Widely separated frequencies in coupled oscillators with energy-preserving quadratic nonlinearity

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Received 13 June 2002; received in revised form 24 February 2003; accepted 26 March 2003

Communicated by C.K.R.T. Jones

Abstract
In this paper we present an analysis of a system of coupled oscillators suggested by atmospheric dynamics. We make two assumptions for our system. The first assumption is that the frequencies of the characteristic oscillations are widely separated and the second is that the nonlinear part of the vector field preserves the distance to the origin. Using the first assumption, we prove that the reduced normal form of our system has an invariant manifold which exists for all values of the parameters. This invariant manifold cannot be perturbed away by including higher order terms in the normal form. Using the second assumption, we view the normal form as an energy-preserving three-dimensional system which is linearly perturbed. Restricting ourselves to a small perturbation, the flow of the energy-preserving system is used to study the flow in general. We present a complete study of the flow of the energy-preserving system and its bifurcations. Using these results, we return to the dissipative system and provide the condition for having a Hopf bifurcation of one of the two equilibria of the perturbed system. We also numerically follow the periodic solution created via the Hopf bifurcation and find a sequence of period-doubling and fold bifurcations, also a torus (or Neimark–Sacker) bifurcation.

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Keywords: High-order resonances; Singular perturbation; Bifurcation

1. Introduction
High-order resonances in a system of coupled oscillators tend to get less attention rather than the lower-order ones. In fact, as noticed in [10], the tradition in engineering is to neglect the effect of high-order resonances in a system. However, the results of Broer et al. [1,2], Langford and Zhan [14,15], Nayfeh et al. [16,17], Tuwankotta and Verhulst [21], etc. showed that in the case of widely separated frequencies, which can be seen as an extreme type of high-order resonances, the behavior of the system is different from the expectation.

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Think of a system
\[ \ddot{x} + \omega^2 x = f(\dot{x}, x, y, t), \]
\[ \ddot{y} + \omega^2 y = g(\dot{y}, x, y, t), \]
where \( \omega_x \) and \( \omega_y \) are assumed to be positive real numbers, and \( f \) and \( g \) are sufficiently smooth functions. If there exists \( k_1, k_2 \in \mathbb{N} \) such that \( k_1 \omega_x - k_2 \omega_y = 0 \), we call the situation resonance. If \( k_1 \) and \( k_2 \) are relatively prime and \( k_1 + k_2 < 5 \) we call this low-order resonance (or, also called genuine or strong resonances).

One of the phenomena of interest in a system of coupled oscillators is the energy exchanges between the oscillators. It is well known that in low-order resonances, this happens rather dramatically compared to higher-order ones. For systems with widely separated frequencies, the behavior is different from the usual high-order resonances in the following sense. In [10, 16, 17], the authors observed a large scale of energy exchanges between the oscillators. In the Hamiltonian case, the results in [1, 2, 21] show that although there is no energy exchange between the oscillators, there are important phase interactions occurring on a relatively short time-scale.

1.1. Motivations

In this paper we study a system of coupled oscillators with widely separated frequencies. This system is comparable with the systems which are considered in [10, 14–17]. However, we are mainly concerned with the internal dynamics, i.e. autonomous framework. Thus, in comparison with [10, 16, 17], there is no time-dependent forcing term in our system. Our goal is to describe the dynamics of the model using normal form theory. This analysis can be considered as a supplement to [14, 15] which are concentrated on the unfolding of the trivial equilibrium and its bifurcation. In general, the trivial equilibrium has a degeneracy of codimension three. In this paper, we are more concerned with the existence of a nontrivial equilibrium and its bifurcation.

Another motivation for studying this system comes from the applications in atmospheric research. In [4], a model for ultra-low-frequency variability in the atmosphere is studied which represents a novel approach to the long time behavior (weeks or more) of the atmosphere. In such a study, one usually encounters a system with a large number of degrees of freedom, which is a projection of the Navier–Stokes equation to a finite-dimensional space. The projected system in [4] is 10-dimensional and the projection is done using so-called Empirical Orthogonal Functions (see the reference in [4] for an introduction to the EOF-approach). In that projected system, the linearized system around an equilibrium has two (among five) pairs of eigenvalues which are \( \lambda_1 = -0.00272154 \pm 0.438839 \) and \( \lambda_2 = 0.00165548 \pm 0.0353438 \). One can see that \( \Im(\lambda_1)/\Im(\lambda_2) = 12.4163 \ldots \), which is clearly not a strong resonance.

A study of the dynamics of the two modes mentioned above is done in [4]. The author found that the flow collapses into a nontrivial equilibrium. In this paper we consider a wider, physically relevant, range of parameters in the system which allow us to study the bifurcation of the nontrivial equilibrium. The existence of a homoclinic or heteroclinic orbit in a dynamical system often follows by the existence of chaotic dynamics for the nearby parameter values. This is interesting for application in atmospheric research since it provides an explanation for the occurrence of very long time-scales in the system. Motivated by this, in this paper we will look for such an orbit.

One of the main differences between the analysis in this paper and in [4] is the bifurcation parameter. In [4] the bifurcation parameter is one of detuning. In this paper, the frequencies are fixed. There are two bifurcation parameters which represent the strength of the coupling between the two modes and the strength of self-interaction term within the slow oscillator.

In fluid dynamics the model usually has a special property, namely the nonlinear part of the vector field (the advection term) preserves the energy while the linear part makes it dissipative. We assume the same holds in our system. We take the simplest representation of the energy, i.e. the distance to the origin, and assume that the flow of the nonlinear part of the vector field preserves the distance to the origin.
1.2. Summary of the results

Let us consider a system of first-order ordinary differential equations in $\mathbb{R}^4$ with coordinate $z = (z_1, z_2, z_3, z_4)$. We add the following assumptions to our system:

(A1) The system has an equilibrium: $z_e \in \mathbb{R}^3$ such that the linearized vector field around $z_e$ has four simple eigenvalues $\lambda_1, \lambda_1, \lambda_2,$ and $\lambda_2$, where $\lambda_1, \lambda_2 \in \mathbb{C}$. Furthermore, we assume that $\text{Im}(\lambda_1)$ is much larger in size compared to $\text{Im}(\lambda_2)$, $\text{Re}(\lambda_1)$ and $\text{Re}(\lambda_2)$.

(A2) The nonlinear part of the vector field preserves the energy which is represented by the distance to the origin.

In Section 2, we will re-state these assumptions in a more mathematically precise manner.

We use normal form theory to construct an approximation for our system. In Theorem 3.1, we show that the normal form, truncated up to any finite degree, exhibits an invariant manifold which exists for all values of parameters. This invariant manifold coincides with the linear eigenspace corresponding to the pair of eigenvalues $\lambda_1$ and $\lambda_2$. In [10], the same invariant manifold is found. In this paper, we prove that this invariant manifold exists for a slightly more general system. Furthermore, it exists in the truncated up to any finite degree of the normal form of that system.

We start with a simpler situation where $\text{Re}(\lambda_1) = \text{Re}(\lambda_2) = 0$. In this situation the system preserves the energy. The phase space of such a system is fibered by the energy manifolds, which are spheres in our case. By restricting the flow of the normal form to each of these spheres, we reduce the normal form to a two-dimensional system of differential equations parameterized by the value of the energy, which is the radius of the sphere. As a consequence, each equilibrium that we find on a particular sphere (which is nondegenerate on that particular sphere) can be continued to some neighboring spheres. This gives us a manifold of equilibria of the normal form for $\text{Re}(\lambda_1) = \text{Re}(\lambda_2) = 0$. In fact we have two of such manifolds in our system. This analysis is presented in Sections 5 and 6.

For small values of $\text{Re}(\lambda_1)$ and $\text{Re}(\lambda_2)$, the normal form can be considered as an energy-preserving three-dimensional system which is linearly perturbed. Note that the linear perturbation removes the conservation of energy from the system. The dynamics consists of slow–fast dynamics. The fast dynamics corresponds to the motion on two-spheres described in the above paragraph. The slow dynamics is the motion from one sphere to another along the direction of the curves of critical points.

In [7], Fenichel proved the existence of an invariant manifold where the slow dynamics takes place. This slow manifold is actually a perturbation of the manifold of equilibria which exists for the unperturbed case. The conditions that have to be satisfied are that the unperturbed manifold should be normally hyperbolic and compact. Since both such curves in our system, fail to satisfy these condition, we cannot conclude that there exists an invariant slow manifold. For an introduction to geometric singular perturbation, see [12].

For a thorough treatment on the theory of invariant manifolds, see [11] and also [22]. The dynamics however, is similar apart from the fact that the slow motion is funnelling into a very narrow tube along the curve instead of following a unique manifold.

The linear perturbation is governed by two parameters: $\mu_1 (\approx \text{Re}(\lambda_1))$ and $\mu_2 (\approx \text{Re}(\lambda_2))$. If $\mu_1\mu_2 > 0$, the system becomes simple in the sense that we have only one equilibrium, the trivial one. The flow of the normal form collapses to the trivial equilibrium either in positive or negative time, which implies the nonexistence of any other limit set. In the opposite case: $\mu_1\mu_2 < 0$, the trivial equilibrium is unstable. In general, if $\mu_1\mu_2 < 0$ we have two critical points: the trivial one and the nontrivial one. Note that the nontrivial critical point is not branching out of the trivial critical point. The reason for this is since we assume a fixed ratio between $\mu_1$ and $\mu_2$.

There are two situations where the nontrivial equilibrium fails to exist. The first situation is when we have no interaction between the dynamics of $(z_1, z_2)$ and $(z_3, z_4)$. The other situation corresponds to a particular instability balance between the modes. For a large part of the parameter space, we prove that the solutions are bounded (see...
Combining the information of the energy-preserving flow (Section 5) and its bifurcations (Section 6), we can derive a lot of information of the dynamics of the normal form for small $\mu_1$ and $\mu_2$.

