The Relevance of Wavy Beds as Shoreline Protection

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Abstract We study the relevance of wavy beds as shoreline protection through the Bragg scattering mechanism for surface waves over impermeable sinusoidal beds. We take into account the presence of a current and a hard-wall beach on the right of the wavy bed. As well known, a relatively small amplitude of sinusoidal beds can reduce the amplitude of incident waves effectively, due to Bragg resonance. In the case of no current and only outgoing waves on the transmission side, Bragg resonance occurs when the wave number of sinusoidal seabed is twice the wave number of incident wave. Using asymptotic expansion method, we obtain that when there is a current, there are two wave numbers of monochromatic wave that lead to Bragg resonance. These two wave numbers reduces to the Bragg resonance wave number in the no current case. The effect of beach on the right of the sinusoidal bed is simulated numerically. It shows that the presence of hard-wall beach on the right of sinusoidal patch will increase the incident wave amplitude that hit the shore much higher, and hence increase the hazard to the shore. The situations will less severe if the shore can absorb the wave partially.

Keywords Bragg resonance, multiple scale asymptotic expansion, impermeable sinusoidal beds, hard-wall beaches.

1. Introduction

When an incident wave propagating over a wavy bed, it produces reflection and transmission waves. Assuming only outgoing waves on the transmission side, the amplitudes of both the incident and reflected waves are found to decrease monotonically over the bar patch in the shoreward direction. The most significant decrease of incident wave amplitude happen when the wave number of sinusoidal seabed is twice the wave number of incident wave. This phenomenon is called Bragg resonance and it is a stable phenomena. This motivates the idea of artificially constructing bars to protect a beach from incident waves. Literatures such as [1, 3, 4, 7, 8] study its relevance by incorporating currents, reflected beaches, bottom slopes, and permeable sea beds. Experimental study that motivates many researches in this area was done by Heathershaw [2] in 1982. C.L. Ting et.al. in [7] study Bragg scattering of surface waves over permeable rippled beds with current. They obtained that the current velocity affects the position of maximum reflection coefficient. However, he did not give an exact formula for monochromatic wave number that leads to Bragg resonance.

In this paper we will take into account the effect of current and a hard-wall beach on the right of sinusoidal bar. We apply the asymptotic expansion method for the linear shallow water equation for sinusoidal beds in the case when there is a current. The procedure is similar to those applied by C.C. Mei et. al. in [6, 8]. Similar method is applied by Philip L.-F. Liu in [5] for water waves in a channel with corrugated boundaries. We obtain that there are two wave numbers that may lead to Bragg resonance. These two wave numbers reduce to the Bragg resonance wave number in the no current case. We also obtained a time dependent linear equations for the incident (right-running) and reflected (left-running) wave amplitudes. The equations are coupled and they maintain transfer energy between the right and left running waves until steady states of Bragg resonance has been achieved. The effect of a hard-wall beach on the right of sinusoidal beds are simulated using these equations. We simulate transfer energy between right and left running waves until the steady state is achieved. In this way, we obtain the most hazardous situation of Bragg resonance, and also its dependence on current velocity.

2. Bragg resonance conditions with current

We focus on finding Bragg resonance condition for surface wave above a sinusoidal bed in the case there is a current. The method used is similar with the one used by C.C. Mei in [6] which is basically the multiple scale asymptotic expansion. Consider the SWE equation with current \( \bar{U} \) for a bottom topography \( h(\hat{x}) = h_0 (1 + \varepsilon \delta \sin \bar{K} \hat{x}) \) as follows

\[
\begin{align*}
\eta_t + [(\eta + h)(U + u)]_{\hat{y}} &= 0 \\
u_t + (U + u)u_{\hat{x}} + g(\eta + h)_{\hat{x}} &= 0.
\end{align*}
\]

with \( \eta(\hat{x},\hat{t}) \) is surface displacement, and \( u(\hat{x},\hat{t}) \) is horizontal component of fluid particles.

