^{-1} \frac{(2^kR)^{\alpha-n+n/s} (2^kR)^{n/s'}}{(1+2^kR)^\gamma} \end{aligned}$$

By using Hölder inequality, we obtain

$$|I_1(x)| \le C_2 Mf(x) \left(\sum_{k=-\infty}^{-1} \frac{(2^kR)^{(\alpha-n)s+n}}{(1+2^kR)^{\gamma s}} \right)^{1/s} \left(\sum_{k=-\infty}^{-1} (2^kR)^n \right)^{1/s'}.$$

But $\sum_{k=-\infty}^{-1} \frac{(2^kR)^{(\alpha-n)s+n}}{(1+2^kR)^{\gamma s}} \lesssim \int_{0<|x|<R} K_{\alpha,\gamma}^s(x)\, dx$, and so we get

$$\begin{aligned} |I_1(x)| &\le C_2 Mf(x) \left(\int_{0<|x|<R} K_{\alpha,\gamma}^s(x)\, dx \right)^{\frac{1}{s}} R^{n/s'} \\ &\le C_2 \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} Mf(x)\, \sigma(R)\, R^n \\ &\le C_3 \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} Mf(x)\, R^{n-\alpha}. \end{aligned}$$

Next, we estimate I_2 as follows:

$$\begin{aligned} |I_2(x)| &\le C_4 \sum_{k=0}^{\infty} \frac{(2^kR)^{\alpha-n}}{(1+2^kR)^\gamma} \int_{2^kR \le |x-y| < 2^{k+1}R} |f(y)|\, dy \\ &\le C_4 \sum_{k=0}^{\infty} \frac{(2^kR)^{\alpha-n}}{(1+2^kR)^\gamma} (2^kR)^{n/p_1'} \left(\int_{2^kR \le |x-y| < 2^{k+1}R} |f(y)|^{p_1} dy \right)^{1/p_1} \\ &\le C_5 \|f\|_{L^{p_1,\phi}} \sum_{k=0}^{\infty} \frac{(2^kR)^\alpha \phi(2^kR) (2^kR)^n}{(1+2^kR)^\gamma} \frac{\left(\int_{2^kR \le |x-y| < 2^{k+1}R} dy \right)^{1/s}}{(2^kR)^{n/s}} \\ &\le C_6 \|f\|_{L^{p_1,\phi}} \sum_{k=0}^{\infty} (2^kR)^{n-\alpha+\beta} \frac{\left(\int_{2^kR \le |x-y| < 2^{k+1}R} K_{\alpha,\gamma}^s(x-y)\, dy \right)^{1/s}}{\sigma(2^kR)\, (2^kR)^{n/s}} \end{aligned}$$

Because $\frac{\int_{2^k R\le|x-y|<2^{k+1}R} K^s_{\alpha,\gamma}(x-y)dy}{\sigma\left(2^kR\right)\left(2^kR\right)^n} \lesssim \|K_{\alpha,\gamma}\|^s_{L^{s,\sigma}}$ for every $k = 0, 1, 2, \ldots$, we obtain

$$\begin{aligned} |I_2(x)| &\le C_6 \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} \|f\|_{L^{p_1,\phi}} \sum_{k=0}^{\infty} \left(2^kR\right)^{n-\alpha+\beta} \\ &\le C_7 \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} \|f\|_{L^{p_1,\phi}} R^{n-\alpha+\beta}. \end{aligned}$$

It follows from the above estimates for I_1 and I_2 that

$$|I_{\alpha,\gamma}f(x)| \le C_8 \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} \left(Mf(x)R^{n-\alpha} + \|f\|_{L^{p_1,\phi}} R^{n-\alpha+\beta}\right)$$

for every $x \in \mathbb{R}^n$. Now, for each $x \in \mathbb{R}^n$, choose $R > 0$ such that $R^\beta = \frac{Mf(x)}{\|f\|_{L^{p_1,\phi}}}$, whence

$$|I_{\alpha,\gamma}f(x)| \le C_9 \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} \|f\|^{(\alpha-n)/\beta}_{L^{p_1,\phi}} Mf(x)^{1+(n-\alpha)/\beta}.$$