The nontrivial equilibrium that we mentioned above is a continuation of one of the equilibria of the fast system. Although we have the explicit expression for the location of the nontrivial equilibrium, to derive the stability result using linearization is still cumbersome. Using geometric arguments, the stability result and also the bifurcations of this nontrivial equilibrium can be achieved easily. In [14,15] the bifurcation of the nontrivial equilibrium is not covered because they are concerned with the unfolding of the trivial equilibrium.

We show in this paper that the only possible bifurcation for the nontrivial equilibrium is Hopf bifurcation. This Hopf bifurcation can be predicted analytically. This result is presented in Section 8. We also study the bifurcation of the periodic solution which is created via the Hopf bifurcation of the nontrivial equilibrium. However, this is difficult to do analytically. Using the continuation software AUTO[5], we present the numerical bifurcation analysis of this periodic solution in Section 9. Numerically, we find torus (Neimark–Sacker) bifurcation and a sequence of period-doubling and fold bifurcations.

### 1.3. The layout

In Section 2 the system is introduced. The small parameter in the system is the frequency of one of the oscillators and it is called $\tilde{\varepsilon}$. Using averaging we normalize the system and reduce it to a three-dimensional system of differential equations. The normalized system is analyzed in Section 3. We complete the analysis of the case where $\mu_1 \mu_2 > 0$ in this section and assume that $\mu_1 \mu_2 < 0$ in the rest of the paper. In Section 4, we re-scale $\mu_1$ and $\mu_2$ using a new small parameter $\varepsilon$. By doing this we formulate the normal form as a perturbation of an energy-preserving system in three-dimensional space. There are two continuous sets of equilibria of the energy-preserving part of the system and they are analyzed in Section 5. In Section 6, we use the fact that the phase space of the energy-preserving part of the system is fibered by invariant half spheres, to project the unperturbed system to a two-dimensional system of differential equations. The stability results derived in Section 5 are applied to study the bifurcation in the projected system. In Section 7, we turn on our perturbation parameter: $\varepsilon \neq 0$. Using geometric arguments, we derive the stability results for the nontrivial equilibrium. Furthermore, in Section 8 we use a similar argument to derive the condition for Hopf bifurcation of the nontrivial equilibrium. The bifurcation of the periodic solution which is created via Hopf bifurcation is studied numerically in Section 9.

### 2. Problem formulation and normalization

Let $0 < \tilde{\varepsilon} \ll 1$ be a small parameter. Consider a system of ordinary differential equations in $\mathbb{R}^4$ with coordinates $z = (z_1, z_2, z_3, z_4)$, defined by:

$$
\dot{z} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} z + F(z),
$$

where $A_j, j = 1, 2$ are $2 \times 2$ matrices, with eigenvalues: $\tilde{\varepsilon} \mu_1 \pm i$, and $\tilde{\varepsilon} \mu_2 \pm i \tilde{\omega}$ respectively, and $\mu_1$, $\mu_2$, and $\tilde{\omega}$ are real numbers. We assume that $\mu_1$ and $\mu_2$ are bounded and $\tilde{\omega}$ is bounded away from zero and infinity. The nonlinear function $F$ is a quadratic, homogeneous polynomial in $z$ satisfying $z \cdot F(z) = 0$. Thus, the flow of the system $\dot{z} = F(z)$ is tangent to the sphere: $z_1^2 + z_2^2 + z_3^2 + z_4^2 = R^2$, where $R$ is the radius. We view the system (2.1) also as a coupled oscillators system. It is easy to see that in the case $F(z) = 0$, then system (2.1) is equivalent to the system of two oscillators with dissipation.
We re-scale the variables by \( z \mapsto \tilde{\varepsilon} z \). By doing this we formulate the system (2.1) as a perturbation problem, i.e.

\[
\dot{z} = \begin{pmatrix} \tilde{A}_1 & 0 \\ 0 & 0 \end{pmatrix} z + \tilde{\varepsilon} \tilde{F}(z),
\]

(2.2)

with

\[
\tilde{A}_1 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

Note that \( \tilde{F} \) is no longer homogeneous; it contains linear terms. We normalize (2.2) with respect to the actions defined by the flow of the unperturbed vector field of (2.2) (that is for \( \tilde{\varepsilon} = 0 \)). This can be done by applying the transformation

\[
z_1 \mapsto r \cos (t + \phi), \quad z_2 \mapsto -r \sin (t + \phi), \quad z_3 \mapsto x, \quad z_4 \mapsto y
\]

to (2.2) and then average the resulting equations of motion with respect to \( t \) over \( 2\pi \). See [18] for details on the averaging method.

The averaged equations are of the form

\[
\dot{\phi} = \tilde{\varepsilon} G_1(r, x, y) + O(\tilde{\varepsilon}^2),
\]

\[
\dot{r} = \tilde{\varepsilon} G_2(r, x, y) + O(\tilde{\varepsilon}^2),
\]

\[
\dot{x} = \tilde{\varepsilon} G_3(r, x, y) + O(\tilde{\varepsilon}^2),
\]

\[
\dot{y} = \tilde{\varepsilon} G_4(r, x, y) + O(\tilde{\varepsilon}^2),
\]

where \( G_j, j = 1, \ldots, 4 \) are at most quadratic. Thus, we can reduce the system to a three-dimensional system of differential equations by dropping the equation for \( \phi \). This reduction is typical for an autonomous system. We note that by applying the averaging method, we can preserve the energy-preserving nature of the nonlinearity. Furthermore, by rotation we can choose a coordinate system such that the equation for \( r \) is of the form \( \dot{r} = \tilde{\varepsilon} G_2(r, x) + O(\tilde{\varepsilon}^2) \).

We omit the details of the computations and just write down the reduced averaged equations (or normal form) after rescaling time by \( t \mapsto \tilde{\varepsilon} t \), i.e.

\[
\begin{pmatrix} \dot{r} \\ \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \mu_1 & 0 & 0 \\ 0 & \mu_2 & 0 \\ 0 & 0 & \mu_2 \end{pmatrix} \begin{pmatrix} r \\ x \\ y \end{pmatrix} + \begin{pmatrix} \delta r \Omega(x, y)y - \delta r^2 \\ \delta r \Omega(x, y)x \end{pmatrix},
\]

(2.3)

where \( \Omega(x, y) = \omega + \alpha x + \beta y \), \( \mu_1, \mu_2, \alpha, \beta, \omega \), and \( \delta \) are real numbers. It is important to note that up to this order, the small parameter \( \tilde{\varepsilon} \) is no longer present in the normal form, by time reparameterization.

To facilitate the analysis we introduce some definitions. Let a function \( G : \mathbb{R}^3 \to \mathbb{R}^3 \) be defined by

\[
G(\xi) = \begin{pmatrix} \delta r \Omega(x, y)y - \delta r^2 \\ \delta r \Omega(x, y)x \end{pmatrix},
\]

(2.4)

where \( \xi = (r, x, y)^T, \delta \Omega(x, y) = \omega + \alpha x + \beta y \). We also define a function \( S : \mathbb{R}^3 \to \mathbb{R} \) by

\[
S(\xi) = r^2 + x^2 + y^2.
\]

(2.5)

Note that \( dS/dt = 0 \) along the solution of \( \dot{\xi} = G(\xi) \). Lastly, we define

\[
S(R) = |\xi|^2 + x^2 + y^2 = R^2, \quad R \geq 0,
\]

(2.6)

which is the level set \( S(\xi) = R^2 \).
Remark 2.1 (Symmetries in the system). We consider two types of transformations: transformation in the phase space $\Phi_j : \mathbb{R}^3 \to \mathbb{R}^3$, $j = 1, 2$ and in the parameter space: $\Psi : \mathbb{R}^6 \to \mathbb{R}^6$. Consider $\Phi_1(r, x, y) = (-e, x, y)$, which keeps the system (2.3) invariant. This immediately reduces the phase space to $D = \{ r \geq 0 | e \in \mathbb{R} \} \times \mathbb{R}^3$. Another symmetry which turns out to be important is a combination between $\Phi_2(r, x, y) = (e, -x, -y)$ and $\Psi(a, b, \delta, \omega, \mu_1, \mu_2) = (-a, -b, -\delta, \omega, \mu_1, \mu_2)$. System (2.3) is invariant if we transform the variables using $\Phi_2$ and also the parameters using $\Psi$. It implies that we can reduce the parameter space by fixing a sign for $\beta$. We choose $\beta < 0$. One can also consider a combination involving time-reversal symmetry. We are not going to take this symmetry into account because this symmetry changes the stability of all invariant structures in the system. Thus, we assume: $\omega > 0$.

3. General invariant structures

System (2.3) has exactly two general invariant structures in the sense that they exist for all values of the parameters. They are the trivial equilibrium $(r, x, y) = (0, 0, 0)$ and the invariant manifold $r = 0$. The linearized system around the trivial equilibrium has eigenvalues $\mu_1, \mu_2 \neq 0$. We have three cases: $\mu_1 \mu_2 > 0$, $\mu_1 \mu_2 < 0$ or $\mu_1 \mu_2 = 0$.

If $\mu_1 \mu_2 > 0$, along the solutions of system (2.3), we have $\dot{S} = \mu_1 r^2 + \mu_2 (x^2 + y^2)$ (see (2.5) for the definition of $S$) is positive (or negative) semi-definite if $\mu_1 > 0$ (or $\mu_1 < 0$, respectively). Thus, $S$ is a globally defined Lyapunov function. As a consequence, all solutions collapse into the neighborhood of the trivial equilibrium for positive (or negative) time, if $\mu_1 < 0$ (or $\mu_1 > 0$, respectively). Moreover, there is no other invariant structure apart from this trivial equilibrium and the invariant manifold $r = 0$. This completes the analysis for this case.

For $\mu_1 \mu_2 < 0$ the trivial equilibrium is unstable. In the case where $\mu_1 > 0$, the equilibrium has one-dimensional unstable manifold and two-dimensional stable manifold. The stable manifold is the invariant manifold $r = 0$. The situation is reversed in the case $\mu_1 < 0$. The global dynamics in this case is not clear at the moment. We will come back to this question in Sections 7-9.