2.1 Multiple scale expansion method

Here we apply the multiple scale expansion method in order to find an analytical solution of (1), in the case of near resonance, which is valid for all \( \hat{x} \) and \( \hat{t} > 0 \). Let we
introduce fast and slow variables in space and time as follows
\[ x = \hat{x}, \; \bar{x} = \varepsilon \hat{x} \]
\[ t = \hat{t}, \; \bar{t} = \varepsilon \hat{t}. \]
The relation between partial derivatives are
\[ \partial_\bar{t} = \partial_t + \varepsilon \partial_\hat{t}, \quad \partial_\bar{x} = \partial_x + \varepsilon \partial_\hat{x}. \]
Next we expand
\[ \eta(x, \bar{x}; t, \bar{t}) = \eta_0(x, \bar{x}; t, \bar{t}) + \varepsilon \eta_1(x, \bar{x}; t, \bar{t}) + \ldots, \]
\[ u(x, \bar{x}; t, \bar{t}) = u_0(x, \bar{x}; t, \bar{t}) + \varepsilon u_1(x, \bar{x}; t, \bar{t}) + \ldots, \]
with \( \varepsilon > 0 \) is a small parameter. Substituting (5) and (6) into (1) gives us the following series of equations
\[ O(1) : \partial_\bar{t} \eta_0 - \left( U^2 + g \hat{h}_0 \right) \partial_\bar{x} \eta_0 - 2U \hat{h}_0 \partial_\bar{x} u_0 = 0, \]  
\[ O(\varepsilon) : \partial_\bar{t} \eta_1 - \left( U^2 + g \hat{h}_0 \right) \partial_\bar{x} \eta_0 - 2U \hat{h}_0 \partial_\bar{x} u_1 = -2 \left( \varepsilon \partial_\bar{t} \eta_0 - \left( U^2 + g \hat{h}_0 \right) \partial_\bar{x} \eta_0 - 2U \hat{h}_0 \partial_\bar{x} u_0 \right) \]
\[ + g \hat{h}_0 E \sin K \hat{x} \partial_\bar{x} \eta_0 + g \hat{h}_0 E K \cos K \partial_\bar{x} \eta_0 \]
\[ + 2U \hat{h}_0 E \sin K \partial_\bar{x} \eta_0 + 3U \hat{h}_0 E K \cos K \partial_\bar{x} \eta_0. \]
Note that equation (7) is equivalent with the linear form of (1) for flat bottom \( \hat{h}_0 \). Solutions \( \eta_0 \) and \( u_0 \) that can be obtained through the Riemann invariant form of (7) are superposition of monochromatic waves with two wave numbers \( k_1 \), \( k_2 \) that satisfy the dispersion relation
\[ \omega = \pm c_1, \quad \omega = \pm c_2, \]
where \( \omega \) is the wave frequency, and \( c_1 = \sqrt{g \hat{h}_0 - U} \) and \( c_2 = \sqrt{g \hat{h}_0 + U} \) are phase velocity of each monochromatic wave. Following the procedure of multiple scale asymptotic expansion method, we assume that solutions of (1) are also superposition of two monochromatic waves with amplitudes to be determined
\[ \eta_0 = \frac{1}{2} \sqrt{\hat{h}_0} \left( A(\bar{x}, \bar{t}) e^{i(k_1 x_0 \omega t)} + B(\bar{x}, \bar{t}) e^{-i(k_1 x_0 \omega t)} + c.c. \right) \]
\[ - C(\bar{x}, \bar{t}) e^{i(k_2 x_0 \omega t)} - D(\bar{x}, \bar{t}) e^{-i(k_2 x_0 \omega t)} + c.c. \right) \]
\[ u_0 = \frac{1}{2} \left( A(\bar{x}, \bar{t}) e^{i(k_1 x_0 \omega t)} + B(\bar{x}, \bar{t}) e^{-i(k_1 x_0 \omega t)} + c.c. \right) \]
\[ + C(\bar{x}, \bar{t}) e^{i(k_2 x_0 \omega t)} + D(\bar{x}, \bar{t}) e^{-i(k_2 x_0 \omega t)} + c.c. \right) \]
with c.c denotes their complex conjugate. Note that \( \eta_0 \) and \( u_0 \) are superposition of waves with two wave numbers \( k_1 \) and \( k_2 \), which now their amplitudes are functions of \( \bar{x} \) and \( \bar{t} \). Here \( A(\bar{x}, \bar{t}) \) and \( C(\bar{x}, \bar{t}) \) are amplitudes of right running monochromatic wave with wave number \( k_1 \) and \( k_2 \), respectively. The others, \( B(\bar{x}, \bar{t}) \) and \( D(\bar{x}, \bar{t}) \) are amplitudes of left running monochromatic wave with wave number \( k_1 \) and \( k_2 \), respectively. The amplitudes \( A(\bar{x}, \bar{t}), \]
\( B(\bar{x}, \bar{t}), \]
\( C(\bar{x}, \bar{t}) \) and \( D(\bar{x}, \bar{t}) \) are complex functions to be determined.
Next, we look for solution of the order-\( \varepsilon \) equation (8) which are \( \eta_1 \) and \( u_1 \). Substituting (10) and (11) into the right hand side of (8) will yield exponent terms with wave number \( \pm k_1, \pm k_2, \pm (K - k_1), \pm (K - k_2) \). When \( K = 2k_1 \), a case that will lead to Bragg resonance, the exponent terms on the r.h.s. with wave number \( \pm k_1, K - k_1, -K + k_1 \) have the same wave number with the natural mode \( \exp(i(k_1 x \pm \omega t)) \). To avoid unbounded resonance of \( \eta_1 \) and \( u_1 \), we can simply equate to zero the coefficients of those terms, and get the following equations
\[ A_\tau + c_1 A_\tau = \beta B \]
\[ B_\tau - c_1 B_\tau = -\beta A. \]
with
\[ \beta = \frac{c_0 E k_1}{2c_1} \left( \frac{c_0}{2} - 2U \right), \quad c_0 = \sqrt{g \hat{h}_0}. \]
Note that \( \beta \) has dimension of frequency. System of equations (12) can be separated into equations for each \( A(\bar{x}, \bar{t}) \) and \( B(\bar{x}, \bar{t}) \)
\[ A_\tau - c_1^2 A_\tau + \beta^2 A = 0, \]
\[ B_\tau - c_1^2 B_\tau + \beta^2 B = 0. \]
Equations (14) are known as the Klein-Gordon equations. From (12) we can show that
\[ (\partial_\tau + c_1 \partial_\bar{\omega}) \frac{1}{2} | A |^2 = \beta A \]
\[ (\partial_\tau - c_1 \partial_\bar{\omega}) \frac{1}{2} | B |^2 = -\beta A. \]
Quantities \( \frac{1}{2} | A |^2 \) and \( \frac{1}{2} | B |^2 \) represent, respectively, energy of the right and left running waves. Adding equations (15) and (16) we obtain
\[ \partial_\tau \left( | A |^2 + | B |^2 \right) + c_1 \partial_\bar{\omega} \left( | A |^2 - | B |^2 \right) = 0, \]
(17) which means, as long as \( \partial_\bar{\omega} \left( | A |^2 - | B |^2 \right) \neq 0 \), the total energy \( | A |^2 + | B |^2 \) still changing with time(\( \partial_\tau \)) is energy flux quantity \( | A |^2 - | B |^2 \) must reduce to a constant over the sinusoidal patch at the steady state. Moreover, equations (15,16) can be interpreted as follows: if \( \beta A > 0 \) along the sinusoidal bar patch, the right running wave \( A \) gains energy from the left running wave \( B \). Transfer energy will continue until the steady state is
attained, i.e. when \(|A|^2 - |B|^2\) is constant over the bar patch. This energy flux quantity will be calculated in numerical simulation, to indicate when the steady state has been achieved.