Put $p_2 := \frac{\beta p_1}{\beta+n-\alpha}$. For arbitrary $a \in \mathbb{R}^n$ and $r > 0$, we have

$$\left(\int_{|x-a|<r} |I_{\alpha,\gamma}f(x)|^{p_2} dx\right)^{1/p_2} \le C_9 \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} \|f\|^{1-p_1/p_2}_{L^{p_1,\phi}} \left(\int_{|x-a|<r} |Mf(x)|^{p_1} dx\right)^{(1/p_2)}.$$

Divide the both sides by $\phi(r)^{p_1/p_2} r^{n/p_2}$ to get

$$\begin{aligned} \frac{\left(\int_{|x-a|<r} |I_{\alpha,\gamma}f(x)|^{p_2}dx\right)^{1/p_2}}{\psi(r)\, r^{n/p_2}} &= \frac{\left(\int_{|x-a|<r} |I_{\alpha,\gamma}f(x)|^{p_2}dx\right)^{1/p_2}}{\phi(r)^{p_1/p_2} r^{n/p_2}} \\ &\le C_9 \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} \|f\|^{1-p_1/p_2}_{L^{p_1,\phi}} \frac{\left(\int_{|x-a|<r} |Mf(x)|^{p_1}dx\right)^{(1/p_2)}}{\phi(r)^{p_1/p_2} r^{n/p_2}}, \end{aligned}$$

where $\psi(r) := \phi(r)^{p_1/p_2}$. Finally, take the supremum over $a \in \mathbb{R}^n$ and $r > 0$ to obtain

$$\|I_{\alpha,\gamma}f\|_{L^{p_2,\psi}} \le C_{10} \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} \|f\|^{1-p_1/p_2}_{L^{p_1,\phi}} \|Mf\|^{p_1/p_2}_{L^{p_1,\phi}}.$$

Because the maximal operator is bounded on generalized Morrey spaces (Nakai's Theorem), we conclude that $\|I_{\alpha,\gamma}f\|_{L^{p_2,\psi}} \le C_{p_1,\phi} \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} \|f\|_{L^{p_1,\phi}}$. ∎

3. CONCLUDING REMARKS

The results presented in this paper, namely Theorems 2.1, 2.2, and 2.4, extend the results on the boundedness of Bessel-Riesz operators on Morrey spaces [7]. Similar to Gunawan-Eridani's result for I_α, Theorems 2.1, 2.2, and 2.4 ensures that $I_{\alpha,\gamma} : L^{p_1,\phi}(\mathbb{R}^n) \to L^{p_2,\phi^{p_1/p_2}}(\mathbb{R}^n)$. Notice that if we have $\sigma : \mathbb{R}^+ \to \mathbb{R}^+$ such that for $t \in \left(\frac{n}{n+\gamma-\alpha}, \frac{n}{n-\alpha}\right)$, then $R^{-n/t} < \sigma(R)$ holds for every $R > 0$, then Theorem 2.4 gives a better estimate than Theorem 2.2. Now, if we define $\sigma(R) := \left(1 + R^{n/t_1}\right) R^{-n/t}$ for some $t_1 > t$, then $\|K_{\alpha,\gamma}\|_{L^{s,\sigma}} < \|K_{\alpha,\gamma}\|_{L^{s,t}}$. By Theorem 2.2 and the inclusion property of Morrey spaces, we obtain

$$\begin{aligned} \|I_{\alpha,\gamma}f\|_{L^{p_2,\psi}} &\le C \|K_{\alpha,\gamma}\|_{L^{s,\sigma}} \|f\|_{L^{p_1,\phi}} \\ &< C \|K_{\alpha,\gamma}\|_{L^{s,t}} \|f\|_{L^{p_1,\phi}} \\ &\le C \|K_{\alpha,\gamma}\|_{L^{t}} \|f\|_{L^{p_1,\phi}}. \end{aligned}$$

We can therefore say that Theorem 2.4 gives the best estimate among the three. Furthermore, we have also shown that, in each theorem, the norm of Bessel-Riesz operators on generalized Morrey spaces is dominated by that of Bessel-Riesz kernels.

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