For $\mu_1 \mu_2 = 0$, we have again three different possibilities: $\mu_1 = 0$ or $\mu_2 = 0$ or $\mu_1 = \mu_2 = 0$. For the purpose of this paper, we consider only the most degenerate case: $\mu_1 = \mu_2 = 0$. In this case, $S = 0$ which means $S(R)$ is invariant under the flow of (2.3). Thus, the trivial equilibrium is neutrally stable. The phase space of system (2.3) is fibered by invariant sphere $S(R)$ and hence the flow reduces to a two-dimensional flow on these spheres.

The second invariant is the invariant manifold $r = 0$. The following theorem gives us the existence of this manifold in more general circumstances than for (2.3), where it is trivial.

Theorem 3.1 (The existence of an invariant manifold). Consider system (2.1), i.e.

$$
\dot{z} = \begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix} z + F(z),
$$

with $z \in \mathbb{R}^4$, $F : \mathbb{R}^4 \to \mathbb{R}^4$ is sufficiently smooth with properties: $F(0) = 0$ and $D_z F(0)$ is a zero matrix. The eigenvalues of $A_1$ are: $\pm i \omega j$ while for $A_2$ are: $\pm i \omega j \delta$, where $\omega, \mu_1, \mu_2 \in \mathbb{R}$, $j = 1, 2$ and $0 < \omega \ll 1$. Let $\tilde{z} = F(z)$ be a normal form for (3.1), up to an arbitrary finite degree $k$. The flow of the normal form keeps the plane $M = \{ |z_1| = 0 = z_2 \}$ invariant.

Proof. Let us transform the coordinate by $z \mapsto \tilde{z} \tilde{z}$. System (3.1) is transformed to

$$
\dot{\tilde{z}} = \text{diag}(A_1, 0) \tilde{z} + \tilde{F}(\tilde{z}; \tilde{\eta}).
$$
where $\tilde{F}$ contains also linear term. Consider the algebra of vector fields in $\mathbb{R}^4, X(\mathbb{R}^4)$. Note that we can view the vector field $X$ as a map $X: \mathbb{R}^4 \to \mathbb{R}^4$. The Lie bracket in this algebra is the standard commutator between vector fields, i.e.
\[
[X_1, X_2](t) = dX_1(t) \cdot X_2(t) - dX_2(t) \cdot X_1(t),
\]
where $X_1, X_2 \in X(\mathbb{R}^4)$ and $t \in \mathbb{R}^4$. Let the unperturbed vector field of (2.2) be denoted by $X$. It defines a linear rotation in $(z_1, z_2)$-plane. This action keeps all points in the manifold $M = \{z_1 = 0, z_2 \}$ invariant. We normalize the vector field corresponding to the system $\tilde{z} = \tilde{F}(z)$ with respect to this rotation. The resulting normalized vector field truncated to a finite order $k$, commutes with $X$. Thus $[X, X] = 0$. In particular, for every $m \in M$ \[
0 = [X, X](m) = dX(m) \cdot X(m) - dX(m) \cdot X(m) = dX(m) \cdot X(m) \cdot X(m).
\]
This implies $X(m) \in \ker(dX(m)) = M$. \hfill $\square$

The dynamics in this invariant manifold gives us only a partial information of the flow. In the next section we re-write (2.3) as a perturbation of a system with a first integral.

4. The re-scaled system

Recall that if $\mu_1 = \mu_2 = 0$, system (2.3) has an integral, i.e. $S(t)$. Let $\epsilon$ be a small parameter. We re-scale: $\mu_1 = \epsilon k_1$ and $\mu_2 = -\epsilon k_2$, where $k_1 > 0$. System (2.3) becomes
\[
\begin{align*}
\dot{r} &= \delta r + \epsilon k_1 r, \\
\dot{x} &= \delta y - \delta z - \epsilon k_2 x, \\
\dot{y} &= -\delta x - \epsilon k_2 y,
\end{align*}
\]
where $\delta = \omega + \alpha x + \beta y$. We have assumed that $\omega > 0$ and $\beta < 0$.

**Lemma 4.1.** There exists a bounded domain $B$ in phase space such that all solutions of system (4.1) with $\delta > 0$, $k_1 > 0$, and $k_2 > 0$ enter a bounded domain $B$ and remain there forever after.

**Proof.** Consider a function $F(t) = r^2 + \delta^2 + z^2 - 2\delta x \cdot y$, where $\delta$ is a parameter to be determined later. The level set of $F$, i.e. $F(r, x, y) = c$ is a sphere, centered at $(r, x, y) = (0, \eta_0, -\eta_0)$ with radius $\sqrt{c + \eta_0^2(\alpha^2 + \beta^2)}$. The derivative of $F$ along a solution of system (4.1) is
\[
L_F F = (2\epsilon k_1 + 2\beta \delta) r^2 - \epsilon k_2 (x^2 + y^2) - 2\delta (\alpha x + \beta y)^2 - L(x, y),
\]
where $L(x, y)$ is a polynomial with degree at most one. Since $k_1 > 0, k_2 > 0$ and $\delta > 0$, we have $2\epsilon k_1 + 2\beta \delta < 0$ if and only if $\eta > -\epsilon k_1/(\beta \delta) > 0$. This means under the conditions in this Lemma, we can always choose $\eta$ in such a way that the quadratic part of $L_F F$ is negative definite.

Let us fix $\eta$ so that the quadratic part of $L_F F$ is negative definite. Consider $(x, y) \in \mathbb{R}^2$ and a real number $c \in \mathbb{R}$. From equation $r^2 + (x - \eta_0)^2 + (y + \eta_0)^2 = c + \eta_0^2(\alpha^2 + \beta^2)$ we can compute $r$ which solves the equation, as a function of $x, y$, and $c$: $r(x, y, c)$. Let us define $G: \mathbb{R}^2 \to \mathbb{R}$, by assigning to $(x, y)$ the value of $L_F F(r(x, y), c, x, y)$. One can check that $G(x, y)$ has a unique maximum and $\partial G/\partial c$ does not depend on $x$ or $y$. Thus, we can solve $\partial G/\partial x = 0$ and $\partial G/\partial y = 0$ for $(x, y)$, and the solution is independent of $c$. Let $(x_0, y_0)$ be the solution of $\partial G/\partial x = 0$ and $\partial G/\partial y = 0$. We can solve the equation $G(x_0, y_0, c) = 0$ if $F(t) > c_0$. It follows that every solution enters the ball $B = \{(x, y)^2 + (x - \eta)^2 + (y + \eta)^2 \leq c_0 + \eta^2(\alpha^2 + \beta^2)\}$ and remains there forever after. \hfill $\square$
We cannot apply the same arguments as above if \( \delta < 0 \). In Section 7 we will derive the conditions for bounded solutions in this case. If \( \delta = 0 \), then the dynamics of \( r \) is decoupled from the rest. Moreover, \( r \) grows exponentially with a rate: \( \epsilon \kappa_1 \). Thus, we conclude that all solutions except for those in \( r = 0 \), eventually run off to infinity. If \( \kappa_1 < 0 \) and \( \kappa_2 < 0 \), in the invariant manifold \( r = 0 \) all solutions run off to infinity except for the origin. This motivates us to restrict ourselves to the case where \( \kappa_1 > 0 \) and \( \kappa_2 > 0 \). To understand system (4.1), first we study the case where \( \epsilon = 0 \).

5. Two manifolds of equilibria

Recall that we have assumed that \( \omega > 0 \) and \( \beta < 0 \) (see Remark 2.1). For \( \epsilon = 0 \), system (4.1) becomes

\[
\dot{r} = \delta r x, \quad \dot{x} = \Omega y - \delta r^2, \quad \dot{y} = -\Omega x.
\]

(5.1)

At this point we assume that \( \alpha \neq 0 \), \( \delta \neq 0 \), \( \beta < 0 \), and \( \omega > 0 \).

5.1. A manifold of equilibria in the plane \( r = 0 \)

There are two manifolds of equilibria in system (5.1). One of them is the line: \( \Omega = \omega + \alpha x + \beta y = 0 \) and it lies in the invariant manifold \( r = 0 \). We parameterize this set by \( y = y_c \), i.e.

\[
(r, x, y) = \left(0, -\frac{\beta y_c + \omega}{\alpha}, y_c\right), \quad y_c \in (-\infty, +\infty).
\]

(5.2)

The eigenvalues of system (5.1) linearized around (5.2), are

\[
\lambda_1 = 0, \quad \lambda_2 = -\left(\frac{\beta y_c + \omega}{\alpha}\right) \delta, \quad \lambda_3 = \frac{\alpha \delta^2 + \beta^2}{\alpha} y_c + \beta \omega.
\]

(5.3)

It is clear that \( \lambda_1 \) is the eigenvalue corresponding to the tangential direction to the set (5.2). The behavior of the linearized system around the equilibria in (5.2) is determined by the eigenvalues (5.3). They are presented in Fig. 1.

Remark 5.1. If \( \alpha = 0 \) we parameterize the manifold as \( (r, x, y) = (0, x_c, -\omega/\beta), x_c \in (-\infty, +\infty) \). Each of these equilibria with \( x_c > 0 \) has two positive eigenvalues (and one zero) and those with \( x_c < 0 \) have two negative eigenvalues (and one zero). At \( x_c = 0 \) we have two extra zero eigenvalues.

![Fig. 1. The diagram shows the sign of the eigenvalues (5.3) for \( \alpha < 0 \).](image)
5.2. A manifold of equilibria in the plane \( x = 0 \)

The other manifold of equilibria of system (5.1) lies in the plane \( x = 0 \). The manifold is a curve defined by equation \( br^2 - \beta(y + \omega)(2\beta r) = -\alpha^2 /(4\beta) \), which is an ellipse if \( \delta > 0 \), or hyperbola if \( \delta < 0 \). This curve (manifold) of equilibria intersects \( r = 0 \) at \( y = 0 \) and at \( y = -\omega/\beta \). Note that \( y = -\omega/\beta \) is also the intersection point with the line \( z = 0 \) which explains why we have an extra zero eigenvalue if \( y_c = -\omega/\beta \) in (5.3).