Analogously when \(K = 2k_i\), to avoid unbounded resonance of \(\eta_1\) and \(u_1\) caused by terms with wave number \(k_i\) we get

\[
\begin{align*}
C_t + c_2 C_x &= \frac{c_0 E k_2}{2c_2} \left( \frac{c_0}{2} - 2U \right) D, \\
D_t - c_2 D_x &= -\frac{c_0 E k_2}{2c_2} \left( \frac{c_0}{2} - 2U \right) C.
\end{align*}
\]

(18)

All results available for \(A\) and \(B\) hold analogously for \(C\) and \(D\).

We conclude here that solutions (1) are (10) and (11), with \(A, B, C, \) and \(D\) are governed by (12) and (18). For the flat bottom case \(E = 0\) equations (12) and (18) reduce to transport equations for each \(A, B, C, \) and \(D\), as we expect.

Further, in the case of flat depth \(h_0\) and current \(U\), Bragg resonance will happen when the wave number \(K\) of sinusoidal beds equals \(K = 2k_i\), \(i = 1, 2\), where wave number of monochromatic wave \(k_i\) satisfies dispersion relation (9). Therefore, the two wave numbers \(k_1\) and \(k_2\) that may lead to Bragg resonance are

\[
k_1 = \frac{\omega}{\sqrt{gh_0 - U}}, \quad k_2 = \frac{\omega}{\sqrt{gh_0 + U}}.
\]

(19)

In the no current case \(U = 0\) the two wave numbers \(k_i\) reduces to the well-known wave number for Bragg resonance: \(k = \omega / \sqrt{gh_0}\), as in [6]. Further, equations (12) and (18) reduce to corresponding equations for the no current case as obtained by C.C.Mei in [6].

