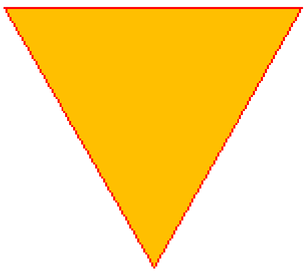
Applications of Series

We have seen several examples of infinite series by now, and you may be wondering why we should be concerned with these mathematical excursions into "the infinite". On this page we study an example that illustrates how infinite series arise in the investigation of recursions and fractals, and why it is useful to know how to treat them.

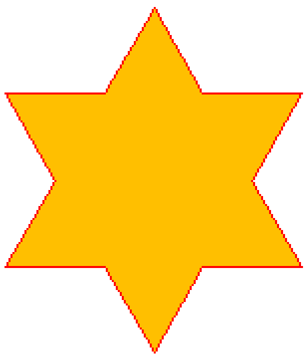
The Koch Snowflake: finite area but infinite perimeter

The Koch snowflake is a geometric shape created by a repeated set of steps. The shape itself is called a fractal, and has some remarkable properties. One of these properties is "self -similarity". This refers to the fact that small parts of the shape are very similar to the whole shape itself. The process of creating this snowflake is, in principle, infinite, resulting in some peculiarities that we will explore below.

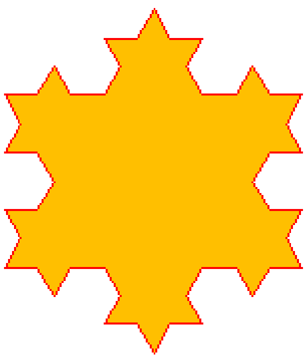
Steps in creating the Koch Snowflake:



Start with an equilateral triangle



Add smaller copies of the triangle to each side



Add yet smaller copies of the triangle to each side again

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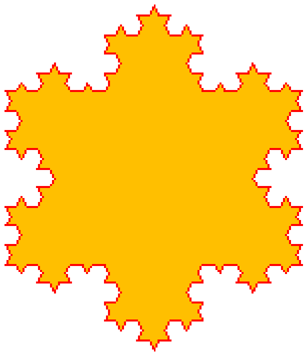
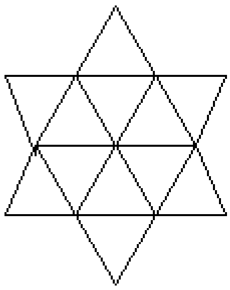


Repeat the process...

Area contained inside the Koch Snowflake:

To compute the area of the snowflake we will sum up the areas contained in each of the little triangles that we have added. Suppose that the initial area of the triangle is A . We first observe that the area of each of the pieces added at the next stage is 1/9 A. This can be seen from the diagram below.



Since three pieces are added, the area in increased by 3(1/9) = 1/3.
The triangles at the next stage each contain 1/9 of the area of the triangle used in the previous step, but now we add 4 of these per side, for a total of 3(4)=12 around the snowflake. At the next stage still, we will be adding 3(16) triangles with area again scaled by a factor 1/9. Summarizing these stages, we have:

Step 1:

$$\text{Area } = A$$

Step 2:

$$\text{Area } = A + 3(\frac{A}{9}) = A(1+\frac{3}{9})$$

Step 3:

$$\text{Area } = A(1+\frac{3}{9}+3(4)(\frac{1}{9})(\frac{1}{9}))$$

Step 4:

$$\text{Area } = A(1+3\frac{1}{9}+3(4)(\frac{1}{9})(\frac{1}{9})+3(16)(\frac{1}{9})(\frac{1}{9})(\frac{1}{9}))$$

.
Step n:

$$\text{Area } = A(1+3\frac{1}{9}+3(4)(\frac{1}{9})^2+3(4)^2(\frac{1}{9})^3+\ldots+3(4)^n(\frac{1}{9})^n)$$

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Continuing this process for increasing n leads to a series whose terms are of the form

$$\text{Area } = A(1 + \frac{3}{9}[1 + \frac{4}{9} + (\frac{4}{9})^2 + ...])$$

$$\text{Area } = A(1 + (\frac{3}{9}) \sum_{n=0}^{\infty} (\frac{4}{9})^n)$$

The series above is just a simple geometric series with the base 4/9, and we know that since this number is less than 1, the series converges. Indeed,

$$\text{Area } = A(1 + (\frac{3}{9})[\frac{1}{1 - 4/9}])$$

$$= A(1 + (\frac{1}{3})[\frac{1}{5/9}])$$

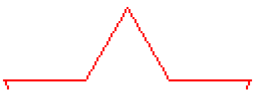
$$= A(1 + (\frac{1}{3})\frac{9}{5}) = A(1 + \frac{3}{5}) = \frac{8A}{5}$$

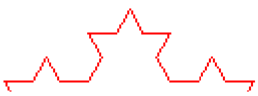
Thus, we found that the area inside the Koch snowflake is finite, and that it sums up to eight fifths of the area in the original triangle. Since the shape is generated by an infinite recursion (repetition of "the same" geometric manipulation over and over again), we had to sum an infinite series to obtain our result.

Perimeter of the Koch Snowflake:

To simplify the matter, let us describe what happens to one side of the triangle as the recursion is repeated. Suppose that the original length of one side is L . Then we go through the following steps:

Step 1:  1 segment of length L.

Step 2:  4 segments, each of length L/3. Total length of side is (4/3)L.

Step 3:  4 times 4 segments, each of length (1/3) L/3. Total length of side is (4/3)(4/3) = $(4/3)^2$.

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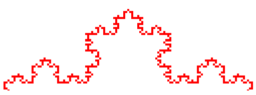
Step n:  Total length = $(4/3)^n$.

The lengths are not added in this case. Rather, at each stage of the process, the length of one of the original sides of the triangle is expanded by a factor of (4/3). This, as we have seen, leads to a sequence (not a series, just a list of numbers) of the form

$$L = (\frac{4}{3})^n$$

But this sequence does not converge. Its terms are powers of a number greater than 1, and thus these terms grow without bound. Thus, the perimeter of the Koch snowflake is infinite, even though its area is finite.

The demonstration below allows you to experiment with your own Koch snowflake. Move the red dot to observe the changes in this intricate fractal as the number of iterations, n, increases.