5.2.1. An ellipse of critical points

Let us now look at the case of \( \delta > 0 \) where we have an ellipse of critical points. We parameterize the ellipse by \( y_c \), i.e.

\[
(x, y) = \left( \frac{y_c(\omega + \beta y)}{\delta}, 0, y_c \right).
\]

(5.4)

where \( 0 \leq y_c \leq -\omega/\beta \). The linearized system of system (5.1) around each of these equilibria has eigenvalues

\[
\lambda_1 = 0, \quad \lambda_2 = 1/2(\omega \pm \sqrt{\delta}), \quad \lambda_3 = 0.
\]

(5.5)

The following lemma gives the stability results for these critical points.

**Lemma 5.2.** Let \( \alpha < 0 \).

1. If \( \delta \geq -\beta/2 \) then Re(\( \lambda_{2,3} \)) < 0 for all except the two end points of the set of equilibria (5.4).
2. If \( 0 < \delta < -\beta/2 \), then at the equilibrium

\[
(x_c, y_c, z_c) = \left( \frac{\omega}{2(\delta + \beta)}, \frac{\beta + 2\omega}{\delta}, 0, \frac{-\omega}{2(\delta + \beta)} \right).
\]

(5.6)

\( \lambda_2 = 2\omega y_c < 0 \) and \( \lambda_3 = 0 \). Moreover, for the equilibria in (5.4) with \( 0 < y_c < y_c \), Re(\( \lambda_{2,3} \)) < 0, while for the other equilibria \( (y_c, y_c, -\omega/\beta, \lambda_2 < 0 \) and \( \lambda_3 > 0 \).

**Proof.** Consider \( D \) in (5.5) as a quadratic function in \( y_c \). If \( D(y_c) > 0 \) for \( 0 < y_c < -\omega/\beta \) then the Lemma holds. Let \( D \) > 0 and define a function \( L(y_c) = (ay_c)^2 - b(\omega + \beta y_c) + c \). Note that \( L(0) = \omega^2 > 0 \) and \( L(-\omega/\beta) = 0 \). If \( \delta \geq -\beta/2 \) we have \( L(-\omega/\beta) = -(\beta + \omega)0 = 0 \). Thus we conclude that \( D(y_c) > (ay_c)^2 \), for \( 0 < y_c < -\omega/\beta \). If \( \delta > -\beta/2 \), then \( L(-\omega/\beta) \) < 0. Thus, there exists \( 0 < y_c < -\omega/\beta \) such that \( L(y_c) = 0 \). From the definition of \( L(y_c) \) we conclude that \( y_c = -\omega/(2(\delta + \beta)) \). Since \( L(y_c) = 0 \) we have \( D(y_c) = (ay_c)^2 \) so that either \( \lambda_2 = 0 \) or \( \lambda_3 = 0 \). Moreover, \( L(y_c) < 0 \) so that for \( 0 < y_c < y_c, L(y_c) > 0 \). \( \square \)

5.2.2. A hyperbola of critical points

For the case \( \delta < 0 \), the set of equilibria (5.4) is a hyperbola with two branches. We call the branch of the hyperbola with \( y_c > -\omega/\beta \) the positive branch and the one with \( y_c < 0 \) the negative branch. Recall that the eigenvalues of these equilibria are

\[
\lambda_1 = 0, \quad \lambda_2 = \frac{\omega \pm \sqrt{D}}{2}, \quad \lambda_3 = \frac{\omega - \sqrt{D}}{2}.
\]

where \( D = (ay_c)^2 - 4(\omega + \beta y_c)(2\delta + \beta y_c + \omega) \). One can see that \( D \) is a quadratic function in \( y_c \). It is easy to check that \( \lambda_2 = 0 \) or \( \lambda_3 = 0 \) if and only if \( y_c = -\omega/\beta \) of \( y_c = -\omega/(2\delta + \beta) \). However, for \( \delta < 0 \) we have
In this figure, the limit sets of system (5.1) are presented. The left plot is for the case of \( \delta > 0 \) and the right is for \( \delta < 0 \).

Thus, we conclude that these equilibria cannot have an extra zero eigenvalue except for \( y_0 = -\omega/\beta \). Thus, at one of the branches, \( \text{Re}(\lambda_{2,3}) \) are always negative while at the other branches positive. If \( \alpha^2 < 8\beta(\delta + \beta) \) then for a large value of \( y_0 \), the eigenvalues form a complex pair (see Fig. 2).

6. Bifurcation analysis of the energy-preserving system

Since \( S(R) \) is invariant under the flow of system (5.1), we reduce it to a two-dimensional flow on a sphere. Moreover, the upper half of the sphere \( S(R) \) is invariant under the flow of system (5.1). Thus we can define a bijection which maps orbits of system (5.1) to orbits of a two-dimensional system defined in a disc \( D(0, R) = \{ (x, y) | x^2 + y^2 \leq R^2 \} \). This bijection is nothing but a projection from the upper half of the sphere \( S(R) \) to the horizontal plane. The transformed system is

\[
\dot{x} = \Omega y - \delta(x^2 + y^2), \quad \dot{y} = -\Omega x, \tag{6.1}
\]

where \( \Omega = \omega + \alpha x + \beta y \). Note that the boundary of \( D(\Theta, R) \): \( x^2 + y^2 = R^2 \) is invariant under the flow of system (6.1). We call this boundary the equator.

Let \( R_p = -\omega/\beta, \) \( R_h = \omega/\sqrt{2\delta + \beta} \) and \( R_s = -\omega^2(\delta + \beta)/\sqrt{2\beta + 3\delta} \).

These points are bifurcation points of system (6.1), as we vary \( R \). It is easy to see that \( R_h < R_p < R_s \) if all parameters are nonzero (recall that we have chosen \( \beta < 0 \)).

6.1. On the periodic solution of the projected system

For \( R < R_h \), the equator is a periodic solution. The period of this periodic solution \( \omega \) at the equator is

\[
T(R) = 4 \int_0^R \frac{1}{(\omega + \alpha(x^2 + y^2) + \beta y\sqrt{R^2 - y^2})} \, dy. \tag{6.2}
\]

To study the stability of the periodic solution we transform to polar coordinate \( (\rho, \theta) \) in the usual way. System (6.1) is transformed to

\[
\dot{\rho} = \delta(\rho^2 - R^2) \cos(\theta), \quad \dot{\theta} = -\omega + \alpha \rho \cos(\theta) - \delta \frac{\rho}{\rho}. \]
We then compute \( x(t) \) and \( y(t) \) for a maximal subset of \( \mathbb{J} \) by solving the differential equations of \( \rho' = \rho(\theta) \phi \), linearized around \( \rho = R \), to have a first-order differential equation of the form \( \rho' = A(\theta)\rho \). Near the periodic solution (i.e., \( \rho = R \), \( \theta = t \)) is monotonically increasing. Thus, \( \rho' = A(\theta)\rho \) can be approximated by \( \rho' = A' \rho \) where

\[
A' = \int_0^{2\pi} A(\theta) d\theta = \alpha x + \frac{4\pi^2 \omega (-1 + p^2 + \sqrt{1 - p^2})}{(a^2 + b^2)^2} \tag{6.3}
\]

and \( p = R \sqrt{a^2 + b^2} \). Thus the periodic solution \( x^2 + y^2 = R^2 \) is unstable if \( a \beta < 0 \) and stable if \( a \beta > 0 \), respectively. If \( \alpha \neq 0 \), then this periodic solution is the only periodic solution in the projected system \( (6.1) \).

**Theorem 6.1.** If \( \alpha \neq 0 \), system \( (6.1) \) has no periodic solution in the interior of \( D(0, R) \).

**Proof.** Let us fix \( R < R_p \). Then there is a unique equilibrium of system \( (6.1) \) in the interior of \( D(0, R) \), namely: \( (0, y) \). Define \( \mathcal{G} = \{(0, y)\} \) and \( \mathcal{J} = \{(0, y)\} - R \leq y < R \). We write \( \nu(x, y) \) for the velocity vector field corresponds to system \( (6.1) \). If \( \Phi(t; (x, y)) \) is the flow of system \( (6.1) \) at time \( t \) with initial condition \( (x, y) \), we want to show that

for all \( P \in \mathcal{J} \), there exists \( t \in (0, \infty) \) such that \( \Phi(t; P) = \mathcal{I} \).

Let \( \mathcal{J}' \) be a maximal subset of \( \mathcal{J} \) with such a property. Clearly \( \mathcal{J}' \neq \emptyset \) since \( \Phi(T; (0, 0)) = (0, t) \) is in \( \mathcal{I} \) where \( T < \infty \) is defined in \( (6.2) \). Take \( (0, \bar{y}) \in \mathcal{J}' \). Clearly \( \mathcal{J}' \neq \emptyset \) since \( \Phi(T; (0, \bar{y})) = (0, t) \) is in \( \mathcal{I} \) where \( T < \infty \) is defined in \( (6.2) \). Take \( (0, \bar{y}) \in \mathcal{J}' \). Clearly \( \mathcal{J}' \neq \emptyset \) since \( \Phi(T; (0, \bar{y})) = (0, t) \) is in \( \mathcal{I} \) where \( T < \infty \) is defined in \( (6.2) \).

Thus, \( \mathcal{J}' \) is open in \( \mathcal{J} \). This is the trajectory \( (x(t), y(t)) \) that \( \Phi(t; (x, y)) = (x(t), y(t)) \), there exists \( \tilde{t} \) such that 

\[
x(\tilde{t}) = 0, \quad y(\tilde{t}) = 0, \quad s(\tilde{t}) \neq 0 \quad \text{(otherwise the equilibrium is not unique). By the Implicit Function Theorem we have: for an open neighborhood \( \nu(x, y) \) there exists \( \tilde{t} \) in the neighborhood of \( \tilde{t} \) such that \( x(\tilde{t}) = 0 \). Thus, \( \mathcal{J}' \) is open in \( \mathcal{J} \). This is also closed by uniqueness of the equilibrium and the fact that \( (0, R) \in \mathcal{J}' \).