3. Steady state of Bragg resonance

Next we study the relevance of sinusoidal bottom in reducing the amplitude of an incident monochromatic wave when there is some reflection from the shore. Imagine an incident monochromatic wave train with wave number \(k_1\), amplitude \(A_0\) coming from the left, enters a region of an inhomogeneity with wavelength \(2k_1\), which is assumed to be confined in \(0 < \bar{x} < L\). Here, we will only discuss a situation when the monochromatic wave has the detuned Bragg resonance frequency \(\omega = k_1 c_1\). We note that in the case of slightly detuned from Bragg resonance, situation will be qualitatively the same, and Bragg resonance is a stable phenomenon. Here we will simulate the effect of reflecting shore on the right of sinusoidal bar patch. Some wave energy is transmitted beyond the inhomogeneity \(\bar{x} > L\) and some is reflected backward \(\bar{x} < 0\). To allow reflected waves we formulate the wave profile at any time as

\[
\eta = A(\bar{x}, \bar{t}) e^{i(k_1 \bar{x} - \omega \bar{t})} + cc + B(\bar{x}, \bar{t}) e^{i(k_1 \bar{x} - \omega \bar{t})} + cc.,
\]

(20)

where \(A(\bar{x}, \bar{t})\) and \(B(\bar{x}, \bar{t})\) satisfy (12) for \(0 < \bar{x} < L\). To the right and to the left of the bars, the equations are simply the corresponding transport equations for \(A\) and \(B\).

Rewriting equation (12) back in physical variables \(\hat{x}\) and \(\hat{t}\), and after erasing all hats for simplicity, we get

\[
A_t + c_1 A_x = \epsilon \beta B
\]

(21)

\[
B_t - c_1 B_x = -\epsilon \beta A.
\]

(22)

From now on, we are working with physical variables and they are denoted by \(x, t\), without \(\hat{\cdot}\). Next, we will numerically solve equations (21) and (22) with a hard-wall beach on the right side. This type of beach is the most hazardous according to Yu & Mei in [8]. The energy flux that reduced to a constant over the sinusoidal bar patch will determine whether the steady state situation has been achieved. Equation for \(A\) in (21) with the left boundary condition \(A(0, t) = A_0\) is descritized using FTBS.

Equation for \(B\) in (22) with the right boundary condition \(B(L, t) = A(L, t)\) (a hard-wall beach boundary condition), is descritized using FTFS. The grid size \(\Delta x\) and \(\Delta t\) are chosen so that \(\Delta x = c_1 \Delta t\). With this choice, numerical computation will yield analytical results.

Numerical simulation of Bragg resonance will be made using the following data: flat depth \(h_0 = 10\) m wave number of incident wave \(k_1 = \pi / m\), the sinusoidal bar patch has amplitude \(0.8\) m (or \(\varepsilon L = 0.08\)), wave number \(2k_1 = 2\pi / m\), and length \(L = 10\) m. Here we take \(U = -0.5\) m/s.

Figure 1 presents numerical computation for the case of hard-wall beach. We present them in the same way as Yu & Mei in [8]. Here, we recall some important argument for the sake of clarity. At early stages, say for \(t < 2\) s, the wavefront has not yet reached the shoreline \(x = L\), therefore ahead of the wave front, both \(A\) and \(B\) are zero. The magnitude of \(A\) is reduced except at the entry and at the wave front, where \(B = 0\), which shows the Bragg scattering mechanism. This scenario continues up to \(t = 2\)
s when the front reaches the shoreline at \( x = L \). Afterwards, for \( t > 2.5 \) s the magnitude of \( A \) increases tremendously due to shoreward energy flux, and finally reach a value \( A(L, \infty) = 3.1139 \), see Figure 1. Which means, in this case, the incident wave amplitude that hit the hard-wall beach has increased more than three times higher. However, when \( B(L, t) = 0 \) (a fully absorbing beach) or when \( B(L, t) = -A(L, t) \), the magnitude of \( A(L, t) \) is steadily reduced until the steady state is reached.

### Table 1: Values of \( A(L, \infty) \) for three different \( U \) and several values of \( \varepsilon \).

<table>
<thead>
<tr>
<th>( \varepsilon )</th>
<th>( U = -0.5 )</th>
<th>( U = 0 )</th>
<th>( U = 0.5 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.08</td>
<td>1.9802</td>
<td>1.8734</td>
<td>1.7448</td>
</tr>
<tr>
<td>0.1</td>
<td>2.3338</td>
<td>2.1809</td>
<td>1.9990</td>
</tr>
<tr>
<td>0.12</td>
<td>2.7198</td>
<td>2.5176</td>
<td>2.2780</td>
</tr>
<tr>
<td>0.14</td>
<td>3.1139</td>
<td>2.8673</td>
<td>2.5729</td>
</tr>
</tbody>
</table>

The main result of this study is that when there is current, there are two wave numbers that lead to Bragg resonance. We simulate the growth of the incident wave amplitude that hit a hard-wall beach until the steady state has been achieved. The presence of current change the steady state situation quantitatively.

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### References