Let us define a map \( X : \mathbb{R}^2 \to \mathbb{R}^2 \) by \( (x, y) \to (x, y) \). Consider \( \Gamma(t) \) which is the trajectory \( (x(t), y(t)) \) of \( \Phi(t; P) \), \( 0 \leq t \leq s \), where \( P \in \mathcal{J} \). Consider \( \theta \) such that \( \Gamma(t) = (x, y) \) with \( x \neq 0 \). We have

\[
\begin{align*}
\frac{d}{dt} X(\Gamma(t)) &\big|_{t \in \mathbb{I}} = X \left( \frac{d}{dt} \Phi(t; P) \right) = X(\nu(x(t), y(t))) \\
&= -\nu(x(t), y(t)) + 2ax(x(t), y(t))^T
\end{align*}
\]

Thus the vector field of \( (6.1) \) is nowhere tangent to \( \Gamma(t) \) because of \( \alpha = 0 \).

Finally, consider the domain with boundary \( \Gamma(\mathbb{I}) \). The flow of system \( (6.1) \) is either flowing into the domain, or flowing out of the domain. We can make the domain as small as we want by choosing \( P \) close enough to \( (0, y) \) or as big as possible by choosing \( P \) close enough to \( (0, R) \). We conclude that there is no other limit cycle in the interior of \( D(0, R) \).

Let \( R > R_p \). If \( \beta > -\beta/2 \), we can use Poincaré’s theorem that the interior of a periodic orbit for a planar vector field always contains an equilibrium point, see [19]. He proved this by observing that the index of the vector field along the periodic orbit is equal to one. This was the first application of his invention, in the same paper, of the concept of the index of a vector field, which was one of the founding ideas of algebraic topology. System \( (6.1) \) has no other critical points apart from those in the equator. Thus, the limit cycle could not exist. This idea also applicable in the case \( 0 < \beta < -\beta/2 \) and \( R > R_p \). If \( R_p < R < R_p \) is similar with the case \( R < R_p \).

**Corollary 6.2.** If \( \alpha = 0 \) all but the critical solution of \( (6.1) \) for \( R < R_b \) are periodic.

**Remark 6.3.** (The Bounded-Quadratic-Planar systems). In 1966, Coppel proposed a problem of identifying all possible phase portrait of the so-called Bounded-Quadratic-Planar systems. A Bounded-Quadratic-Planar system is a...
system of two autonomous, ordinary, first-order differential equations with quadratic nonlinearity where all solutions are bounded. The maximum number of limit cycles that could exist is one of the questions of Coppel. This problem turns out to be very interesting and not as easy as it seems. In fact, the answer to this problem contains the solution to the 16th Hilbert problem which is unsolved up to now (see \[6\]). System (6.1) is a Bounded-Quadratic-Planar system. From this point of view, Theorem 6.1 is an important result for our systems. This result enables us to compute all possible phase portraits of system (6.1).

From the previous section, one could guess that there are three situations for system (6.1), i.e. if \(\delta > -\beta/\omega\), \(0 < \delta < -\omega/2\), and \(\delta < 0\). For \(R\) close to zero but positive, the phase portrait of system (6.1) is similar in all three situations. The equator is an unstable periodic solution and there is only one equilibrium of system (6.1). There are three possible bifurcations of the equilibria in system (6.1), namely simultaneous saddle-node and homoclinic bifurcation, pitchfork bifurcation and saddle-node bifurcation.

### 6.2. A simultaneous saddle-node and homoclinic bifurcation

If \(R\) passes the value \(R_h\), system (6.1) undergoes a simultaneous saddle-node and homoclinic bifurcation (also called Andronov–Leontovich bifurcation, see [13, pp. 250–252]). If \(R < R_h\) the equator is a periodic solution. The period of this periodic solution goes to infinity as \(R\) approaches \(R_h\) from below. Exactly at \(R = R_h\), the limit cycle becomes an homoclinic to a degenerate equilibrium (with one zero eigenvalue). This degenerate equilibrium is created via a saddle-node bifurcation. This is clear since after the bifurcation (that is when \(R > R_h\)) we have two equilibria in the equator and the homoclinic orbit vanishes.

This bifurcation occurs in all three situations of system (6.1). The difference is that, in the case of \(\delta > 0\), the limit cycle at the equator is stable while if \(\delta < 0\) is unstable. This difference has a consequence for the stability type of the two equilibria at the equator after the saddle-node bifurcation.

### 6.3. A pitchfork bifurcation

The second bifurcation which occurs also in all three situation of system (6.1) is a pitchfork bifurcation, which is natural due to the presence of the symmetry \(\Phi_1 : (r, x, y) \mapsto (-r, x, y)\). However, there is a difference between the cases of \(\delta > -\beta/\omega\), \(0 < \delta < -\omega/2\) and the cases of \(\delta < 0\). In the first cases, the equilibrium which is inside the domain, collapses into the saddle-type equilibrium at the equator when \(R = R_p\). After the bifurcation (\(R > R_p\)) a stable (with two negative eigenvalues) equilibrium is created at the equator. The flow of system (6.1) after this bifurcation is then simple. We have two equilibria at the equator, one is stable with two-dimensional stable manifold and one is unstable with two-dimensional unstable manifold. The flow simply moves from one equilibrium to the other. This is the end of the story for the case \(\delta > -\beta/\omega\).

In the second cases (\(0 < \delta < -\omega/2\)), a saddle-type equilibrium branches out of the saddle-type equilibrium at the equator, at \(R = R_p\). The equilibrium at the equator then becomes a stable equilibrium with two-dimensional stable manifold.

In the third cases (\(\delta < 0\)), a stable focus branches out of the saddle-type equilibrium at the equator. After the bifurcation, we have four equilibria, two at the equator and two inside the domain. Both of the equilibria at the equator are of the saddle-type. One of the equilibria inside the domain is a stable focus while the other is unstable focus. There is no other bifurcation in the cases where \(\delta < 0\) (Fig. 3).

### 6.4. A saddle-node bifurcation

In the cases where \(0 < \delta < -\beta/\omega\) we have an extra bifurcation, i.e. a saddle-node bifurcation. Recall after a pitchfork bifurcation, inside the domain there is a saddle-type equilibrium. There is also a stable focus
Fig. 3. In the upper part of this figure, we present the phase portraits of system (6.1) as $R \to \infty$ for the case where $\delta > -\beta/2$. Passing through $R_p$, one of the equilibria of system (6.1) undergoes a pitchfork bifurcation. As $R$ passes through $R_h$ we have a saddle-node bifurcation which happens simultaneously with a homoclinic bifurcation. In the middle part of this figure, we draw the phase portraits of system (6.1) as $R \to \infty$ for the case where $\delta < -\beta/2$. In this case, before pitchfork bifurcation, there is a saddle-node bifurcation at $R = R_s$. In the lower part of the figure, there are the phase portraits of system (6.1) in the case $\delta < 0$. The two equilibria, collapse to each other in a degenerate equilibrium, if $R = R_s$. When $R > R_s$, the degenerate equilibrium vanishes. Therefore, we have a saddle-node bifurcation. We note that the location of the degenerate equilibrium plays an important role in the analysis of the normalized system (i.e., for $0 < \varepsilon \ll 1$). In the neighborhood of that point we find a Hopf bifurcation, see Section 8.

After the bifurcation, the phase portrait of system (6.1) is again similar with the cases where $\delta > -\beta/2$. We are left with two equilibria at the equators, one is stable, with two-dimensional stable manifold, and the other is unstable, with two-dimensional unstable manifold. The phase portraits of system (6.1) are plotted in Fig. 3.

6.5. Some degenerate cases

To complete the bifurcation analysis of system (6.1), let us turn our attention to the degenerate cases. We have three cases, i.e., $\alpha = 0$, $\beta = 0$, and $\delta = 0$. We only present the analysis for $\alpha = 0$. Note that if $\alpha = 0$, the vector field corresponding to system (5.1) is symmetric with respect to the y-axis. Instead of redoing the whole calculation again, we can also draw the conclusion by looking at Fig. 3 and make $x$-symmetric pictures out to them. If $\alpha = 0$, we have $R_s = R_p$ which means that the linearized system of system (5.1) around the equilibrium in the equator is zero when $R = R_p$. See Fig. 4 for the phase portraits of the system (6.1).

Remark 6.4. It is quite remarkable to have open domains where all solutions are periodic and, for some values of the parameters, a complementary open domains in which all solutions run from a source node to a sink.
Fig. 4. The phase portraits of system (6.1) as $R \to \infty$ for $\alpha = 0$ where we have an extra structure namely mirror symmetry (see also Remarks 6.4 and 6.5). The upper figure is for the case where $\delta \geq -\beta / 2$, the middle figure is for $0 < \delta < -\beta / 2$, while the lower figure is for $\delta < 0$.

node. In the domain of the periodic solutions one can construct an integral, an analytic function which is constant along the orbits and which separates the orbits, whereas in the complementary domain every integral is constant, equal to its value at the limiting sink and source point. In this way the $x$-symmetry leads to an integrability which is only valid in a part of the phase space. In the periodic domain the solutions of the perturbed system can be analyzed further by means of the averaging method, which may be an interesting project for future investigation.

Remark 6.5. Another point to state is how stable this structure is. The symmetric picture in Fig. 4 might change if we include higher order terms in our analysis. Furthermore, by adding higher order terms we can draw a relation between the situations in Fig. 3 (for $\alpha \neq 0$) and Fig. 4 (where $\alpha = 0$). Generically, when $\alpha$ passes zero, the critical points in the interior of the circle $x^2 + y^2 = R^2$ might undergo a Hopf bifurcation. If that is the case then the system (5.1) has a periodic solution either for $\alpha > 0$ or $\alpha < 0$. As a consequence, Fig. 3 also has to be revised. This might also be an interesting project for future investigation.

Our next goal is to turn on the perturbation $\epsilon$ to be nonzero. An immediate consequence of this is that $S(r, x, y) = R^2$ is no longer invariant under the flow of system (4.1).

7. The isolated nontrivial equilibrium

Let us now consider system (4.1) for $\epsilon \neq 0$, with $k_1 > 0$ and $k_2 > 0$. Recall that $\dot{S} = 2\epsilon(k_1r^2 - k_2(x^2 + y^2))$. Putting $\dot{S} = 0$ gives us an equation which defines a cone in $D$. This cone separates the phase space $D$ into two parts: the inner part where $S > 0$ and the outer part where $S < 0$. If an equilibrium of system (4.1) exists, then it must lie on the cone.
The location of the nontrivial equilibrium of system (4.1) is

\[
\begin{align*}
\xi_0(c) &= \frac{(1 + \beta/\delta) - \beta v}{(\beta v^2)} \frac{\epsilon}{\delta}, \\
x_0(c) &= -\frac{x_0}{\delta}, \\
y_0(c) &= \frac{(\epsilon v - \beta v)\xi}{(\beta v^2)}.
\end{align*}
\]

(7.1)

One can immediately see that (7.1) exists if and only if \((\beta v^2) \neq 0\).

To facilitate the analysis, let us write (7.1) as \((\pi_0(c), x_0(c), y_0(c))\) and correspondingly, the variables \(\xi = (c, x, y)\). In the variable \(\xi\) the system (4.1) is written as \(\dot{\xi} = H(\xi, \epsilon)\). Let us also name the cone \(\mathcal{S} = k_1 r^2 - k_2 (x^2 + y^2) = 0\) as \(C\) and the manifold of critical points (5.4) as \(E\).

Assuming that \(D_H(\xi, 0))\) has only one eigenvalue with zero real part, by the Center Manifold Theorem, there exists a coordinate system such that around \(\xi_0(0)\), system (4.1) can be written as

\[
\begin{align*}
\left(\begin{array}{c}
\xi_0(c) \\
\xi_0(c)
\end{array}\right) &= \left(\begin{array}{c}
A(c)\xi_0(c) \\
\lambda(c)\xi_0(c)
\end{array}\right) + \text{higher-order term},
\end{align*}
\]

(7.2)

where \(A(\epsilon)\) has no eigenvalue with zero real part and \(\lambda(0) = 0\). Let us choose \(\epsilon_1\) small enough such that the real part of the eigenvalues of \(A(c)\) remain nonzero for \(0 < \epsilon \leq \epsilon_1\).

Let \(W_0\) be the invariant manifold of system (7.2) which is tangent to \(E_{\xi_0(0)}\), where \(E_{\xi_0(0)}\) is the linear eigenspace corresponding to \(\lambda(c)\). We note that the Center Manifold Theorem gives the existence of \(W_0\). Also, \(W_0\) is the center manifold of \(\xi_0(0)\), which is, in our case, uniquely defined and tangent to \(E\) at \(\xi_0(0)\). Since \(E\) intersects \(C\) at \(\xi_0(0)\) transversally, for small enough \(\epsilon_2\) we have \(W_0(\epsilon)\) intersect \(C\) at \(\xi_0(c)\) transversally for \(0 < \epsilon \leq \epsilon_2\).

Lastly, \(E\) also intersects \(S(\mathcal{R})\) transversally, for \(|\mathcal{R} - \{\xi_0(0)\}| < c\) for some positive number \(c\). This follows from the assumption that \(D_H(\xi_0(0))\) has only one zero eigenvalue. Thus, there exists \(\epsilon_3\), small enough, such that \(W_0(\epsilon)\) intersects \(S(\mathcal{R})\) transversally for \(|\mathcal{R} - \{\xi_0(c)\}| < c\) and \(0 < \epsilon \leq \epsilon_3\). Choosing \(\epsilon^* = \min[\epsilon_1, \epsilon_2, \epsilon_3]\), we have proven the following lemma.

Lemma 7.1. Let us assume that \(D_H(\xi_0(0))\) has only one zero eigenvalue. There exists \(0 < \epsilon^* \in \epsilon; \) such that, for \(\epsilon \in (0, \epsilon^*)\), the system (7.2) has an invariant manifold \(W_0\) which is tangent to \(E_{\xi_0(0)}\) at \(\xi_0(c)\). This invariant manifold intersects the cone \(\mathcal{S} = 0\) transversally at \(\xi_0(c)\). It also intersects the sphere \(S(\mathcal{R})\) transversally, for all \(\mathcal{R} \neq \{\xi_0(0)\} \). The system (7.2), the conclusion we draw by invoking the analysis in Section 4. For the sign of \(\lambda(c)\) we have the following lemma.

Lemma 7.2. Consider the system (7.2). For \(\epsilon \in (0, \epsilon^*)\), we have \(\lambda(c) > 0\) if

1. \(\delta < \alpha\) and \(\epsilon^* > \alpha > 0\), or \(\alpha < \alpha\) and \(\epsilon^* > \alpha > 0\),
2. \(\delta - \beta v/\delta > \beta/\delta\).

Also for \(\epsilon \in (0, \epsilon^*)\), \(\lambda(c) < 0\) if

1. \(\delta < \alpha\) and \(\epsilon^* > \alpha > 0\), or \(\alpha < \alpha\) and \(\epsilon^* > \alpha > 0\),
2. \(\delta < -\beta/\delta\), or \(\epsilon^* < \alpha > 0\), or \(\alpha < \alpha\) and \(\epsilon^* > \alpha > 0\).

Proof. We only prove the first case of the first part of the lemma. The other cases can be proven in the same way.

From Lemma 7.1, we conclude that \(W_0\) intersects the cone \(C\) transversally. The situation for \(\delta < \alpha\) and \(\delta > \alpha\) is drawn in Fig. 5. The three concentric circles, marked by \(\mathcal{R}\), \(\mathcal{R}\) and \(\mathcal{R}\) are the intersection between the
Fig. 5. The continuous set of critical points $E$ for $\delta < 0$ and $\beta < 0$ is plotted on the figure above. The dashed lines represent the cone $C$. It separates the phase space into two parts, the expanding part (the shadowed area) and the contracting part. There are also three concentric circles drawn in this figure. The radius of these circles satisfies: $R' < R < R''$.

sphere $S(R')$, $S(R)$, and $S(R')$ with the plane $x = 0$, respectively. Note that $\max(|R' - R|, |R'' - R|, |R' - R|) < \varepsilon$.

As $\varepsilon$ becomes positive, an open subset of $E$ which contains $\xi^o$ can be continued with $\varepsilon$ and form the invariant slow manifold $W_\varepsilon$ with properties described in Lemma 7.1. Thus, we conclude that inside the shadowed area, the dynamics is moving from $S(R)$ to $S(R'')$. On the other side, the dynamics is moving from $S(R)$ to $S(R')$, i.e. $\lambda(\varepsilon) > 0$.

In Section 4 we left out a question whether the solutions of system (4.1) are bounded in the case $\delta < 0$. Using the same arguments as in Lemma 7.1 and the proof of Lemma 7.2, for $\varepsilon$ small enough we have the following result.

**Corollary 7.3.** If $\delta < 0$, $\alpha < 0$ and $\kappa_2^2 < \kappa_1$ then the solution of (4.1) is bounded.

**Proof.** If $\delta < 0$, $E$ is a hyperbola with two branches: the negative and positive branches. The negative branch is the one that passes through the origin. For $\alpha < 0$, the positive branch is attracting. Moreover, the positive branch is in the interior of $S < 0$. This ends the proof.

In the next section we are going to study the behavior near the boundary $\kappa_2/\kappa_1 = (\beta(2\delta + \beta))/\delta$.

8. Hopf bifurcations of the nontrivial equilibrium

The most natural thing to start with in doing the bifurcation analysis is to follow an equilibrium while varying one of the parameters in system (4.1). However, the analysis in the previous sections shows that we have no possibility of having more than one nontrivial critical point. Thus, we have excluded the saddle-node bifurcation of the nontrivial equilibrium of our system. Let us fix all parameters but $\delta$. We will use this parameter as our continuation parameter.

Recall that we have fixed $\beta < 0, \alpha < 0, \omega > 0$ and $\kappa_j > 0, j = 1, 2$.

Let $\delta > -\beta_1/(2\alpha_1 + \kappa_2)$ and consider the system (7.2). By Lemma 5.2, considering the chosen value of parameters: $\beta < 0, \alpha < 0, \omega > 0$ and $\kappa_j > 0$, $j = 1, 2$, we conclude that $\Re(\lambda_{1,2}) < 0$, where $\lambda_{1,2}$ are the eigenvalues of $A(0)$. Using Lemma 7.1, for small enough $\varepsilon$, $\Re(\lambda_{1,2}(\varepsilon)) < 0$, where $\lambda_{1,2}(\varepsilon)$ are the eigenvalues of $A(\varepsilon)$. If $\delta < -\beta_1/(2\alpha_1 + \kappa_2)$, by Lemma 7.1 we have $\lambda(\varepsilon) > 0$ and by Lemma 5.2, we have $\lambda_1 > 0$.

See the left figure of Fig. 6 where we have drawn an illustration for this situation. At $\delta = -\beta_1/(2\alpha_1 + \kappa_2)$ we have the situation where system (4.1) near $E(0)$ has a two-dimensional center manifold $W_c$. Locally, $W_c$ intersect

Fig. 6. In the left figure we draw the illustration for the situation in Theorem 8.1. \( W^0 \) is the center manifold of \( \xi^0(0) \). The three curves labeled by \( R_1, R_2 \) and \( R_3 \) are the intersection between the center manifold \( W_\varepsilon \) with \( S(R) \), where the label is the value of \( R \). In the right figure, we plot the two parameters numerical continuation of the Hopf point that we found if \( \delta > 0 \). The numerical data for this continuation are: \( \beta = -6, \omega = 3, \kappa_1 = 5, \kappa_2 = 1 \) and \( \varepsilon = 0.01 \).

\( S(R_j) \) transversally (thus, so does \( W_\varepsilon \) for small enough \( \varepsilon \)). At \( S(R_2) \), the analysis in the Section 6 shows that there are only two equilibria which are at the equator. It is easy to check that the dynamics is as depicted in Fig. 6. At \( S(R_1) \), as an equilibrium of system (6.1), \( \xi^0(0) \) has undergone a saddle-node bifurcation. Thus, it splits up into one stable equilibrium and one saddle-type equilibrium, which are drawn using a solid line and a dashed line, respectively. Again, the dynamics at \( S(R_1) \) is then verified. For \( \varepsilon \neq 0 \) but small, all of the dynamics is preserved. As an addition, we pick up a slow dynamics moving from one sphere to the other which is separated by the cone \( C \) which is the straight line in Fig. 6. This geometric arguments show that in the center manifold \( W_\varepsilon \), around \( \xi^0(\varepsilon) \), we have rotations. Thus, as \( \delta \) passes \(-\beta \kappa_1/(2\kappa_1 + \kappa_2)\), generically the nontrivial equilibrium undergoes a Hopf bifurcation.

**Theorem 8.1** (Hopf bifurcation I). Keeping \( \beta < 0, \omega < 0, \omega > 0 \) and \( \kappa_j > 0, j = 1, 2 \) fixed, the nontrivial equilibrium (7.1) undergoes a Hopf bifurcation in the neighborhood of \( \delta = -\beta \kappa_1/(2\kappa_1 + \kappa_2) \).

**Remark 8.2.** It is suggested by this study that if we singularly perturbed a saddle-node bifurcation we get a Hopf bifurcation. One could ask a question how generic is this phenomenon. The answer to this question can be found in the paper of Stiefenhofer [20]. Using blown-up transformations with different scaling (this is typical in singular perturbation problems), it is proved that this phenomenon is generic.

We check this with numerical computation for the parameter values: \( \alpha = -2, \beta = -6, \omega = 3, \kappa_1 = 5, \kappa_2 = 1 \) and \( \varepsilon = 0.01 \). We found Hopf bifurcation in the neighborhood of \( \delta = 2.81 \) while our analytical prediction is 2.73. We have to note that from our analysis it seems that the parameter \( \alpha \) does not play any role. However, the location of the nontrivial equilibrium depends on \( \alpha \). This might be the explanation for the rather large deviation of our analytical prediction of the bifurcation value \( \delta \), compared to the numerical result.

We can also vary \( \kappa_1 \) while keeping \( \delta \) fixed. Again, we find an agreement with our analytical prediction. In this experiment, we kept \( \alpha = -2, \beta = -6, \omega = 3, \kappa_2 = 1 \) and \( \varepsilon = 0.01 \). For \( \delta = 2 \) we found Hopf bifurcation if
Another Hopf bifurcation happens in the neighborhood of \( \alpha = 0 \). This is obvious from the bifurcation analysis of the system (6.1). We have the following result.

**Theorem 8.3 (Hopf bifurcation II).** If \( \delta < 0 \) or if \( \delta > -\beta / 2 \), keeping all other parameter fixed but \( \alpha \), the nontrivial equilibrium (7.1) undergoes a Hopf bifurcation in the neighborhood of \( \alpha = 0 \).

On the left figure of Fig. 6, we have plotted a two parameters continuation of the Hopf point in \((\alpha, \delta)\)-plane. One can see that for a large value of \( \delta \), a Hopf bifurcation occurs in the neighborhood of \( \alpha = 0 \). This is in agreement with Theorem 8.3. For \( \delta < \beta / 2 \approx 3 \) in our experiment, the Hopf curve is almost independent of \( \alpha \) just as it is predicted by Theorem 8.1. We find also another Hopf bifurcation close to \( \delta = 0 \). This branch actually belongs to the same curve. However, to see this bifurcation we need to re-scale the parameter which results in a different asymptotic ordering. We are not going into the details of this.

9. Numerical continuations of the periodic solution

In this section we present a one parameter continuation of the periodic solution created via Hopf bifurcation of the nontrivial critical point. This is in general a difficult task to do analytically. Using the numerical continuation software AUTO [5], we compute the one parameter continuations of the periodic solution.

9.1. A sequence of period-doubling and fold bifurcations

The numerical data that we use are the same as in the previous section: \( \omega = -2, \beta = -6, \omega = 3, \kappa_1 = 5, \kappa_2 = 1 \) and \( \epsilon = 0.01 \). We start with a stable equilibrium found for \( \delta = 4 \) and follow it with decreasing \( \delta \). Recall that in the neighborhood of \( \delta = 2.81 \) we find a Hopf bifurcation where a stable periodic solution is created.

We follow this periodic solution with the parameter \( \delta \). The periodic solution undergoes a sequence of period-doubling and fold bifurcations. In Fig. 7 we plot \( \delta \) against the period of the periodic solution. Also we attached the graph of the periodic solutions. For \( \delta \) in the neighborhood of 1.15, the periodic solution is unstable (except probably in some very small intervals of \( \delta \)). Moreover, the trivial and the nontrivial equilibria are also unstable. Since the solution is bounded, by forward integration we will find an attractor. We plotted the attractor and the Poincaré section of the attractor in the same figure. The attractor that we found by forward integrating is nonchaotic. It is not clear at the moment whether the attractor is periodic or not. The Poincaré section that we draw suggests that this is not a periodic solution.

Although a sequence of period-doubling and fold bifurcations usually leads to chaos, it seems that in our system it is not the case. In order to understand this, we do a two parameters continuation of the Hopf point. The parameters that we use are \( \delta \) and \( \epsilon \). Recall that we have fixed \( \alpha = -2, \beta = -6, \kappa_1 = 5, \) and \( \kappa_2 = 1 \).

In Fig. 7, we also plotted the result of the two parameters continuation of the Hopf point using \( \delta \) and \( \epsilon \). One can see that as the value of \( \epsilon \) increases, the distance between two Hopf bifurcations in parameter space becomes smaller. The stable periodic solution that comes out of the nontrivial equilibrium via the first Hopf bifurcation, collapses back into the nontrivial equilibrium via another Hopf bifurcation. For several values of \( \epsilon \) we plot the one parameter continuation of the periodic solution. This result gives us an indication that the sequence of period-doubling and fold bifurcations in our case is not an infinite sequence. We remark though that it is still possible that for \( \epsilon \) small enough, we might still find an infinite sequence of these bifurcations. We do not have that for \( \epsilon \geq 0.025 \).
Fig. 7. On the upper-left part of this figure we plot the sequence of period-doubling and fold bifurcations of the periodic solution. There also we have attached the periodic solution for four decreasing values of $\delta$. The attractor for $\delta = 1.1$ is drawn in the lower-left part of this figure while in the upper-right part is the Poincaré section of the attractor. The numerical data that we use are $\alpha = -2$, $\beta = -6$, $\omega = 3$, $\kappa_1 = 5$, $\kappa_2 = 1$ and $\epsilon = 0.01$. On the lower-right part of the figure, we plot the two parameters continuation of the Hopf point using $\epsilon$ and $\delta$. For several values of $\epsilon$, we do one parameter continuation of the resulting periodic solution.

Remark 9.1. It is also interesting to note that, based on these numerical studies, there is an indication that the behavior of the system (4.1) is actually much simpler if $\mu_1$ and $\mu_2$ are large. This observation is based on the fact that for $\epsilon > 0.114$, the nontrivial equilibrium is stable. The flow then collapses into this critical point, except inside the invariant manifold $r = 0$.

9.2. The slow–fast structure of the periodic solution

Let us now try to understand the construction of this periodic solution. From the previous discussion, one can see that exactly at the Hopf bifurcation point, the center manifold of the corresponding equilibrium is not tangent to the sphere $S(\delta \hat{R})$. This means that the periodic solution that is created after the bifurcation is a combination of slow and fast dynamics.

In Fig. 8 we have plotted the projections to the $(y,r)$-plane of the periodic solution for four values of $\delta$. On each plot, there are two dotted lines through the origin. These lines represent the cone $C$. Thus, the location of
the nontrivial equilibrium is in $O(\epsilon)$-neighborhood of the intersection point between one of the lines with the ellipse $\mathcal{E}$.

This periodic solution is created via Hopf bifurcation at $\delta \approx 2.81$. We draw the projection of the periodic solution at four values of $\delta$, i.e., $2.8194$,..., $1.8868$,..., $1.6834$,... and $1.4808$,... We also plotted the ellipse of equilibria and the cone $\dot{S} = 0$ using dotted lines. As $\delta$ decreases, the periodic solution gets more loops which is represented by the spikes in Fig. 8. This fits our analysis in Section 6 (see also Fig. 3). For $\epsilon = 0$ and $R > 0$ small enough, the equator of the sphere $r^2 + x^2 + y^2 = R^2$ is an unstable periodic solution of system (5.1), since $\alpha < 0$. However, the equator becomes less unstable when $\delta$ decreases (recall that the stability of the equator is determined by $\alpha \delta$, see (6.3)). Thus, the smaller $\delta$ is, the longer the periodic solution stays near the invariant manifold $r = 0$.

Recall that as $\delta$ decreases, the periodic solution described above also undergoes a sequence of period-doubling and fold bifurcations. Thus, apart from the periodic solution above, there are also some unstable periodic solutions with much higher period. Moreover, the periodic solution that we plotted in Fig. 8 is not necessarily stable.

9.3. Nonexistence of orbits homoclinic to the origin

In the system (4.1), the condition on the saddle value to have Shilnikov bifurcation can be easily satisfied (see [9] for the condition). However, we cannot have a homoclinic orbit in the normal form. The reason is quite straightforward.

In Theorem 3.1 we prove that the plane $r = 0$ is invariant under the flow of the normal form. It implies that the two-dimensional stable manifold of the equilibrium at the origin is $r = 0$. Thus, there is no possibility of having an orbit homoclinic to this critical point. Moreover, we cannot perturb the manifold away by including the higher order terms in the normal form. The existence of an orbit homoclinic to the origin in the full system is still an open question, which is not treated in this paper. Another possibility is to add some term that perturbed the invariant manifold $r = 0$ away. This can be done by introducing time-dependent perturbation, for instance: periodic forcing term or parametrically excited term. These are subjects of our further research.
Fig. 9. In this figure, on the left part we plot the torus that we find by continuing a periodic solution. This periodic solution is created via Hopf bifurcation at $\delta = 2.81$ and $\alpha = -2$. The torus is computed for the value of $\alpha = 6.8$. On the right-hand side, we plot the two parameters continuation of the torus (or Neimark–Sacker) bifurcation point, Hopf point and one branch of the period-doubling point. The vertical axis is $\alpha$ while the horizontal is $\delta$.

9.4. Neimark–Sacker bifurcation

Another interesting bifurcation that happens in system (2.3) is a torus bifurcation. Recall that the numerical data that we use are $\alpha = -2$, $\beta = -6$, $\omega = 3$, $\kappa_1 = 5$, $\kappa_2 = 1$ and $\epsilon = 0.01$. At $\delta = 2.81$ we find a Hopf bifurcation, and if we continue the periodic solution by varying $\delta$, we get a sequence of fold and period-doubling bifurcations as drawn in Fig. 7. Instead of following the stable periodic solution with $\delta$, we now follow it using $\alpha$. Around $\alpha = -0.9$, the periodic solution becomes unstable via period-doubling bifurcation. Around $\alpha = -0.2$, the periodic solution regains its stability by the same bifurcation. Around $\alpha = 6.7$, the periodic solution becomes neutrally stable. After this bifurcation, an attracting torus is created and it is drawn in Fig. 9 on the left. This is also known as Secondary Hopf or Neimark–Sacker bifurcation [13].

To complete the bifurcation analysis, in the same figure but on the right, we plot the two parameters continuation of the torus bifurcation, the Hopf bifurcation and the two period-doubling bifurcations mentioned above. Note that the two period-doubling bifurcations are actually connected. On that diagram we have indicated the region where we have a stable nontrivial critical point. Above the torus curve (the curve where the periodic solution becomes neutrally stable) we shaded a small domain. In that domain, we can expect to compute the torus numerically. Further away from the curve, the torus get destroyed and a new attractor is formed.

The torus curve ends in a codimension two point $Cd_2$, since its location is determined by two equations (which are represented by the two curves). There is still a lot of work that has to be done to be able to say something more about the behavior near this point. We are not going to do that in this paper. Also, near this point there is a lot of period-doubling and fold curves which are close to each other in the $(\delta, \alpha)$-plane. It is indeed interesting to devote some studies to the neighborhood of the point $Cd_2$.

Remark 9.2. In doing the numerical continuation, we found that to compute the two-dimensional torus in our system is cumbersome. The computation become less cumbersome if the value of $\kappa_1/\kappa_2$ is not large. For instance, our computation whose results are plotted in Fig. 9 is for $\kappa_1/\kappa_2 = 5$. If we decrease this value, it is easier to compute the torus since it survives in a larger set of parameters.
9.5. A heteroclinic connection

For \( \delta < 0 \), the nontrivial equilibrium undergoes a Hopf bifurcation in the neighborhood of \( \alpha = 0 \). Continuing this periodic solution using \( \alpha \) as the continuation parameter, we find a Neimark–Sacker bifurcation. Apart from this bifurcation, we do not find another codimension one bifurcation of the periodic solution. If \( \alpha < 0 \), in the previous analysis we show that negative branch of the hyperbola is repelling. If we choose, \( \kappa_2 \delta < \kappa_1 \beta \), by Corollary 7.3 we conclude that the solutions of system (2.3) are bounded. The trivial and the nontrivial equilibria are both unstable of the saddle-type. The trivial equilibrium has two-dimensional stable manifold \( W^s_0 \) (which is \( r = 0 \)) and one-dimensional unstable manifold which is exponentially close to the negative branch. On the other hand, the nontrivial equilibrium has two-dimensional unstable manifold \( W^u_\ell \) which is locally transversal to the negative branch, and one-dimensional stable manifold which is exponentially close to the negative branch. Generically, \( W^s_0 \) intersects \( W^u_\ell \) transversally in a one-dimensional manifold. This one-dimensional manifold lies in \( r = 0 \). However, in our system there is no other limit set in \( r = 0 \) apart from the origin. Thus, we conclude that the two manifolds do not intersect each other. Since the solutions are bounded, we conclude that \( W^u_\ell \) does not span to infinity. By these arguments, we numerically find an attracting periodic solution to where \( W^s_0 \) is attracted to. Moreover, the one-dimensional unstable manifold of the origin is connected with the one-dimensional stable manifold of the nontrivial critical point. We illustrate the situation in Fig. 10.

10. Concluding remarks

We have discussed in this paper the dynamics of a four-dimensional system of coupled oscillators with widely separated frequencies. In combination with an energy-preserving nonlinearity, it creates a system with rich dynamics of the slow–fast type in three-dimensional space. We have presented an analysis of the system in the case no unbounded solutions are possible. The case where we allow some solutions to be unbounded will be treated in the sequel to this paper.
We have completed the analysis for the energy-preserving part of the normal form. Although in a sense it is very special, we note that the energy-preserving part can be viewed as a Bounded-Quadratic-Planar system which has been extensively studied but in general still contains a lot of open problems. Extending this analysis for small perturbations, we can get a lot of information of the dissipative normal form.

10.1. On the energy exchanges

Although we leave out the forcing terms, there is energy exchange between the characteristic modes of our normal form. The main ingredient that we need for this energy exchange is \( \mu_1 \mu_2 < 0 \). Physically, this means one of the modes should be damped while the other is excited. This, however, is not a restrictive condition since if both modes are damped (or excited), clearly one would need an energy source (or an absorber) to have energy exchange.

In Fig. 11, we have plotted the Euclidean norm of two different solutions which can be viewed as the plot of the energy, against time. The left figure corresponds to the attractor which is plotted in Fig. 7 while the other is for the torus solution. In both figures, we see energy exchanges between one oscillator and the other. The size of the energy which is exchanged is large compared to \( \varepsilon \) which is 0.01.

Recall that the 1st oscillator is the excited one while the second is the damped one. Reading the plot of the left figure of Fig. 11, we see that the 2nd oscillator looses its energy. As a consequence the total energy of the system also decreases. Although it is not clear from the figure, the energy in the 1st oscillator is very small but nonzero (if it would be zero, it will stay zero forever). After some time, the energy of the 1st oscillator is strong enough (which implies the distance to the origin is far enough) for the nonlinear part of the vector field to play its role. Since the nonlinearity does not add nor take away energy (it is energy-preserving), the energy is then transferred to the 2nd oscillator. In Fig. 11 this is indicated by the sudden growth of energy of the 2nd oscillator. After a short while, the 1st oscillator looses almost all of its energy while the second regains its energy; then the process repeats itself. The behavior of the energy function for the torus solution does not differ much from the above description.

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**Fig. 11.** In this figure we plot the Euclidean norm of two different type of solutions. The graph which is labeled as “Total” is the graph of \( r^2 + x^2 + y^2 \). The graph which is labeled as “1st oscillator” is the graph of \( r^2 \) while “1st oscillator” is the graph of \( x^2 + y^2 \). In the left figure we plot the solution corresponding to the attractor in Fig. 7 while the right figure corresponds to the torus in Fig. 9.
10.2. Back to the atmospheric research

It is interesting to translate back the results in this paper to the atmospheric model. The model used in [4] is derived using EOFs as its basis functions. Thus, the state variables in this paper are actually the coefficients of these basis functions. The basis functions itself can be interpreted as dominant patterns in the atmosphere. An asymptotically stable nontrivial equilibrium that we find in this paper means an attracting pattern in the atmosphere. This pattern is a (nontrivial) linear combination of the EOF basis functions. A periodic solution would mean a periodic evolution from one pattern to the other and back.

Although in [4] the dynamics of the two modes with wide separation in the frequencies does not show interesting behavior, in this paper we show that the dynamics of such a system in general is rich. One of the conclusions of [4] is that the homoclinic orbit found in the five-modes system might be responsible for the long time-scale behavior in the system. The analysis in our paper gives an indication that although the homoclinic orbit is not there, a long time-scale behavior may appear because of the slow–fast structure in the system.

In relation with the results in [8,23] on how to prove that a three-dimensional system of differential equations is nonchaotic, we note that our system is more complex than theirs. The studies in [8,23] are concentrated on nonlinear three-dimensional systems having only at most five terms. Our normal form contains 11 terms. So far in our analysis we do not find chaotic behavior. It is evident in our system that we cannot have homoclinic orbits. This excludes the Shilnikov’s scenario for a route to chaos. Thus, whether our system is chaotic or not is still an open question.

It is also interesting to note that torus (or Neimark–Sacker) bifurcation usually is followed by a lot of chaos in the system in the presence of homoclinic tangencies (see for instance [3]). This may provide us with a way to find chaotic behavior in our system.

We left out several interesting questions from our analysis. Below we have listed several open questions.

The invariant manifold $r = 0$ can be perturbed away by perturbing the systems with small periodic forcing term or a parametrical excitation term. In the absence of this invariant manifold, we might find a homoclinic orbit that could lead to a lot of interesting dynamics. The complication is that we have to analyze a four-dimensional normal form.

The behavior (dynamics) of the system near the codimension two point: $C_{d2}$ is not analyzed in this paper. This type of codimension two point is treated carefully in the book by Kuznetsov [13]. One could for instance follow the periodic solution around the point $C_{d2}$ and compare the result with the cases studied in [13].

The global dynamics in the case of the absence of the nontrivial equilibrium is a very interesting case. This will be treated in a sequel to this paper.

Acknowledgements

J.M. Tuwankotta wishes to thank KNAW and CICAT TUDelft for financial support. He wishes to thank Hans Duistermaat, Ferdinand Verhulst (both from Universiteit Utrecht), Henk Broer (Rijksuniversiteit Groningen) and Daan Crommelin (Universiteit Utrecht and KNMI) for many discussions during this research; also Yuri Kuznetsov, Bob Rink, Thijs Ruigrok and Lennaert van Veen (all from Universiteit Utrecht) for many comments. He also thanks Santi Goenarso for her support in various ways.

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